

A description of collineations-groups of an affine plane

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Abstract: Based on the following very interesting work in the past [2], [3], [4], [9], [12], this article becomes a description of collineations in the affine plane [10]. We are focusing at the description of translations and dilatations, and we make a detailed description of them. We describe the translation group and dilatation group in affine plane [11]. A detailed description we have given also for traces of a dilatation. We have proved that translation group is a normal subgroup of the group of dilatations, wherein the translation group is a commutative group and the dilatation group is just a group. We think that in this article have brings about an innovation in the treatment of detailed algebraic structures in affine plane.

Keywords: Affine plane, collineations, translation, dilatation, trace of points, dilatation group.

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1 The collineation group

Definition 1.1. Let $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an affine plane and $\mathcal{S} = \{\psi : \mathcal{P} \rightarrow \mathcal{P} \mid \text{where } \psi \text{—is bijection}\}$ set of bijections to set points $\mathcal{P}$ on yourself. Collineation of affine plane $\mathcal{A}$ called a bijection $\psi \in \mathcal{S}$, such that

$$\forall \ell \in \mathcal{L}, \psi(\ell) \in \mathcal{L}, \quad (1.1)$$

Otherwise, a collineation of the affine plane $\mathcal{A}$ is a bijection of set $\mathcal{P}$ on yourself [14], that preserves lines. It is known that the set of bijections to a set over itself is a group on associated with the binary action " $\circ$ " of composition in it, which is known as total group or symmetric groups [1], [6], [7], [8], [13].

In a collineation ψ of an affine plans, image $\psi(P)$ to a point P to plans often mark briefly P' .

Proposition 1.2. *Every bijections of set to points $\mathcal{P}$ on yourself to affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a his collineation.*

Proof. Let it be a bijection $\psi : \mathcal{P} \rightarrow \mathcal{P}$ and an line $\ell \in \mathcal{L}$. From [11],[12] , the line ℓ is incident with two different points $P, Q \in \mathcal{P}$, which the bijection ψ leads in two different points $P' = \psi(P)$ and $Q' = \psi(Q)$. From axiom A1 of the affine plane [10], [11] these points define a single line $P'Q' \in \mathcal{L}$, namely, $P'Q' = \psi(PQ) = \psi(\ell) \in \mathcal{L}$. $\square$

Corollary 1.3. *Collineation set $\mathbf{Col}_{\mathcal{A}}$ of the affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ forms symmetrical groups about the composition "o", namely $(\mathbf{Col}_{\mathcal{A}}, \circ)$ is symmetric group.*

It is clear that identical bijections $id_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{P}$ is a collineation of the affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, we call identical collineations of $\mathcal{A}$. In this collineation, every point of $\mathcal{P}$ passes itself, as well as every line $\mathcal{L}$ passes itself.

Definition 1.4. An point P of the affine plan $\mathcal{A}$ called fixed point his associated with a collineation δ , if coincides with the image itself $\delta(P)$, briefly when

$$P = \delta(P).$$

According to this definition, we have this

Proposition 1.5. *Every point of the affine plane, is a fixed point related to his identical collineation.*

2 The dilatation group

Definition 2.1. [3], [4] Dilatation of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ called a its collineation δ such that

$$\forall P \neq Q \in \mathcal{P}, \delta(PQ) \parallel PQ \quad (2.1)$$

According to axiom A1 of the affine plane definitions [10], [11] , [12]. , the line that passes through two different points P, Q we have written PQ . From the fact that dilatations δ is bijection, worth implication

$$P \neq Q \Leftrightarrow \delta(P) \neq \delta(Q) \quad (2.2)$$

therefore line $\delta(PQ)$ also written $\delta(P)\delta(Q)$ and definition 2.1, in these circumstances, takes the view

$$\forall P \neq Q \in \mathcal{P}, \delta(P)\delta(Q) \parallel PQ \quad (2.3)$$

It is clear that identical collineations $id_{\mathcal{P}}$ of an affine plan $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, is an dilatation of his, who called his identical dilation.

However, by being bijection an dilatation δ of an affine plane, the his inverse δ^{-1} is also bijection. It's a dilatation his affine plan the bijections δ^{-1} ? By (2.3), easily shown that is true this

Proposition 2.2. *Inverse bijections δ^{-1} of a dilatations δ of an affine plan is also an dilatation of that plan.*

Proposition 2.3. *Composition of two dilatations of a affine plan is agin an dilatations of his.*

Proof. Let's be δ_1 and δ_2 two dilatations of the affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$. For any two points of the plan, according to (2.2) have:

$$P \neq Q \iff \delta_1(P) \neq \delta_1(Q)$$

and

$$\delta_1(P) \neq \delta_1(Q) \iff \delta_2(\delta_1(P)) \neq \delta_2(\delta_1(Q))$$

From the first equivalence according to (2.3), we have

$$\delta_1(P) \delta_1(Q) \parallel PQ$$

and from the second we have

$$\delta_2(\delta_1(P)) \delta_2(\delta_1(Q)) \parallel \delta_1(P) \delta_1(Q)$$

the parallelism relation in $\mathcal{A}$ is a equivalence relation [4], [10], [11] therefore by its transition properties take:

$$\delta_2(\delta_1(P)) \delta_2(\delta_1(Q)) \parallel PQ$$

otherwise,

$$(\delta_2 \circ \delta_1)(P) (\delta_2 \circ \delta_1)(Q) \parallel PQ$$

In conclusion,

$$\forall P \neq Q \in \mathcal{P}, (\delta_2 \circ \delta_1)(P) (\delta_2 \circ \delta_1)(Q) \parallel PQ$$

$$\forall P \neq Q \in \mathcal{P}, (\delta_2 \circ \delta_1)(P) (\delta_2 \circ \delta_1)(Q) \parallel PQ$$

which according to (2.3), indicates that the composition $\delta_2 \circ \delta_1$ of dilatations δ_2 and δ_1 of affine plane $\mathcal{A}$ is also its dilation. $\square$

Let it be $\mathbf{Dil}_{\mathcal{A}} = \{\delta \in \mathbf{Col}_{\mathcal{A}} | \delta - \text{is a dilatation of } \mathcal{A}\}$ the dilatation set of affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$. As such it is the subset of collineations $\mathbf{Col}_{\mathcal{A}}$. Propositions 2.3, indicates that $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ is a sub-structure of the symmetric group $(\mathbf{Col}_{\mathcal{A}}, \circ)$ of collineations of the affine plane $\mathcal{A}$. Propositions 2.2, indicates that this sub-structure is a sub-group of the group $(\mathbf{Col}_{\mathcal{A}}, \circ)$, [1], [6], [7], [8], [13]. Is obtained in that way this

Theorem 2.4. *The dilatation set $\mathbf{Dil}_{\mathcal{A}}$ of affine plane $\mathcal{A}$ forms a group with respect to composition $\circ$.*

Definition 2.5. Let it be δ an dilatation of affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, P his one point. Lines that passes by P and $\delta(P)$, called trace of points P regarding dilatations δ .

If $P \neq \delta(P)$, then according to axiom A1, trace $P\delta(P)$ is the only. And when $P = \delta(P)$, trace the point P regarding δ will mark $P\delta(P)$. In this case the trace $P\delta(P)$ is not only after, according to a proposition in the affine plane [5], [10], each point is incident with at least three lines, which have from this, the point P has at least three traces associated with dilatation δ .

Theorem 2.6. *For an point P to an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, not fixed related to an his dilatation δ , is true propositions*

$$\forall Q \in \mathcal{P} - \{P\}, Q \in P\delta(P) \implies \delta(Q) \in P\delta(P).$$

Otherwise every point of a traces of a not-fixed point, to an affine plane associated with its dilation has its own image associated with that dilation in the same traces.

Proof. Given that $Q \neq P$ and δ is a dilatation of plane $\mathcal{A}$, then according to (2.2), $PQ \parallel \delta(P)\delta(Q)$. From condition, $Q \in P\delta(P)$, that implies $Q, P, \delta(P) \in PQ$. By parallelism of lines $PQ, \delta(P)\delta(Q)$ and by the fact that points $\delta(P)$ is the common point of their, results:

$$PQ = \delta(P)\delta(Q)$$

that implies

$$\delta(Q) \in PQ = \delta(P)\delta(Q)$$

□

Now formulate a constructive character theorem of an dilatations for affine plane. Mark with ℓ_m^P the line who fulfills the conditions $P \in \ell_m^P$ and $\ell_m^P \parallel m$.

Theorem 2.7. [3] *If two different assigned points P, Q of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, are defined their image $P' = \delta(P)$ and $Q' = \delta(Q)$ by an his dilatations $\delta \neq id_{\mathcal{A}}$, then image $R' = \delta(R)$ of an other points $R \in \mathcal{P} - \{P, Q\}$ determined as follows:*

$$\begin{aligned} R \notin PQ &\implies \delta(R) = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'} \\ R \in PQ &\implies \exists S \in \mathcal{P}, S \notin PQ, \delta(R) = \ell_{RP}^{P'} \cap \ell_{RS}^{S'} \end{aligned}$$

Proof. **I)** Examine the first case when the point R is not line incidents PQ : ($R \notin PQ$). Distinguish three sub-cases.

Case 1: $R \notin PP'$ and $R \notin QQ'$. In this case we have $P \neq P'$ and $Q \neq Q'$, after that, if the accept for example $Q = Q'$, then $RQ = RQ'$ and by (2.2) we have $R'Q' \parallel RQ$. Hence:

$$R'Q' \parallel RQ \xrightarrow{A.1} R'Q' = RQ = RQ \implies R \in R'Q'.$$

contrary to condition.

Constructing in P' the line $\ell_{RP}^{P'}$, that is parallel to RP . From (2.2), $R'P' \parallel RP$. Hence, the axiom A2 of affine plane we have that $\ell_{RP}^{P'} = R'P'$. Consequently $R' \in \ell_{RP}^{P'}$. Constructing now Q' the line $\ell_{RQ}^{Q'}$, that is parallel with RQ .

Same as above comes out that $R' \in \ell_{RQ}^{Q'}$. Well,

$$R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'}$$

Results in this manner the true proposition

$$\forall R \in \mathcal{P} - \{P, Q\}, R \notin PQ \implies \delta(R) = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'} \quad (2.4)$$

indicating that $R' = \delta(R)$, defined as cutting points of the lines:

$$\ell_{RP}^{P'} = \delta(RP) \quad \text{and} \quad \ell_{RQ}^{Q'} = \delta(RQ).$$

Case 2: $R \in PP'$ or $R \in QQ'$, concrete terms with the first. From (2.2), $R'P' \parallel RP$, implicates that $R' \in RP = \ell_{RP}^{P'}$. As in the case 1, shows that $R' \in \ell_{RQ}^{Q'}$. Consequently we have $R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'}$.

Case 3: R is incident with both tracks PP', QQ' . Since $R \notin PQ$, then PP', QQ' are different, therefore, according to A1, they meet at point $R : R = PP' \cap QQ'$. Just like the case 2 proved that

$$R \in PP' \implies R' \in RP = \ell_{RP}^{P'}$$

and

$$R \in QQ' \implies R' \in RQ = \ell_{RQ}^{Q'}$$

hence

$$R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'} = RP \cap RQ = R. \quad (2.5)$$

II) Now examine another case, when the point R is incident with the line PQ : $R \in PQ$. According to axiom A3, in affine plane has a point S , non-incident with PQ . Is clear that $R \notin PS$ (otherwise PS and PQ they will be coincide according to Axiom A1, because they will pass by two different points P and R , which would implicate the wrong conclusion that $S \in PQ$).

Since $S \notin PQ$, according to (2.4), constructed $S' = \ell_{SP}^{P'} \cap \ell_{SQ}^{Q'}$, $\delta(S) = S'$. But also $R \notin PS$, hence according to (2.4) next constructed the $R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'}$. In summary the construction of point $R' = \delta(R)$ in these conditions is presented in the form

$$\forall R \in \mathcal{P} - \{P, Q\}, R \in PQ \implies \exists S \notin PQ, S' = \ell_{SP}^{P'} \cap \ell_{SQ}^{Q'} \text{ and } R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'} \quad (2.6)$$

□

Is clear that when the tracks $P\delta(P), Q\delta(Q)$ the two points P, Q the affine plane are cutting, then definitely $P \neq Q$. Therefore, from (2.5) and this proves true

Corollary 2.8. *If the tracks $P\delta(P), Q\delta(Q)$ the two points P, Q of an affine plane are expected, then their cutting points $P\delta(P) \cap Q\delta(Q)$ is a fixed point related to his dilatations δ .*

Corollary 2.9. *If an point Q of affine plans is to trace $P\delta(P)$ to an his point P , then the its image $\delta(Q)$ locates at the a trace.*

Otherwise, in an affine plan, line, which is an a trace of his points by an dilatation, is a trace for every other points of it

Proof. We distinguish two cases.

Case 1: The point Q is fixed-point in connection with dilatation δ . In this case we have to:

$$\delta(Q) = Q$$

Since from corollary condition, we have to point $Q \in P\delta(P)$, we have to

$$\delta(Q) \in P\delta(P)$$

Case 2: The point Q is not fixed-point in connection with dilatation δ . The corollary of proof we have from Theorem 2.6.

□

Corollary 2.10. *Two dilatations $\delta_1 \neq id_{\mathcal{P}}$ and $\delta_2 \neq id_{\mathcal{P}}$ of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, are equal if and only if, when two points $P \neq Q \in \mathcal{P}$ are simultaneously true equations*

$$\delta_1(P) = \delta_2(P) \text{ and } \delta_1(Q) = \delta_2(Q) \quad (2.7)$$

Otherwise, an dilatation $\delta \neq id_{\mathcal{P}}$ of an affine plane is completely determined by giving his image according to two different points of the plans.

Proof. For two dilatations δ_1 and δ_2 , valid the implication

$$\delta_1 = \delta_2 \iff \forall R \in \mathcal{P}, \delta_1(R) = \delta_2(R) \quad (2.8)$$

after being reflections, for two reflections $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$, valid the implication [8]

$$f = g \iff \forall x \in X, f(x) = g(x). \quad (2.9)$$

From here, when dilations $\delta_1 \neq id_{\mathcal{P}}$ and $\delta_2 \neq id_{\mathcal{P}}$ are equally, particularly for points $P \neq Q \in \mathcal{P}$, we have

$$\delta_1(P) = \delta_2(P) \text{ and } \delta_1(Q) = \delta_2(Q).$$

Conversely, let's have $\delta_1(P) = \delta_2(P)$ and $\delta_1(Q) = \delta_2(Q)$, and prove to $\delta_1 = \delta_2$. From the condition, equation is true for $R = P$ and for $R = Q$. On a different point, i.e a point $R \in \mathcal{P} - \{P, Q\}$, in case when $R \notin PQ$, according to (2.4) and (2.6), we have

$$\delta_1(R) = \ell_{RP}^{\delta_1(P)} \cap \ell_{RQ}^{\delta_1(Q)} = \ell_{RP}^{\delta_2(P)} \cap \ell_{RQ}^{\delta_2(Q)} = \delta_2(R)$$

but also in case when $R \in PQ$, according to (2.5), (2.6) and to above, we have

$$\exists S \notin PQ, \delta_1(S) = \ell_{SP}^{\delta_1(P)} \cap \ell_{SQ}^{\delta_1(Q)} = \ell_{SP}^{\delta_2(P)} \cap \ell_{SQ}^{\delta_2(Q)} = \delta_2(S)$$

Well, $\forall R \in \mathcal{P}, \delta_1(R) = \delta_2(R)$, that according to (2.7), shows that $\delta_1 = \delta_2$. $\square$

Theorem 2.11. *For every dilatation $\delta \neq id_{\mathcal{P}}$ of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, exists in the plane least two not fixed points about what dilatation.*

Proof. The fact that dilatation $\delta \neq id_{\mathcal{P}}$ imply the existence of at least one point P in plane $\mathcal{A}$ that is not the fixed-point connected to δ , namely $\delta(P) \neq P$. Of course exist also another point $Q \in \mathcal{P}$, such that $\delta(Q) \neq Q$.

On the contrary, if $\forall Q \in \mathcal{P} - \{P\}$ we would have $\delta(Q) = Q$, then

$$\delta(PQ) \parallel PQ \implies \delta(P) = P,$$

in contradiction the fact that the point P is not the fixed-point. $\square$

Theorem 2.12. *If an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, has two fixed points about an dilatation then he dilatation is identical dilatation $id_{\mathcal{P}}$ of his.*

Proof. Let's be P, Q two fixed points, in relation to an dilatation δ and R another point of affine plane $\mathcal{A}$. If $R \notin PQ$, according to (2.4), the image of her R' is

$$R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'} = \ell_{RP}^P \cap \ell_{RQ}^Q,$$

because P, Q are the fixed points, in relation to an dilatation δ . But $\ell_{RP}^P \cap \ell_{RQ}^Q = R$, that imply $R' = R$. If the point $R \in PQ$, from axioms A2, exist a point S such that $S \notin PQ$. According to (2.4), image of her S' is

$$S' = \ell_{SP}^{P'} \cap \ell_{SQ}^{Q'} = \ell_{SP}^P \cap \ell_{SQ}^Q = S,$$

that imply $S' = S$. Then, according to (2.5),

$$R' = \ell_{RP}^P \cap \ell_{RS}^S = R.$$

□

According to this theorem, if related to an dilatation $\delta \neq id_{\mathcal{P}}$ of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, plan has a fixed point, he can not have any other fixed point, because otherwise, it would be identical dilatation $id_{\mathcal{P}}$. So we have this

Corollary 2.13. *For every dilatation $\delta \neq id_{\mathcal{P}}$ to an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ in the plan has an fixed point with respect to that dilatation, then it is only.*

Theorem 2.14. *For every dilatation $\delta \neq id_{\mathcal{P}}$ to an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, which has a fixed point V associated with it dilatation, is true propositions*

$$\forall P \in \mathcal{P}, V \in P\delta(P),$$

otherwise, all the tracks regarding with dilatation δ crossed in the point V .

Proof. The proposition (1.5) is evident for $P = V$. For $P \neq V$ have $P \neq \delta(P)$, because, according to corollary, the fixed-point V associated with that dilatation δ is the only in the plane $\mathcal{A}$. Whereas, the fact that δ is a dilatation, imply $VP \parallel V\delta(P)$, because $V = \delta(V)$. The lines VP and $V\delta(P)$ have in common point V , therefore $VP = V\delta(P)$, that imply $V \in P\delta(P)$. □

Theorem 2.15. *An affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, has not fix point related to an dilatation $\delta \neq id_{\mathcal{P}}$ then and only then, when all the tracks $P\delta(P)$ for all $P \in \mathcal{P}$ are parallel between themselves.*

Proof. If the tracks by dilation δ in plane $\mathcal{A}$, are parallel between them, then the plane has not fixed point by dilatation δ , when on the contrary, by corollary of Theorem 2.12, those would not be parallel.

Conversely, we accept that the plan $\mathcal{A}$ has not fixed point by dilatation $\delta \neq id_{\mathcal{P}}$ and prove that all the tracks $P\delta(P)$ for all $P \in \mathcal{P}$ are parallel between them. Let's be P, Q two random points of the affine plane $\mathcal{A}$. The case where $P = Q$ is evident. And when $P \neq Q$, again their tracks are parallel, because if we accept that $P\delta(P) \nparallel Q\delta(Q)$, then their cross cutting, according to corollary 2.8, of the Theorem 2.7, will be the fixed point of $\mathcal{A}$, associated with dilations δ , that is in contradiction with the condition. Further, the parallelism between all traces derived from the fact that the parallel lines of a affine plane, is equivalence relation. $\square$

This Theorem we can give also this wording:

For every dilatation $\delta \neq id_{\mathcal{P}}$ of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, worth propositions

$$"\mathcal{A} \text{ has not fixed point by } \delta" \iff \forall P, Q \in \mathcal{P}, P\delta(P) \parallel Q\delta(Q) \quad (2.10)$$

The last two theorems summarized in this

Proposition 2.16. *In an affine plane related to dilatation $\delta \neq id_{\mathcal{P}}$ all traces $P\delta(P)$ for all $P \in \mathcal{P}$, or cross the by a single point, or are parallel between themselves.*

3 The traslations groups

By Propositions 2.16, in an affine plane all traces related an dilatation of his or cross the by a single point, or are parallel between themselves. This fact leads us to this

Definition 3.1. [3],[4],[9],[12], Translation of an affine plans $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, called identical dilatation $id_{\mathcal{P}}$ his and every other of its dilatation, about which heaffine plane has not fixed points.

If σ is an translation different from identical translation $id_{\mathcal{P}}$, then, by Theorems 2.15, all traces related to σ form the a set of parallel lines. According to a proposition in the affine plane [4], [10], which have from this, at every point $P \in \mathcal{P}$ pass at least three lines out $\mathcal{L}$, among which only one is its a trace of translations σ . Because $\parallel$ parallelism relation on $\mathcal{L}$, is an equivalence relation, see [10], then $\pi = \mathcal{L}/\parallel$ is an a cleavage of $\mathcal{L}$ in the equivalence classes by parallelism, see [10], [11], [12]. Each class has representative an line that passes from of random point P .

Definition 3.2. For one translation $\sigma \neq id_{\mathcal{P}}$, equivalence classes of the cleavage $\pi = \mathcal{L}/\parallel$, which contained tracks by σ of points of the plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ called the direction of his translations σ and marked π_{σ} .

So, for $\sigma \neq id_{\mathcal{P}}$, the direction π_{σ} represented by single the trace by σ every point $P \in \mathcal{P}$, for translation $id_{\mathcal{P}}$. We say that there are undefined direction. Otherwise we say that has the same the direction with every other translations σ of the plane $\mathcal{A}$, namely true accept the propositions:

$$\text{For every translation } \sigma \text{ of the plane } \mathcal{A}, \pi_{id_{\mathcal{P}}} = \pi_{\sigma} \quad (3.1)$$

Subject of review at this point would be the set of translations of the affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$:

$$\mathbf{Tr}_{\mathcal{A}} = \{\sigma \in \mathbf{Dil}_{\mathcal{A}} \mid \sigma \text{ is translation of } \mathcal{A}\}.$$

Let it be $\alpha : \mathbf{Tr}_{\mathcal{A}} \rightarrow \mathbf{Tr}_{\mathcal{A}}$, an whatever application of $\mathbf{Tr}_{\mathcal{A}}$, on yourself. For every translation σ , its image $\alpha(\sigma)$ is again an translation, that can be $\alpha(\sigma) = id_{\mathcal{P}}$ or $\alpha(\sigma) \neq id_{\mathcal{P}}$. So there is a certain direction or indefinitely. The first equation, in the case where $\sigma = id_{\mathcal{P}}$, takes the view $\alpha(id_{\mathcal{P}}) = id_{\mathcal{P}}$, and the second $\alpha(\sigma) \neq id_{\mathcal{P}}$, that it is not possible to α application. To avoid this, yet accept that for every application $\alpha : \mathbf{Tr}_{\mathcal{A}} \rightarrow \mathbf{Tr}_{\mathcal{A}}$, is true equalization:

$$\alpha(id_{\mathcal{P}}) = id_{\mathcal{P}}. \quad (3.2)$$

Theorem 3.3. If a certain point P of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$. Its image is determined $P' = \sigma(P)$ according to an his translations $\sigma \neq id_{\mathcal{P}}$, then image $Q' = \sigma(Q)$ a other point $Q \in \mathcal{P} - \{P\}$ determined as follows:

$$Q \notin PP' \implies \sigma(Q) = \ell_{PP'}^Q \cap \ell_{PQ}^{P'} \quad (3.3)$$

$$Q \in PP' \implies \exists S \notin PP', S' = \ell_{PP'}^S \cap \ell_{PS}^{P'} \text{ and } \sigma(Q) = PP' \cap \ell_{SQ}^{S'} \quad (3.4)$$

Proof. First consider the case when the point Q is not the point of the traces PP' of points P according to σ . Then, by Theorem 2.15, the QQ' its trace is parallel to PP' , therefore $QQ' = \ell_{PP'}^Q$, indicates that $Q' \in \ell_{PP'}^Q$. But by being translations, σ is a dilatation therefore PQ is parallel to $P'Q' = \ell_{PQ}^{P'}$, also indicates that $Q' \in \ell_{PQ}^{P'}$. Results so that the image Q' is a cross cutting of the lines $\ell_{PP'}^Q$ and $\ell_{PQ}^{P'}$ so $Q' = \ell_{PP'}^Q \cap \ell_{PQ}^{P'}$.

Consider now another case, when the point Q is a point of trace PP' of points P according to σ . According to axiom A3, in affine plane, see [4], [5], [10], [11], [12] exists a point S no incidents with trace PP' . According to (3.3), constructed its image $S' = \ell_{PP'}^S \cap \ell_{PS}^{P'}$. But also Q not is a incidents with trace SS' , therefore according to (3.3), further constructed also $Q' = \ell_{SS'}^Q \cap \ell_{SQ}^{S'}$. In summary the construction of image $Q' = \sigma(Q)$, in these conditions is presented in the form:

$$Q \in PP' \implies \exists S \notin PP', S' = \ell_{PP'}^S \cap \ell_{PS}^{P'} \text{ and } \sigma(Q) = PP' \cap \ell_{SS'}^{Q'}. \quad \square$$

Corollary 3.4. *Two translations $\sigma_1 \neq id_P$, $\sigma_2 \neq id_P$, of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, are equal only when for a point $P \in \mathcal{P}$, is true equalization*

$$\sigma_1(P) = \sigma_2(P) \quad (3.5)$$

Proof. From the definition of translations that have: since the translations σ_1 and σ_2 are different from the identical translation, then these translations not have the fixed points. If have which $\sigma_1 = \sigma_2$ it is obvious which $\forall P \in \mathcal{P}, \sigma_1(P) = \sigma_2(P)$. *Conversely:* Let's have a point $P \in \mathcal{P}$, to which is true equalization $\sigma_1(P) = \sigma_2(P)$. Now take another whatever point $Q \in \mathcal{P}$, by Theorem 3.3, we have which $\sigma_1(Q)$ and $\sigma_2(Q)$ defined by the equations (3.3) and (3.4).

In the case where $Q \notin P\sigma_1(P) = P\sigma_2(P)$, by the equations (3.4) we have:

$$\sigma_1(Q) = \ell_{P\sigma_1(P)}^Q \cap \ell_{PQ}^{\sigma_1(P)} = \ell_{P\sigma_2(P)}^Q \cap \ell_{PQ}^{\sigma_2(P)} = \sigma_2(Q) \implies \sigma_1(Q) = \sigma_2(Q)$$

In the case where $Q \in P\sigma_1(P) = P\sigma_2(P)$, by the equations (3.4) we have:

$$Q \in P\sigma_1(P) = P\sigma_2(P) \implies \exists S \notin P\sigma_1(P) = P\sigma_2(P),$$

$$S' = \ell_{P\sigma_1(P)}^S \cap \ell_{PS}^{\sigma_1(P)} = \ell_{P\sigma_2(P)}^S \cap \ell_{PS}^{\sigma_2(P)},$$

and

$$\sigma_1(Q) = P\sigma_1(P) \cap \ell_{SS'}^{Q'} = P\sigma_2(P) \cap \ell_{SS'}^{Q'} = \sigma_2(Q).$$

So we have which:

$$\forall Q \in \mathcal{P}, \sigma_1(Q) = \sigma_2(Q) \implies \sigma_1 = \sigma_2. \quad \square$$

Otherwise, one translations $\sigma \neq id_P$ of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, is completely determined by giving her the likeness to an point according to the plane.

Proposition 3.5. *The inverse translation σ^{-1} of an translation σ to an affine plane $\mathcal{A}=(\mathcal{P},\mathcal{L},\mathcal{I})$, is also an translation in the affine plane.*

Proof. If $\sigma = id_{\mathcal{P}}$, then, by $id_{\mathcal{P}}^{-1} = id_{\mathcal{P}}$ as bijection, it turns out that σ is translation. If $\sigma \neq id_{\mathcal{P}}$, then according to her in the plan has not fixed point. Suppose that σ^{-1} is not translation. As an dilatation, from the Proposition 2.16, in the plane has a fixed point P according to σ^{-1} , for which we have $\sigma^{-1}(P) = P$. But

$$\sigma(\sigma^{-1}(P)) = \sigma(P) \iff P = \sigma(P),$$

proving that the P is fixed-point also for σ , in contradiction with condition. $\square$

Inasmuch as $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = id_{\mathcal{P}}$, σ^{-1} is the inverse translation of the translation σ .

Corollary 3.6. *For every translation σ to an affine plane $\mathcal{A}=(\mathcal{P},\mathcal{L},\mathcal{I})$, σ and σ^{-1} have the same direction.*

Proof. It is evident that $\sigma \neq id_{\mathcal{P}} \iff \sigma^{-1} \neq id_{\mathcal{P}}$. Well, the translations σ and σ^{-1} they have determined directions when $\sigma \neq id_{\mathcal{P}}$. The trace of a point P according to σ is the line $P\sigma(P)$, which represents the direction π_{σ} , while the trace of point $\sigma(P) = P'$ according to σ^{-1} is the line $P'\sigma^{-1}(P')$, which represents the direction $\pi_{\sigma^{-1}}$. But $\sigma(P) = P' \iff P = \sigma^{-1}(P')$, hence $P'\sigma^{-1}(P') = P\sigma(P)$, which represents the direction $\pi_{\sigma} = \pi_{\sigma^{-1}}$. According to (3.2), this equation is true even when $\sigma = id_{\mathcal{P}}$. $\square$

Proposition 3.7. *The composition of the two translations in affine plane is again a translation of his.*

Proof. Let's be σ_1, σ_2 two translations of a affine plane. Having been bijections, if one from the translations σ_1, σ_2 is equal to $id_{\mathcal{P}}$, then $\sigma_2 \circ \sigma_1 = id_{\mathcal{P}}$. In this case, from the Definition 3.1, that production is a translation. Even if $\sigma_1 \neq id_{\mathcal{P}}$ and $\sigma_2 \neq id_{\mathcal{P}}$, again $\sigma_2 \circ \sigma_1$ is translation. Suppose the *contrary*, that $\sigma_2 \circ \sigma_1$ is not translation. Then, as dilatation, from the proposition 2.16, in the plan has a fixed point P according to $\sigma_2 \circ \sigma_1$, for which we have $(\sigma_2 \circ \sigma_1)(P) = P \iff \sigma_2(\sigma_1(P)) = P \iff \sigma_1(P) = \sigma_2^{-1}(P)$. According to corollary of the Theorem 3.3, results that $\sigma_1 = \sigma_2^{-1}$, that implicates $\sigma_2 \circ \sigma_1 = id_{\mathcal{P}}$, in contradiction with supposition. $\square$

Proposition 3.8. *If translations σ_1 and σ_2 have the same direction with translation σ to a affine plane $\mathcal{A}=(\mathcal{P},\mathcal{L},\mathcal{I})$, then and composition $\sigma_2 \circ \sigma_1$ has the same the direction, otherwise*

$$\forall \sigma_1, \sigma_2, \sigma \in \mathbf{Tr}_{\mathcal{A}}, \pi_{\sigma_1} = \pi_{\sigma_2} = \pi_{\sigma} \implies \pi_{\sigma_2 \circ \sigma_1} = \pi_{\sigma}. \quad (3.6)$$

Proof. According to (3.2) and the corollary of Proposition 3.5, easily indicated veracity of propositions in cases where at least one from the translations $\sigma_1, \sigma_2, \sigma$ is $id_{\mathcal{P}}$ or when $\sigma_1 = \sigma_2^{-1}$. In the case where $\sigma_2 \circ \sigma_1 \neq id_{\mathcal{P}}, \sigma \neq id_{\mathcal{P}}$, from the condition, for a point $P \in \mathcal{P}$, the traces $P\sigma(P), P\sigma_1(P)$ and $P\sigma_2(P)$ are parallel and have a common point P , hence they coincide. This means that in the trace $P\sigma(P)$, are also images of the point P , according σ_1 and σ_2 . We mark $\sigma_1(P) = P_1$, then, on the track $P\sigma(P)$, is also the image

$$\sigma_2(P_1) = \sigma_2(\sigma_1(P)) = (\sigma_2 \circ \sigma_1)(P).$$

This proves that

$$P\sigma(P) = (\sigma_2 \circ \sigma_1)(P),$$

namely

$$\pi_{\sigma_2 \circ \sigma_1} = \pi_{\sigma}.$$

□

Theorem 3.9. *Set $\mathbf{Tr}_{\mathcal{A}}$ of translations to an affine plane $\mathcal{A}$ form a group about the composition $\circ$, which is a sub-group of the group $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ to dilatations of affine plane $\mathcal{A}$.*

Proof. According to Definition 3.1, the translation set $\mathbf{Tr}_{\mathcal{A}}$ of the affine plane $\mathcal{A}$ is sub-set, of $\mathbf{Dil}_{\mathcal{A}}$. The Proposition 3.7, proves that $(\mathbf{Tr}_{\mathcal{A}}, \circ)$ is the sub-structure of the Dilatation group $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ of the affine plane $\mathcal{A}$. Proposition 3.5, proves that this sub-structure is a sub-group of the sub-group $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ of the group $(\mathbf{Col}_{\mathcal{A}}, \circ)$, see [7] and [8]. □

Theorem 3.10. *Group $(\mathbf{Tr}_{\mathcal{A}}, \circ)$ of translations to the affine plane $\mathcal{A}$ is normal sub-group of the group of dilatations $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ of him plane.*

Proof. For this, according to a theorem [7] it suffices to prove that:

$$\forall \delta \in \mathbf{Dil}_{\mathcal{A}}, \forall \sigma \in \mathbf{Tr}_{\mathcal{A}} \text{ have } \delta^{-1} \circ \sigma \circ \delta \in \mathbf{Tr}_{\mathcal{A}}.$$

If $\sigma = id_{\mathcal{P}}$, then

$$\delta^{-1} \circ \sigma \circ \delta = \delta^{-1} \circ (id_{\mathcal{P}} \circ \delta) = \delta^{-1} \circ \delta = id_{\mathcal{P}} \in \mathbf{Tr}_{\mathcal{A}}.$$

If $\sigma \neq id_{\mathcal{P}}$, we mark $\sigma_1 = \delta^{-1} \circ \sigma \circ \delta$, which is a dilatation, while $\delta^{-1}, \sigma, \delta$, are dilatations. If $\sigma_1 = id_{\mathcal{P}}$, then again $\delta^{-1} \circ \sigma \circ \delta \in \mathbf{Tr}_{\mathcal{A}}$. But even if the $\sigma_1 \neq id_{\mathcal{P}}$, $\delta^{-1} \circ \sigma \circ \delta \in \mathbf{Tr}_{\mathcal{A}}$. Suppose the contrary, that in this case $\delta^{-1} \circ \sigma \circ \delta \notin \mathbf{Tr}_{\mathcal{A}}$. Then, as dilatation from Proposition 2.16, in the plan has a fixed point P according to σ_1 , for which we have

$$\sigma_1(P) = P \iff (\delta^{-1} \circ \sigma \circ \delta)(P) = P \iff \delta^{-1}(\sigma(\delta(P))) = P \iff \sigma(\delta(P)) = \delta(P).$$

The last equation proves that the point $\delta(P)$ is the fixed point of the plane relating to σ , in the contradiction with the fact that $\sigma \neq id_P$. $\square$

Corollary 3.11. *For every dilatations $\delta \in \mathbf{Dil}_{\mathcal{A}}$ and for every translations $\sigma \in \mathbf{Tr}_{\mathcal{A}}$ of affine plane $\mathcal{A}=(\mathcal{P},\mathcal{L},\mathcal{I})$, translations σ and $\delta^{-1} \circ \sigma \circ \delta$ of his have the same direction.*

Proof. From the above results for us $\delta \neq id_P, \sigma \neq id_P \iff \delta^{-1} \circ \sigma \circ \delta \neq id_P$. Well translations σ and $\delta^{-1} \circ \sigma \circ \delta$ have determined directions, when $\sigma \neq id_P$ and $\delta \neq id_P$. For judging for its directions π_σ and $\pi_{\delta^{-1} \circ \sigma \circ \delta}$, examine the tracks according σ for points of plane $\delta(P) = P'$ and according $\sigma_1 = \delta^{-1} \circ \sigma \circ \delta$ for its points P . If these traces are two parallel lines of the plan, then these belong to the same equivalence class, namely $\pi_\sigma = \pi_{\delta^{-1} \circ \sigma \circ \delta}$. Trace of point P' according to σ is the line $P'\sigma(P')$, while the trace of point P according to σ_1 is $P\sigma_1(P)$.

For dilatation δ have

$$P = (\delta^{-1} \circ \delta)(P) = \delta^{-1}(\delta(P)) = \delta^{-1}(P'),$$

whereas

$$\sigma_1(P) = (\delta^{-1} \circ \sigma \circ \delta)(P) = \delta^{-1}(\sigma(\delta(P))) = \delta^{-1}(\sigma(P')).$$

But δ^{-1} is dilatation hence the different plane points $P', \sigma(P')$ with image

$$\delta^{-1}(P') = P, \delta^{-1}(\sigma(P')) = \sigma_1(P),$$

We have

$$P'\sigma(P') \parallel P\sigma_1(P).$$

Easily proved, according (3.2), that this equation is true even when at least one of dilatations, δ, σ is equal to id_P . $\square$

According to the understanding of the normal sub-group, see [7], [8], [13], from this Theorem also it shows that there is true the implication

$$\forall (\delta, \sigma) \in \mathbf{Dil}_{\mathcal{A}} \times \mathbf{Tr}_{\mathcal{A}}, \delta \circ \sigma = \sigma \circ \delta.$$

Because $\delta \in \mathbf{Tr}_{\mathcal{A}} \implies \delta \in \mathbf{Dil}_{\mathcal{A}}$, from that comes true the implication

$$\forall (\delta, \sigma) \in \mathbf{Tr}_{\mathcal{A}} \times \mathbf{Tr}_{\mathcal{A}}, \delta \circ \sigma = \sigma \circ \delta. \quad (3.7)$$

This indicates that is true this

Corollary 3.12. *The translations group $(\mathbf{Tr}_{\mathcal{A}}, \circ)$ of an affine plane $\mathcal{A}$ is (abelian) commutative.*

By definition of an Abelian Groups, see [7], [8], this means that besides (3.7), are true even these propositions:

$$\forall \sigma_1, \sigma_2, \sigma_3 \in \mathbf{Tr}_{\mathcal{A}}, (\sigma_1 \circ \sigma_2) \circ \sigma_3 = \sigma_1 \circ (\sigma_2 \circ \sigma_3) \quad (3.8)$$

$$\forall \sigma \in \mathbf{Tr}_{\mathcal{A}}, \sigma \circ id_{\mathcal{P}} = id_{\mathcal{P}} \circ \sigma = \sigma \quad (3.9)$$

$$\forall \sigma \in \mathbf{Tr}_{\mathcal{A}}, \exists \sigma^{-1} \in \mathbf{Tr}_{\mathcal{A}}, \sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = id_{\mathcal{P}} \quad (3.10)$$

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