Libertas Mathematica (new series)
Volume 38(2018), No. 1, 1-13

Finding the Fundamental Solutions of the Pell Equation $x^2 - dy^2 = \pm 1$ by determining the Right Neighbor of $F = (d, 0, -1)$

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Abstract: In this work, we first determine the sets $Aut^+(F)$ and $Aut^*(F)$ of the form $F = (d, 0, -1)$ of discriminant $\Delta = 4d$, where d is any positive integer and then we deduce the fundamental solution of the Pell equation $x^2 - dy^2 = \pm 1$ from the right neighbors of F.

Keywords: Indefinite form, right neighbor, Pell equation, fundamental solution.

MSC2010: 11E16, 11E18, 11D04, 11D09.

1 Introduction.

A real binary quadratic form (or just a form) F is polynomial in two variables x and y of the type

$$F = F(x, y) = ax^2 + bxy + cy^2 \tag{1.1}$$

with real coefficients a, b, c. We denote F briefly by $F = (a, b, c)$. The discriminant of F is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta = \Delta(F)$. A binary quadratic form F is called integral if and only if $a, b, c \in \mathbb{Z}$; positive definite if and only if $\Delta(F) < 0, a, c > 0$; indefinite if and only if $\Delta(F) > 0$.

Let $\mathrm{GL}(2, \mathbb{Z})$ be the multiplicative group of 2×2 matrices $g = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ such that $r, s, t, u \in \mathbb{Z}$ and $\det(g) = \pm 1$. Gauss defined the group action of $\mathrm{GL}(2, \mathbb{Z})$ on the set of forms as

$$gF(x, y) = F(rx + ty, sx + uy) \tag{1.2}$$

for $g = \begin{bmatrix} r & s \\ t & u \end{bmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ and $F = (a, b, c)$. If there exists a $g \in \mathrm{GL}(2, \mathbb{Z})$ such that $gF = G$, then F and G are called equivalent. If $\det(g) = 1$, then F and G are called properly equivalent and if $\det(g) = -1$, then F and G are called improperly equivalent. An element $g \in \mathrm{GL}(2, \mathbb{Z})$ is called an automorphism of F if $gF = F$. If $\det(g) = 1$, then g is called a proper automorphism of F and if $\det(g) = -1$, then

g is called an improper automorphism of F. The set of proper automorphisms of F is denoted by $Aut^+(F)$ and the set of improper automorphisms of F is denoted by $Aut^-(F)$. We also set $Aut^*(F) = \{g \in \mathrm{GL}(2,\mathbb{Z}) : gF = -F,\ \det(g) = -1\}$.

The right neighbor of an indefinite integral form $F = (a, b, c)$ of discriminant Δ is denoted by $R(F)$ is the form (A, B, C) determined by $A = c$, $b+B \equiv 0 (\mathrm{mod}\, 2|A|)$, $\sqrt{\Delta} - 2|A| < B < \sqrt{\Delta}$ and $B^2 - 4AC = \Delta$. It is clear that the right neighbor of F is

$$R(F) = \begin{bmatrix} 0 & -1 \\ 1 & -\delta \end{bmatrix} (a, b, c), \tag{1.3}$$

where $\delta = \frac{b+B}{2c}$ is an integer since $b+B \equiv 0 (\mathrm{mod}\, 2|A|)$ and $A = c$ (for further details on binary quadratic forms see [3, 4, 5, 9]).

2 Automorphisms.

We denote the the inverse of the matrix defined in (1.3) by $T(\delta)$, that is,

$$T(\delta) = \begin{bmatrix} 0 & -1 \\ 1 & -\delta \end{bmatrix}^{-1} = \begin{bmatrix} -\delta & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.1}$$

Set

$$\tau_{F,n} = T(\delta_0)T(\delta_1)\cdots T(\delta_{n-1}). \tag{2.2}$$

Let $R^i(F)$ be the i^{th} right neighbor of F. Then it is known from [5, Theorem 9.4 and Corollary 9.5] that if $R^N(F) = F$, then $\tau_{F,N}$ is an element of $Aut^+(F)$ of infinite order and $Aut^+(F) = \{\pm(\tau_{F,N})^n : n \in \mathbb{Z}\}$. If $R^N(F) = R^M(F)$ for smallest positive integers $N > M$, then $Aut^+(F) = \{\pm(\tau_{F,N}\tau_{F,M}^{-1})^n : n \in \mathbb{Z}\}$.

Now let k be any positive integer and let

$$d = k^2 \pm 1,\ d = k^2 \pm 2 \ \text{ and } \ d = k^2 \pm k.$$

In this section, we determine the sets $Aut^+(F)$ and $Aut^*(F)$ of the form

$$F = (d, 0, -1) \tag{2.3}$$

of discriminant $\Delta = 4d$.

Theorem 2.1. *Let F be the form defined in (2.3).*

1. *If $d = k^2 + 1$ for $k \geq 1$, then*

$$Aut^+(F) = \left\{\pm \begin{bmatrix} -2k^2-1 & 2k^3+2k \\ 2k & -2k^2-1 \end{bmatrix}^n : n \in \mathbb{Z}\right\} \text{ and}$$

$$Aut^*(F) = \left\{\pm \begin{bmatrix} k & k^2+1 \\ 1 & k \end{bmatrix}^{2n+1} : n \in \mathbb{Z}\right\}.$$

2. *If* $d = k^2 - 1$ *for* $k \geq 2$, *then*

$$Aut^{+}(F) = \left\{ \pm \begin{bmatrix} -k & k^2-1 \\ 1 & -k \end{bmatrix}^n : n \in \mathbb{Z} \right\} \text{ and } Aut^{*}(F) = \emptyset.$$

3. *If* $d = k^2 + 2$ *for* $k \geq 1$, *then*

$$Aut^{+}(F) = \left\{ \pm \begin{bmatrix} -k^2-1 & k^3+2k \\ k & -k^2-1 \end{bmatrix}^n : n \in \mathbb{Z} \right\} \text{ and } Aut^{*}(F) = \emptyset.$$

4. *If* $d = k^2 - 2$ *for* $k \geq 2$, *then*

$$Aut^{+}(F) = \left\{ \pm \begin{bmatrix} -3 & 4 \\ 2 & -3 \end{bmatrix}^n : n \in \mathbb{Z} \right\} \text{ and}$$

$$Aut^{*}(F) = \left\{ \pm \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{2n+1} : n \in \mathbb{Z} \right\}$$

for $k = 2$; *and*

$$Aut^{+}(F) = \left\{ \pm \begin{bmatrix} k^2-1 & -k^3+2k \\ -k & k^2-1 \end{bmatrix}^n : n \in \mathbb{Z} \right\} \text{ and } Aut^{*}(F) = \emptyset$$

for $k > 2$.

5. *If* $d = k^2 + k$ *for* $k \geq 1$, *then*

$$Aut^{+}(F) = \left\{ \pm \begin{bmatrix} -3 & 4 \\ 2 & -3 \end{bmatrix}^n : n \in \mathbb{Z} \right\} \text{ and}$$

$$Aut^{*}(F) = \left\{ \pm \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{2n+1} : n \in \mathbb{Z} \right\}$$

for $k = 1$; *and*

$$Aut^{+}(F) = \left\{ \pm \begin{bmatrix} -2k-1 & 2k^2+2k \\ 2 & -2k-1 \end{bmatrix}^n : n \in \mathbb{Z} \right\} \text{ and } Aut^{*}(F) = \emptyset$$

for $k > 1$.

6. *If* $d = k^2 - k$ *for* $k \geq 2$, *then*

$$Aut^{+}(F) = \left\{ \pm \begin{bmatrix} -3 & 4 \\ 2 & -3 \end{bmatrix}^n : n \in \mathbb{Z} \right\} \text{ and}$$

$$Aut^{*}(F) = \left\{ \pm \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{2n+1} : n \in \mathbb{Z} \right\}$$

for $k=2$; *and*

$$Aut^+(F)=\left\{\pm\begin{bmatrix} -2k+1 & 2k^2-2k \\ 2 & -2k+1 \end{bmatrix}^n : n\in\mathbb{Z}\right\} \text{ and } Aut^*(F)=\emptyset$$

for $k>2$.

Proof. **(1)** Let $k\geq 1$ be a positive integer and let $d=k^2+1$. Then for the right neighbors of $F=(k^2+1,0,-1)$, we have Table 1:

Table 1:

i	0	1	2	3
A_i	k^2+1	-1	1	-1
B_i	0	$2k$	$2k$	$2k$
C_i	-1	1	-1	1
δ_i	$-k$	$2k$	$-2k$	

Since $R^3(F)=R^1(F)$, we have $\tau_{F,3}=\begin{bmatrix} -4k^3-3k & -2k^2-1 \\ 4k^2+1 & 2k \end{bmatrix}$ and $\tau_{F,1}=\begin{bmatrix} k & 1 \\ -1 & 0 \end{bmatrix}$. Thus $\tau_{F,3}\tau_{F,1}^{-1}=\begin{bmatrix} -2k^2-1 & 2k^3+2k \\ 2k & -2k^2-1 \end{bmatrix}$. So

$$Aut^+(F)=\left\{\pm\begin{bmatrix} -2k^2-1 & 2k^3+2k \\ 2k & -2k^2-1 \end{bmatrix}^n : n\in\mathbb{Z}\right\}.$$

Similarly we get $(\tau_{F,3})^*(\tau_{F,1}^{-1})^*=\begin{bmatrix} k & k^2+1 \\ 1 & k \end{bmatrix}$ and hence

$$Aut^*(F)=\left\{\pm\begin{bmatrix} k & k^2+1 \\ 1 & k \end{bmatrix}^{2n+1} : n\in\mathbb{Z}\right\}.$$

(2) Let $k\geq 2$ be a positive integer and let $d=k^2-1$. Then for the right neighbors of $F=(k^2-1,0,-1)$, we have Table 2:

Table 2:

i	0	1	2	3
A_i	k^2-1	-1	$2k-2$	-1
B_i	0	$2k-2$	$2k-2$	$2k-2$
C_i	-1	$2k-2$	-1	$2k-2$
δ_i	$1-k$	1	$2-2k$	

Since $R^3(F) = R^1(F)$, we get $\tau_{F,3} = \left[\begin{array}{cc} -2k^2+k+1 & -k \\ 2k-1 & 1 \end{array}\right]$ and $\tau_{F,1} = \left[\begin{array}{cc} k-1 & 1 \\ -1 & 0 \end{array}\right]$. So $\tau_{F,3}\tau_{F,1}^{-1} = \left[\begin{array}{cc} -k & k^2-1 \\ 1 & -k \end{array}\right]$ and hence

$$Aut^+(F) = \left\{ \pm \left[\begin{array}{cc} -k & k^2-1 \\ 1 & -k \end{array}\right]^n : n \in \mathbb{Z} \right\}.$$

Since there is no a $g \in \mathrm{GL}(2,\mathbb{Z})$ such that $gF = -F$ with $\det(g) = -1$, we have $Aut^*(F) = \emptyset$.

(3) Let $k \geq 1$ be a positive integer and let $d = k^2 + 2$. Then for the right neighbors of $F = (k^2+2, 0, -1)$, we have Table 3:

Table 3:

i	0	1	2	3
A_i	k^2+2	-1	2	-1
B_i	0	$2k$	$2k$	$2k$
C_i	-1	2	-1	2
δ_i	$-k$	k	$-2k$	

Since $R^3(F) = R^1(F)$, we have $\tau_{F,3} = \left[\begin{array}{cc} -2k^3-3k & -k^2-1 \\ 2k^2+1 & k \end{array}\right]$ and $\tau_{F,1} = \left[\begin{array}{cc} k & 1 \\ -1 & 0 \end{array}\right]$. So $\tau_{F,3}\tau_{F,1}^{-1} = \left[\begin{array}{cc} -k^2-1 & k^3+2k \\ k & -k^2-1 \end{array}\right]$ and hence

$$Aut^+(F) = \left\{ \pm \left[\begin{array}{cc} -k^2-1 & k^3+2k \\ k & -k^2-1 \end{array}\right]^n : n \in \mathbb{Z} \right\}.$$

There is no a $g \in$GL$(2,\mathbb{Z})$ such that $gF = -F, \det(g) = -1$, we have $Aut^*(F) = \emptyset$.

(4) Let $k = 2$. Then for the right neighbors of $F = (2, 0, -1)$, we have Table 4:

Table 4:

i	0	1	2	3
A_i	2	-1	1	-1
B_i	0	2	2	2
C_i	-1	1	-1	1
δ_i	-1	2	-2	

Since $R^3(F) = R^1(F)$, we have $\tau_{F,3} = \left[\begin{array}{cc} -7 & -3 \\ 5 & 2 \end{array}\right]$ and $\tau_{F,1} = \left[\begin{array}{cc} 1 & 1 \\ -1 & 0 \end{array}\right]$. So $\tau_{F,3}\tau_{F,1}^{-1} = \left[\begin{array}{cc} -3 & 4 \\ 2 & -3 \end{array}\right]$ and hence

$$Aut^{+}(F)=\left\{\pm\begin{bmatrix}-3 & 4\\ 2 & -3\end{bmatrix}^{n}: n\in\mathbb{Z}\right\}.$$

Similarly we get $(\tau_{F,3})^{*}(\tau_{F,1}^{-1})^{*}=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$ and hence

$$Aut^{*}(F)=\left\{\pm\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}^{2n+1}: n\in\mathbb{Z}\right\}.$$

Let $k>2$. Then for the right neighbors of $F=(k^2-2,0,-1)$, we have Table 5:

Table 5:

i	0	1	2	3	4	5
A_i	k^2-2	-1	$2k-3$	-2	$2k-3$	-1
B_i	0	$2k-2$	$2k-4$	$2k-4$	$2k-2$	$2k-2$
C_i	-1	$2k-3$	-2	$2k-3$	-1	$2k-3$
δ_i	$1-k$	1	$2-k$	1	$2-2k$	

Since $R^5(F)=R^1(F)$, $\tau_{F,5}=\begin{bmatrix}2k^3-k^2-3k+1 & k^2-1\\ -2k^2+k+1 & -k\end{bmatrix}$ and $\tau_{F,1}=\begin{bmatrix}k-1 & 1\\ -1 & 0\end{bmatrix}$. So $\tau_{F,5}\tau_{F,1}^{-1}=\begin{bmatrix}k^2-1 & -k^3+2k\\ -k & k^2-1\end{bmatrix}$ and hence

$$Aut^{+}(F)=\left\{\pm\begin{bmatrix}k^2-1 & -k^3+2k\\ -k & k^2-1\end{bmatrix}^{n}: n\in\mathbb{Z}\right\}.$$

There is no a $g\in$GL$(2,\mathbb{Z})$ such that $gF=-F$, $\det(g)=-1$, we have $Aut^{*}(F)=\emptyset$.

(5) Let $k\geq 1$ be a positive integer and let $d=k^2+k$. If $k=1$, then we have $F=(2,0,-1)$. So the result is obvious from Table 4 in (4). Let $k>1$. Then for the right neighbors of $F=(k^2+k,0,-1)$, we have Table 6:

Table 6:

i	0	1	2	3
A_i	k^2+k	-1	2	-1
B_i	0	$2k$	$2k$	$2k$
C_i	-1	k	-1	k
δ_i	$-k$	2	$-2k$	

Since $R^3(F) = R^1(F)$, $\tau_{F,3} = \begin{bmatrix} -4k^2-3k & -2k-1 \\ 4k+1 & 2 \end{bmatrix}$ and $\tau_{F,1} = \begin{bmatrix} k & 1 \\ -1 & 0 \end{bmatrix}$. So $\tau_{F,3}\tau_{F,1}^{-1} = \begin{bmatrix} -2k-1 & 2k^2+2k \\ 2 & -2k-1 \end{bmatrix}$ and hence

$$Aut^+(F) = \left\{ \pm \begin{bmatrix} -2k-1 & 2k^2+2k \\ 2 & -2k-1 \end{bmatrix}^n : n \in \mathbb{Z} \right\}.$$

There is no a $g \in \mathrm{GL}(2,\mathbb{Z})$ such that $gF = -F$, $\det(g) = -1$, we have $Aut^*(F) = \emptyset$.

(6) Let $k \geq 2$ be a positive integer and let $d = k^2 - k$. If $k = 2$, then we have $F = (2, 0, -1)$. So the result is obvious from Table 4 in (4). Let $k > 2$. Then for the right neighbors of $F = (k^2 - k, 0, -1)$, we have Table 7:

Table 7:

i	0	1	2	3
A_i	k^2-k	-1	2	-1
B_i	0	$2k-2$	$2k-2$	$2k-2$
C_i	-1	$k-1$	-1	$k-1$
δ_i	$1-k$	2	$2-2k$	

Since $R^3(F) = R^1(F)$, $\tau_{F,3} = \begin{bmatrix} -4k^2+5k-1 & -2k+1 \\ 4k-3 & 2 \end{bmatrix}$ and $\tau_{F,1} = \begin{bmatrix} k-1 & 1 \\ -1 & 0 \end{bmatrix}$. So $\tau_{F,3}\tau_{F,1}^{-1} = \begin{bmatrix} -2k+1 & 2k^2-2k \\ 2 & -2k+1 \end{bmatrix}$ and hence

$$Aut^+(F) = \left\{ \pm \begin{bmatrix} -2k+1 & 2k^2-2k \\ 2 & -2k+1 \end{bmatrix}^n : n \in \mathbb{Z} \right\}.$$

Since there is no a $g \in \mathrm{GL}(2,\mathbb{Z})$ such that $gF = -F$, $\det(g) = -1$, we have $Aut^*(F) = \emptyset$. □

Remark 2.2. We can also determine the sets $Aut^+(F)$ and $Aut^*(F)$ of the form $F = (2, 0, -1)$ in terms of balancing numbers. Behera and Panda [2] introduced balancing numbers $n \in \mathbb{Z}^+$ as solutions of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{2.4}$$

for some positive integer r which is called balancer or cobalancing number. If n is a balancing number with balancer r, then from (2.4) one has $\frac{(n-1)n}{2} = rn + \frac{r(r+1)}{2}$ and so

$$r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2} \quad \text{and} \quad n = \frac{2r+1+\sqrt{8r^2+8r+1}}{2}. \tag{2.5}$$

Let B_n denote the n^{th} balancing number and let b_n denote the n^{th} cobalancing number. Then from (2.5), we see that B_n is a balancing number if and only if $8B_n^2+1$ is a perfect square and b_n is a cobalancing number if and only if $8b_n^2+8b_n+1$ is a perfect square. So $\sqrt{8B_n^2+1}$ and $\sqrt{8b_n^2+8b_n+1}$ are integers which are called the n^{th} Lucas–balancing number and Lucas–cobalancing number and is denoted by C_n and c_n, respectively, that is, $C_n=\sqrt{8B_n^2+1}$ and $c_n=\sqrt{8b_n^2+8b_n+1}$ (for further details on balancing numbers see [6, 10, 11, 12]).

For the form $F=(2,0,-1)$, we proved in (4) of Theorem 2.1 that

$$Aut^+(F)=\left\{\pm\begin{bmatrix}-3 & 4\\ 2 & -3\end{bmatrix}^n : n\in\mathbb{Z}\right\}, Aut^*(F)=\left\{\pm\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}^{2n+1} : n\in\mathbb{Z}\right\}.$$

It can be proved by induction on n that

$$\begin{bmatrix}-3 & 4\\ 2 & -3\end{bmatrix}^n=\begin{cases}\begin{bmatrix}(-1)^nC_n & 4(-1)^{n+1}B_n\\ 2(-1)^{n+1}B_n & (-1)^nC_n\end{bmatrix} & \text{for } n\geq 0\\ \\ \begin{bmatrix}(-1)^nC_n & 4(-1)^nB_n\\ 2(-1)^nB_n & (-1)^nC_n\end{bmatrix} & \text{for } n<0\end{cases}$$

and

$$\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}^{2n+1}=\begin{cases}\begin{bmatrix}c_{n+1} & 4b_{n+1}+2\\ 2b_{n+1}+1 & c_{n+1}\end{bmatrix} & \text{for } n\geq 0\\ \\ \begin{bmatrix}-c_n & 4b_n+2\\ 2b_n+1 & -c_n\end{bmatrix} & \text{for } n<0.\end{cases}$$

So we have

$$\begin{aligned}Aut^+(F)=&\left\{\pm\begin{bmatrix}(-1)^nC_n & 4(-1)^{n+1}B_n\\ 2(-1)^{n+1}B_n & (-1)^nC_n\end{bmatrix} : n\geq 0\right\}\\ &\cup\left\{\pm\begin{bmatrix}(-1)^nC_n & 4(-1)^nB_n\\ 2(-1)^nB_n & (-1)^nC_n\end{bmatrix} : n<0\right\}\end{aligned}$$

and

$$\begin{aligned}Aut^*(F)=&\left\{\pm\begin{bmatrix}c_{n+1} & 4b_{n+1}+2\\ 2b_{n+1}+1 & c_{n+1}\end{bmatrix} : n\geq 0\right\}\\ &\cup\left\{\pm\begin{bmatrix}-c_n & 4b_n+2\\ 2b_n+1 & -c_n\end{bmatrix} : n<0\right\}.\end{aligned}$$

3 Fundamental Solutions of the Pell Equation $x^2 - dy^2 = \pm 1$.

In this section, we prove that the fundamental solutions of the Pell equation (see [1, 7, 8])

$$x^2 - dy^2 = \pm 1$$

can be deduced from the sets $Aut^+(F)$ and $Aut^*(F)$ of the form $F = (d, 0, -1)$ obtained in Theorem 2.1.

Theorem 3.1. *For the Pell equation $x^2 - dy^2 = \pm 1$, we have*

1. *If $k \geq 2$, then the fundamental solution of the positive Pell equation $x^2 - (k^2 + 1)y^2 = 1$ is (x_1, y_1), where*

$$[-x_1 \quad y_1] = \tau_{F,3}\tau_{F,1}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the fundamental solution of the negative Pell equation $x^2 - (k^2 + 1)y^2 = -1$ is (x_1, y_1), where

$$[x_1 \quad y_1] = (\tau_{F,3})^*(\tau_{F,1}^{-1})^* \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

2. *If $k \geq 2$, then the fundamental solution of the positive Pell equation $x^2 - (k^2 - 1)y^2 = 1$ is (x_1, y_1), where*

$$[-x_1 \quad y_1] = \tau_{F,3}\tau_{F,1}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the negative Pell equation $x^2 - (k^2 - 1)y^2 = -1$ has no integer solutions.

3. *If $k \geq 1$, then the fundamental solution of the positive Pell equation $x^2 - (k^2 + 2)y^2 = 1$ is (x_1, y_1), where*

$$[-x_1 \quad y_1] = \tau_{F,3}\tau_{F,1}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the negative Pell equation $x^2 - (k^2 + 2)y^2 = -1$ has no integer solutions.

4. *If $k \geq 3$, then the fundamental solution of the positive Pell equation $x^2 - (k^2 - 2)y^2 = 1$ is (x_1, y_1), where*

$$[x_1 \quad -y_1] = \tau_{F,5}\tau_{F,1}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the negative Pell equation $x^2 - (k^2 - 2)y^2 = -1$ has no integer solutions.

5. *If* $k \geq 2$, *then the fundamental solution of the positive Pell equation* $x^2 - (k^2 + k)y^2 = 1$ *is* (x_1, y_1), *where*

$$[-x_1 \quad y_1] = \tau_{F,3}\tau_{F,1}^{-1}\left[\begin{array}{c}1\\0\end{array}\right]$$

and the negative Pell equation $x^2 - (k^2 + k)y^2 = -1$ *has no integer solutions.*

6. *If* $k \geq 3$, *then the fundamental solution of the positive Pell equation* $x^2 - (k^2 - k)y^2 = 1$ *is* (x_1, y_1), *where*

$$[-x_1 \quad y_1] = \tau_{F,3}\tau_{F,1}^{-1}\left[\begin{array}{c}1\\0\end{array}\right]$$

and the negative Pell equation $x^2 - (k^2 - k)y^2 = -1$ *has no integer solutions.*

Proof. For the simple continued fraction expansion $\sqrt{d} = [a_0; \overline{a_1, a_2, \cdots, a_l}]$ with period length l, we set $A_{-2} = 0$, $A_{-1} = 1$, $A_m = a_m A_{m-1} + A_{m-2}$ and $B_{-2} = 1$, $B_{-1} = 0, B_m = a_m B_{m-1} + B_{m-2}$ for nonnegative integer m. Then it is given in [9] that $C_m = \frac{A_m}{B_m}$ is the m^{th} convergent of $\sqrt{d}$, and the fundamental solution of $x^2 - dy^2 = 1$ is $(x_1, y_1) = (A_{l-1}, B_{l-1})$ if l is even or (A_{2l-1}, B_{2l-1}) if l is odd. If l is even, then the negative Pell equation $x^2 - dy^2 = -1$ has no integer solutions, but if l is odd, then the the fundamental solution of $x^2 - dy^2 = -1$ is $(x_1, y_1) = (A_{l-1}, B_{l-1})$.

(1) Let $d = k^2 + 1$ for $k \geq 2$. Then it is easily seen that

$$\sqrt{k^2+1} = k + (\sqrt{k^2+1} - k) = k + \frac{1}{\frac{1}{\sqrt{k^2+1}-k}} = k + \frac{1}{\frac{\sqrt{k^2+1}+k}{(\sqrt{k^2+1}-k)(\sqrt{k^2+1}+k)}}$$
$$= k + \frac{1}{\sqrt{k^2+1}+k} = k + \frac{1}{2k + (\sqrt{k^2+1} - k)}.$$

So $\sqrt{d} = [k; \overline{2k}]$ and hence $A_0 = k, A_1 = 2k^2 + 1, B_0 = 1$ and $B_1 = 2k$. Thus the fundamental solution of the positive Pell equation $x^2 - (k^2 + 1)y^2 = 1$ is $(x_1, y_1) = (A_1, B_1) = (2k^2 + 1, 2k)$ and the fundamental solution of the negative Pell equation $x^2 - (k^2 + 1)y^2 = -1$ is $(x_1, y_1) = (A_0, B_0) = (k, 1)$. On the other hand, we proved in (1) of Theorem 2.1 that

$$\tau_{F,3}\tau_{F,1}^{-1} = \left[\begin{array}{cc}-2k^2-1 & 2k^3+2k\\ 2k & -2k^2-1\end{array}\right] \quad \text{and} \quad (\tau_{F,3})^*(\tau_{F,1}^{-1})^* = \left[\begin{array}{cc}k & k^2+1\\ 1 & k\end{array}\right].$$

So the fundamental solution of the positive Pell equation $x^2 - (k^2 + 1)y^2 = 1$ is (x_1, y_1), where

$$[-x_1 \quad y_1] = \tau_{F,3}\tau_{F,1}^{-1}\left[\begin{array}{c}1\\0\end{array}\right] = \left[\begin{array}{c}-2k^2-1\\2k\end{array}\right]$$

and the fundamental solution of the negative Pell equation $x^2-(k^2+1)y^2=-1$ is (x_1,y_1), where

$$[x_1 \quad y_1]=(\tau_{F,3})^*(\tau_{F,1}^{-1})^*\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}k\\1\end{bmatrix}$$

as we wanted.

(2) Let $d=k^2-1$ for $k\geq 2$. Then as in (1), we deduce that $\sqrt{d}=[k-1;\overline{1,2k-2}]$. So $A_0=k-1, A_1=k, B_0=1$ and $B_1=1$. Thus the fundamental solution of the positive Pell equation $x^2-(k^2-1)y^2=1$ is $(x_1,y_1)=(A_1,B_1)=(k,1)$ and the negative Pell equation $x^2-(k^2-1)y^2=-1$ has no integer solutions. Further by (2) of Theorem 2.1, we see that $\tau_{F,3}\tau_{F,1}^{-1}=\begin{bmatrix}-k & k^2-1\\1 & -k\end{bmatrix}$. Thus the fundamental solution of the positive Pell equation $x^2-(k^2-1)y^2=1$ is (x_1,y_1), where

$$[-x_1 \quad y_1]=\tau_{F,3}\tau_{F,1}^{-1}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-k\\1\end{bmatrix}.$$

(3) Let $d=k^2+2$ for $k\geq 1$. Then $\sqrt{d}=[k;\overline{k,2k}]$. So $A_0=k, A_1=k^2+1$, $B_0=1$ and $B_1=k$. Thus the fundamental solution of the positive Pell equation $x^2-(k^2+2)y^2=1$ is $(x_1,y_1)=(A_1,B_1)=(k^2+1,k)$ and the negative Pell equation $x^2-(k^2+2)y^2=-1$ has no integer solutions. Further by (3) of Theorem 2.1, we see that $\tau_{F,3}\tau_{F,1}^{-1}=\begin{bmatrix}-k^2-1 & k^3+2k\\k & -k^2-1\end{bmatrix}$. So the fundamental solution of the positive Pell equation $x^2-(k^2+2)y^2=1$ is (x_1,y_1), where

$$[-x_1 \quad y_1]=\tau_{F,3}\tau_{F,1}^{-1}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}-k^2-1\\k\end{bmatrix}.$$

(4) Let $d=k^2-2$ for $k\geq 3$. Then $\sqrt{d}=[k-1;\overline{1,k-2,1,2k-2}]$. So $A_0=k-1, A_1=k, A_2=k^2-k-1, A_3=k^2-1, B_0=1, B_1=1, B_2=k-1$ and $B_3=k$. Thus the fundamental solution of the positive Pell equation $x^2-(k^2-2)y^2=1$ is $(x_1,y_1)=(A_3,B_3)=(k^2-1,k)$ and the negative Pell equation $x^2-(k^2-2)y^2=-1$ has no integer solutions. Also $\tau_{F,5}\tau_{F,1}^{-1}=\begin{bmatrix}k^2-1 & -k^3+2k\\-k & k^2-1\end{bmatrix}$ by (4) of Theorem 2.1. So the fundamental solution of the positive Pell equation $x^2-(k^2-2)y^2=1$ is (x_1,y_1), where

$$[x_1 \quad -y_1]=\tau_{F,5}\tau_{F,1}^{-1}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}k^2-1\\-k\end{bmatrix}.$$

(5) Let $d=k^2+k$ for $k\geq 2$. Then $\sqrt{d}=[k;\overline{2,2k}]$. So $A_0=k, A_1=2k+1$, $B_0=1$ and $B_2=2$. Thus the fundamental solution of the positive Pell equation $x^2-(k^2+k)y^2=1$ is $(x_1,y_1)=(A_1,B_1)=(2k+1,2)$ and the negative Pell equation

$x^2-(k^2+k)y^2=-1$ has no integer solutions. Also $\tau_{F,3}\tau_{F,1}^{-1}=\begin{bmatrix} -2k-1 & 2k^2+2k \\ 2 & -2k-1 \end{bmatrix}$ by (5) of Theorem 2.1. So the fundamental solution of the positive Pell equation $x^2-(k^2+k)y^2=1$ is (x_1,y_1), where

$$[-x_1 \quad y_1]=\tau_{F,3}\tau_{F,1}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} -2k-1 \\ 2 \end{bmatrix}.$$

(6) Let $d=k^2-k$ for $k\geq 3$. Then $\sqrt{d}=[k-1;\overline{2,2k-2}]$. So $A_0=k-1$, $A_1=2k-1, B_0=1$ and $B_2=2$. Thus the fundamental solution of the positive Pell equation $x^2-(k^2-k)y^2=1$ is $(x_1,y_1)=(A_1,B_1)=(2k-1,2)$ and the negative Pell equation $x^2-(k^2-k)y^2=-1$ has no integer solutions. Also $\tau_{F,3}\tau_{F,1}^{-1}=\begin{bmatrix} -2k+1 & 2k^2-2k \\ 2 & -2k+1 \end{bmatrix}$ by (6) of Theorem 2.1. So the fundamental solution of the positive Pell equation $x^2-(k^2-k)y^2=1$ is (x_1,y_1), where

$$[-x_1 \quad y_1]=\tau_{F,3}\tau_{F,1}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=\begin{bmatrix} -2k+1 \\ 2 \end{bmatrix}.$$

This completes the proof. □

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