

Jeribi essential spectrum

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Abstract: The aim of this paper is to present some results concerning the Jeribi essential spectrum. We use the notion of measure of weak noncompactness to give a formulae for the Jeribi essential spectral radius and we use the class of tauberian and cotauberian operators to present some relationship between the Jeribi essential spectrum and the other essential spectra.

Keywords: Measure of weak noncompactness, Tauberian operators, essential spectra, Fredholm operators.

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1 Introduction

The Jeribi essential spectrum of an operator $T \in \mathcal{L}(X)$, denoted by $\sigma_j(T)$, was defined in [9, 7, 13] as the intersection of the spectra of all $K \in \mathcal{W}_*(X)$ perturbations of T , where $\mathcal{W}_*(X)$ stands for each one of the sets of weakly compact operators $\mathcal{W}(X)$ and strictly singular operators $\mathcal{S}(X)$. The Jeribi essential spectrum always satisfying the inclusion $\sigma_j(T) \subseteq \sigma_{e_5}(T)$, where $\sigma_{e_5}(T)$ designed the Schechter essential spectrum. One of the crucial question about the Jeribi essential spectrum of a bounded linear operator on Banach spaces is the following : is there any relationship between the Jeribi essential spectrum and the other essential spectra (Kato essential spectrum, Gustafson essential spectrum, Wolf essential spectrum, Weidman essential spectrum, and others essential spectra)? A partial answer was given in [8] when X is a reflexive Banach space, then the Jeribi essential spectrum is the smallest essential spectrum in the sense of the inclusion of the Kato, Gustafson, Weidmann, Wolf, Schechter and Browder essential spectra. In $L_1(\Omega, d\mu)$ spaces, where (Ω, Σ, μ) is an arbitrary positive measure space it was proved in A. Jeribi's thesis [7] that the Jeribi essential spectrum coincide with the Schechter essential spectrum. Moreover, if the space X satisfies the Dunford-Pettis property then we have $\sigma_{e_5}(T) = \sigma_j(T)$.

We are interested in this paper by the study of the Jeribi essential spectrum of a bounded linear operator on a Banach space X . By using the De Blasi measure of weak noncompactness, we give a formulae for the Jeribi essential spectral radius

and we present the relationship between the Jeribi essential spectrum and the other essential spectra, in the particular case, when the Banach space X has no reflexive infinite dimensional subspaces.

We organize the paper in the following way: In the next section, we gather some results and notations which we deal in this paper. In section 3, we present the main results.

2 Preliminaries

Let X be a Banach space. We denote by $\mathcal{L}(X)$ the set of all bounded linear operators on X and by $\mathcal{K}(X)$ the subspace of compact operators on X . The set of upper semi-Fredholm operators, denoted by $\Phi_+(X)$, consists of the bounded linear operators on X with closed range and finite dimensional kernel. The set of lower semi-Fredholm operators, denoted by $\Phi_-(X)$, consists of the bounded linear operators on X with finite codimensional in X . Operators in $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ are called semi-Fredholm operators on X while $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ denotes the set of Fredholm operators on X . An operator $T \in \mathcal{L}(X)$ is said to have a left Fredholm inverse, if there exist $T^l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $T^l T = I_X - K$. Similarly, $T \in \mathcal{L}(X)$ is said to have a right Fredholm inverse, if there exist $T^r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $T T^r = I_X - K$. Denote by $\Phi^l(X)$ (respectively $\Phi^r(X)$) the set of linear operators which have a left (respectively right) Fredholm inverse. The index, $\text{ind}(T)$, is given by $\text{ind}(T) = \alpha(T) - \beta(T)$ where $\alpha(T)$ is the nullity of T and $\beta(T)$ is the defect of T .

Definition 2.1. An operator $T \in \mathcal{L}(X)$ is said to be strictly singular if the restriction of T to any infinite dimensional subspace of X is not an homeomorphism.

The concept of strictly singular operators was introduced in [11] as a generalization of the notion of compact operators. The set of all strictly singular operators on X , denoted by $\mathcal{S}(X)$, is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (see [5, 11]).

Definition 2.2. An operator $T \in \mathcal{L}(X, Y)$ is called weakly compact if $T(B)$ is relatively weakly compact in Y , for every bounded subset $B \subset X$.

The set of all weakly compact operators is denoted by $\mathcal{W}(X, Y)$. When $X = Y$, $\mathcal{W}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. For basic properties of these operators we refer to [5, 3].

In order to recall the tauberian and cotauberian operators, let us denote X^{**} the second dual (or bidual) of X , T^* the conjugate of T and T^{**} the second conjugate of T . These classes of operators were introduced and investigated respectively by Wilansky [10] and Tacon [15].

Definition 2.3. An operator $T \in \mathcal{L}(X, Y)$ is said to be tauberian whenever T^{**} preserves the natural embedding of X into its double dual i.e $x \in X^{**}, T^{**}x \in Y$ implies $x \in X$.

Denote by $\mathcal{T}(X, Y)$ the set of tauberian operators. It is immediate that a tauberian operator has the property

$$x \in X^{**}, T^{**}x = 0 \text{ imply } x \in X.$$

This implies that

$$\ker(T) \text{ is reflexive.}$$

So, the upper semi-Fredholm operators are trivial examples of Tauberian operators. In the next proposition we recall some well known properties of tauberian operators.

Proposition 2.4 ([6]). *Let $T \in \mathcal{L}(X, Y)$ and $L \in \mathcal{L}(Y, Z)$. The following statements hold.*

1. *If both T and L are tauberian, then LT is tauberian.*
2. *If LT is tauberian, then T is tauberian.*
3. *T is tauberian and weakly compact if and only if X is reflexive.*
4. *If T is tauberian and $W \in \mathcal{W}(X, Y)$, then $T + W$ is tauberian.*

The following result was established in [10, Corollary 4.3].

Lemma 2.5. *Suppose X has no reflexive infinite dimensional subspace, then for $T \in \mathcal{L}(X, Y)$, T is tauberian if and only if $T \in \Phi_+(X, Y)$.*

Definition 2.6. An operator $T \in \mathcal{L}(X, Y)$ is said to be cotauberian if T^* is tauberian.

Denote by $\mathcal{CT}(X, Y)$ the class of all cotauberian operators.

Proposition 2.7 ([6]). *Let $T \in \mathcal{L}(X, Y)$ and $L \in \mathcal{L}(Y, Z)$. The following statements hold.*

1. *If both T and L are cotauberian, then LT is cotauberian.*
2. *If LT is cotauberian, then L is cotauberian.*
3. *T is cotauberian and weakly compact if and only if Y is reflexive.*
4. *If T is cotauberian and $W \in \mathcal{W}(X, Y)$, then $T + W$ is cotauberian.*

Now, let us recall some definitions of essential spectra of an operator $T \in \mathcal{L}(X)$.

Weidmann	:	$\sigma_{e,1}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_-(X)\}$
Gustafson	:	$\sigma_{e,2}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_+(X)\}$
Kato	:	$\sigma_{e,3}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_{\pm}(X)\}$
Wolf	:	$\sigma_{e,4}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\}$
Schechter	:	$\sigma_{e,5}(T) = \mathbb{C} \setminus \Phi_T(X)$
left Fredholm	:	$\sigma_{el}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi^l(X)\}$
right Fredholm	:	$\sigma_{er}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi^r(X)\}$
Jeribi	:	$\sigma_j(T) = \bigcap_{K \in \mathcal{W}_*(X)} \sigma(T + K),$

where $\Phi_T(X) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi(X) \text{ and } \text{ind}(\lambda I - T) = 0\}$; $\sigma(T + K)$ denote the spectrum of $(T + K)$ and $\mathcal{W}_*(X)$ stands for each one of the sets $\mathcal{S}(X)$ and $\mathcal{W}(X)$. In general, we have the following inclusions

$$\sigma_{e,3}(T) = \sigma_{e,1}(T) \cap \sigma_{e,2}(T) \subseteq \sigma_{e,4}(T) \subseteq \sigma_{e,5}(T), \tag{2.1}$$

$$\sigma_{e,2}(T) \subseteq \sigma_{el}(T) \subseteq \sigma_{e,4}(T), \tag{2.2}$$

$$\sigma_{e,1}(T) \subseteq \sigma_{er}(T) \subseteq \sigma_{e,4}(T), \tag{2.3}$$

and $\sigma_j(T) \subseteq \sigma_{e,5}(T)$. The spectral radius of the essential spectra is defined as

$$r_{e_i}(T) = \sup \{|\lambda| : \lambda \in \sigma_{e_i}(T)\} \quad i = 1, \dots, 5$$

and $r_{e_i}(T) = \lim_{n \rightarrow \infty} (\chi(T^n))^{\frac{1}{n}}$, where $\chi(T)$ is the Hausdorff measure of noncompactness of T (see [4]).

The Hausdorff measure of noncompactness of a bounded subset A of X , denoted by $\chi(A)$, is defined to be the infimum of the set of all reals $\varepsilon > 0$ such that A can be covered by a finite number of balls of radius $< \varepsilon$. The Hausdorff measure of noncompactness of an operator $T \in \mathcal{L}(X, Y)$, denoted by $\chi(T)$, is given by $\chi(T) = \chi(T(B_X))$ with B_X the closed unit ball in X .

Definition 2.8. Let X be a Banach space and $A \subset X$ be a bounded set of X . The measure of weak noncompactness $\omega(A)$ of the set A is the infimum of the numbers $\varepsilon > 0$ such that A has a weakly compact ε -net in X .

This measure of weak noncompactness and its properties have been studied by De Blasi in [2]. The measure of weak noncompactness of an operator $T \in \mathcal{L}(X, Y)$, $\omega(T)$, is given by $\omega(T) = \omega(T(B_X))$. The most useful properties of this measure of weak noncompactness, $\omega(T)$, are related in the following theorem.

Theorem 2.9. *Let X, Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$. Then we have*

1. $\omega(T) = 0$ if and only if $T \in \mathcal{W}(X, Y)$.
2. $\omega(T) = \omega(T + W)$, for all $W \in \mathcal{W}(X, Y)$.
3. $\omega(T) \leq \chi(T)$.

3 Main results

In [12], the authors proved that, if X is a Banach space, then

$$\sigma_{e,5}(T) = \bigcap_{K \in \mathcal{S}(X)} \sigma(T + K).$$

So, in the definition of Jeribi essential spectrum, we restrict K belonging to $\mathcal{W}(X)$ only. We define the spectral radius of the Jeribi essential spectrum as

$$r_j(T) = \sup \{ \lambda \in \mathbb{C} : \lambda \in \sigma_j(T) \}.$$

We start with the following main proposition

Proposition 3.1. *Let $T \in \mathcal{L}(X)$. Then, the following assertions hold:*

- i) For all $W \in \mathcal{W}(X)$, $\sigma_{e_i}(T + W) \subset \sigma_j(T) \quad i = 1, \dots, 5, l, r$.
- ii) $r_j(T^n) = (r_j(T))^n$, for all $n \in \mathbb{N}$.
- iii) $r_j(T) \leq \omega(T)$.

Proof. Let $\lambda \in \mathbb{C}$ such that $\lambda \notin \sigma_j(T)$. Then we have the following implications

$$\begin{aligned} \lambda \notin \sigma_j(T) &\Rightarrow \exists W \in \mathcal{W}(X) : \lambda \in \rho(T + W) \text{ (the resolvent set of } T + W) \\ &\Rightarrow \exists W \in \mathcal{W}(X) : (\lambda I - T - W) \in \Phi(X) \text{ ind}(\lambda I - T - W) = 0 \\ &\Rightarrow \exists W \in \mathcal{W}(X) : \lambda \notin \sigma_{e_5}(T + W). \end{aligned}$$

So, for all weakly compact operator W on X , $\lambda \in \sigma_{e_5}(T + W)$ implies that $\lambda \in \sigma_j(T)$. By the inclusions (2.1), (2.2) and (2.3), it follows that $\sigma_{e_i}(T + W) \subset \sigma_j(T)$ for $i = 1, \dots, 4, l, r$.

$$\text{Let } n \in \mathbb{N} \text{ and } W \in \mathcal{W}(X). \text{ Then } \sigma((T + W)^n) = \sigma(T^n + \sum_{k=0}^{n-1} C_k^n T^k W^{n-k})$$

where C_k^n is the binomial coefficient defined by $C_k^n = \frac{n!}{k!(n-k)!}$. Since $\mathcal{W}(X)$

is a closed two-sided ideal of $\mathcal{L}(X)$, then $W' = \sum_{k=0}^{n-1} C_k^n T^k W^{n-k} \in \mathcal{W}(X)$. So, $\sigma((T + W)^n) = \sigma(T^n + W')$ and

$$\bigcap_{W \in \mathcal{W}(X)} \sigma((T + W)^n) = \bigcap_{W' \in \mathcal{W}(X)} \sigma(T^n + W') = \sigma_j(T^n).$$

Hence

$$\bigcap_{W \in \mathcal{W}(X)} \sigma((T + W)^n) = \sigma_j(T^n). \quad (3.1)$$

On the other hand, it is well known that $\sigma(T^n) = \{\lambda^n : \lambda \in \sigma(T)\}$, then

$$\bigcap_{W \in \mathcal{W}(X)} \sigma((T + W)^n) = \left\{ \lambda^n : \lambda \in \bigcap_{W \in \mathcal{W}(X)} \sigma(T + W) \right\} = \{\lambda^n : \lambda \in \sigma_j(T)\}. \quad (3.2)$$

From the two inequalities (3.1) and (3.2) we see that

$$\sigma_j(T^n) = \{\lambda^n : \lambda \in \sigma_j(T)\}.$$

From the above equality, it follows that $r_j(T^n) = (r_j(T))^n$.

Let $\lambda \in \mathbb{C}$ such that $\chi(T) < |\lambda|$. Then using [4, Theorem 4.4] we get $(\lambda I - T) \in \Phi(X)$ with $\text{ind}(\lambda I - T) = 0$. By the fact that $\omega(T) \leq \chi(T)$, then we have the following implications

$$\begin{aligned} \lambda \in \sigma_{e_5}(T) &\Rightarrow |\lambda| \leq \omega(T) \\ &\Rightarrow r_{e_5}(T) \leq \omega(T). \end{aligned}$$

Hence $r_j(T) \leq \omega(T)$ since $\sigma_j(T) \subseteq \sigma_{e_5}(T)$. □

The following theorem is the main result.

Theorem 3.2. *Let $T \in \mathcal{L}(X)$. Then, the spectral radius of the Jeribi essential spectrum is given by*

$$r_j(T) = \lim_{n \rightarrow \infty} [\omega(T^n)]^{\frac{1}{n}}.$$

Proof. The limits of $[\omega(T^n)]^{\frac{1}{n}}$ exists for all $T \in \mathcal{L}(X)$ (see [1]). From the two assertions ii) and iii) of proposition (3.1), it follows immediately that

$$r_j(T) \leq \lim_{n \rightarrow \infty} [\omega(T^n)]^{\frac{1}{n}}. \quad (3.3)$$

Let us prove the opposite inequality. Let $W \in \mathcal{W}(X)$. From the assertion i) of proposition (3.1), we have

$$\sigma_{e_1}(T + W) \subset \sigma_j(T).$$

Moreover, $r_{e_1}(T + W) \leq r_j(T)$. It is well known that

$$r_{e_1}(T + W) = \lim_{n \rightarrow \infty} [\chi((T + W)^n)]^{\frac{1}{n}},$$

then using the fact that $\omega(T) = \omega(T + W)$ for any weakly compact operator W on X and $\omega(T + W) \leq \chi(T + W)$ we obtain

$$\lim_{n \rightarrow \infty} [\omega(T^n)]^{\frac{1}{n}} \leq r_j(T). \tag{3.4}$$

From the two inequalities (3.3) and (3.4) we deduce that

$$r_j(T) = \lim_{n \rightarrow \infty} [\omega(T^n)]^{\frac{1}{n}}.$$

□

In the next main result, we give a relationship between the Jeribi essential spectrum and the other essential spectra when the Banach space X has no reflexive infinite dimensional subspaces.

Theorem 3.3. *Let X be a Banach space which has no reflexive infinite dimensional subspaces and $T \in \mathcal{L}(X)$. Then we have*

$$\sigma_{e_i}(T) \subset \sigma_j(T) \quad i = 1, \dots, 4, l, r.$$

Proof. Take $\lambda \notin \sigma_j(T)$. Then there exists a weakly compact operator W on X such that $\lambda \in \rho(T + W)$ i.e

$$(\lambda I - T - W) \in \Phi(X) \text{ with } \text{ind}(\lambda I - T - W) = 0.$$

In particular, $(\lambda I - T - W) \in \Phi_+(X)$ which is a tauberian operator. Using the stability of tauberian operators under a weakly compact perturbation ($-W$) (assertion 4 of proposition (2.4)), we get $(\lambda I - T) \in \mathcal{T}(X)$. Since the space X has no reflexive infinite dimensional subspaces, using lemma (2.5), we obtain $(\lambda I - T) \in \Phi_+(X)$. So, $\lambda \notin \sigma_{e_2}(T)$. Hence $\sigma_{e_2}(T) \subset \sigma_j(T)$.

To prove that $\sigma_{e_1}(T) \subset \sigma_j(T)$, let us consider $\lambda \notin \sigma_j(T)$. Then there exists a weakly compact operator W on X such that $\lambda \in \rho(T + W)$ i.e

$$(\lambda I - T - W) \in \Phi(X) \text{ with } \text{ind}(\lambda I - T - W) = 0.$$

In particular, $(\lambda I - T - W) \in \Phi_-(X)$ which is a cotauberian operator. Using the stability of cotauberian operators under a weakly compact perturbation ($-W$) (assertion 4 of proposition (2.7)), we get $(\lambda I - T) \in \mathcal{CT}(X)$. By the definition of cotauberian operator, we have $(\lambda I - T)^* \in \mathcal{T}(X)$. Since the space X has no reflexive

infinite dimensional subspaces, then using lemma (2.5), we obtain $(\lambda I - T)^* \in \Phi_+(X)$. It follows from [14, Theorem 4, p.156] that $(\lambda I - T) \in \Phi_-(X)$ i.e $\lambda \notin \sigma_{e_1}(T)$.

The following two inclusions $\sigma_{e_3}(T) \subset \sigma_j(T)$ and $\sigma_{e_4}(T) \subset \sigma_j(T)$ are immediately since $\sigma_{e_3}(T) = \sigma_{e_1}(T) \cap \sigma_{e_2}(T)$ and $\sigma_{e_4}(T) = \sigma_{e_1}(T) \cup \sigma_{e_2}(T)$.

From the inclusions (2.2) and (2.3), we see that $\sigma_{el}(T) \subset \sigma_j(T)$ and $\sigma_{el}(T) \subset \sigma_j(T)$. \square

As an immediate consequence of Theorem (3.3) we have the following corollary.

Corollary 3.4. *Let X be a Banach space such that X has no reflexive infinite dimensional subspaces and $T \in \mathcal{L}(X)$. Then we have the following inclusions*

$$\sigma_{e_3}(T) = \sigma_{e_1}(T) \cap \sigma_{e_2}(T) \subset \sigma_{e_4}(T) \subset \sigma_j(T) \subseteq \sigma_{e_5}(T).$$

Furthermore, $r_j(T) = \lim_{n \rightarrow \infty} \chi(T^n)^{\frac{1}{n}}$.

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