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Multiple interaction strategies in networks related to graph spectra and dominant sets

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Abstract: An interaction network is a collection of agents with pairwise connections described by an graph. Our objective is to maximize the payoff of the agents simultaneously. In the classical strategic complements or substitutes setup, the objective function has a linear and a quadratic part, and maximized under linear constraints.

To address this task, we use quadratic objective functions on linear or quadratic constraints. We will show how existing results of combinatorial graph theory and spectral clustering can be used to solve the optimization problems, where solutions are closely related to dominant sets or spectral clusters. Our primary focus is on the graph and show how certain model parameters can be built into the edge-weight matrix to get a new objective, thus modifying the interactions between the agents.

Keywords: strategic complements and substitutes, edge-weighted graphs, dominant sets, eigenvalues, spectral clusters.

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1 Introduction

We consider edge-weighted graphs and extend existing results on strategic interactions [1, 9] to them. In the classical papers there are unweighted interactions between the agents, and their actions, we are looking for, are nonnegative real numbers. However, without exact meaning, the scaling and the actual values of these actions do not carry too much information for the physical or economic features of them. In fact, they are rather compared with respect to the agents, and in this way, give important information about agent groups that follow similar strategies, and hence about the overall structure of the network from the point of view of the underlying activity towards which the strategies are considered.

Here we rather investigate the problem from the point of the view of the graph. Based on the spectral properties of a graph based matrix, we are able to tell how many and what kind of strategies can be optimal for the agents, and find agent groups following similar strategies. Since the agents form a social network, the optimal or nearly optimal strategies should inevitably be adapted to the structure of the underlying graph. Together with clustering, we also use evaluation of the vertices and edges, which give optima of potential functions, sometimes related to eigenvectors or weighted characteristic vectors of dominant sets.

The structure of the paper is as follows. In Section 2 we introduce the basic notions, and the classical models of strategic complements and substitutes. In Section 3 we consider quadratic objective function over linear constraints. If we optimize over the standard simplex, we can use the results of Motzkin and Straus [13] to unweighted and those of Pavan and Pelillo [16] to edge-weighted graphs. In this way, unweighted and weighted indicator vectors of maximal cliques and dominant sets enter into the solution. In Section 4 quadratic constraints are considered, under which our quadratic optimization has an explicit solution based on eigenvalues and eigenvectors of graph based matrices. Here we use multiple strategies and spectral clustering tools of [4]. We will show that the existence of large positive eigenvalues makes rise to a complementary, whereas that of outstanding negative eigenvalues to a substitute strategy. Some coordinates of the multi-dimensional strategies of some agents can be negative here, but with appropriate rotations the strategy vectors can be substituted by vectors close to weighted indicator vectors of agent groups. Simulation results on generalized random graphs are also presented. We close the paper with a short discussion in Section 5.

2 Preliminaries

2.1 Notation

Let $G = (V, \mathbf{W})$ be *edge-weighted graph* with vertex-set $V = \{1, \dots, n\}$ and $n \times n$ symmetric edge-weight matrix $\mathbf{W}$ of nonnegative entries and zero diagonal. The vertices correspond to the agents, while the weights represent their pairwise similarity or connectedness. The diagonal is zero, as there are not self-loops at the moment.

Let $d_i = \sum_{j=1}^{n} w_{ij}$ be the *generalized degree* of vertex i; the degrees are sometimes collected in the *degree-vector* $\mathbf{d} = (d_1, \dots, d_n)^T$ or in the diagonal *degree-matrix* $\boldsymbol{D} = \operatorname{diag}(\mathbf{d})$. In the edge-weighted case we assume that $\sum_{i=1}^{n} d_i = 1$, since the normalized edge-weight matrix, $\boldsymbol{D}^{-1/2}\mathbf{W}\boldsymbol{D}^{-1/2}$, is not affected by this normalization. In the unweighted case, $\mathbf{W}$ has 1-0 entries depending on whether two agents are connected or not, so it is the usual adjacency matrix of a simple graph, and is denoted by $\mathbf{A}$.

2.2 Game of strategic complements

Based on [1, 9], the strategic complements setup is the following. We generalize their model to an edge-weighted graph $G = (V, \mathbf{W})$; the agents correspond to the vertices and they act with continuous strategies: $x_i \geq 0$ $(i = 1, \dots, n)$, $\mathbf{x} := (x_1, \dots, x_n)^T$. The payoff of player i is

$$u_i(\mathbf{x}) = \alpha x_i - \frac{1}{2}x_i^2 + \phi \sum_{j=1}^{n} w_{ij} x_i x_j,$$

where α and ϕ are given positive constants. The first term is the benefit of agent i using strategy x_i, the second is the cost of agent i, and the last term is the utility (under strategic complementarity in efforts), i.e., the payoff due to his/her collaboration with the neighbors (the neighbors of i are vertices of the set $\{j : w_{ij} > 0\}$, and they are connected to i with strengths proportional to the edge-weights). The players are equivalent, only their network positions differ.

Agents want to maximize their payoffs at the same time, but they can rule only their own strategies. Therefore, we have to maximize $u_i(\mathbf{x})$ with respect to x_i for $i = 1, \dots, n$. Via

$$\frac{\partial u_i(\mathbf{x})}{\partial x_i} = \alpha - x_i + \phi \sum_{j=1}^{n} w_{ij} x_j = 0, \quad i = 1, \dots, n,$$

for the optimal $\mathbf{x}^*$ we have $\mathbf{x}^* = \alpha\mathbf{1} + \phi\mathbf{W}\mathbf{x}^*$, or equivalently, $(\mathbf{I} - \phi\mathbf{W})\mathbf{x}^* = \alpha\mathbf{1}$, where $\mathbf{1}$ is the all 1's vector, and the vectors are column vectors. Consequently,

$$\mathbf{x}^* = \alpha(\mathbf{I} - \phi\mathbf{W})^{-1}\mathbf{1} \tag{2.1}$$

is a unique and inner solution (equilibrium) if $\mathbf{I} - \phi\mathbf{W}$ is positive definite, see also the forthcoming potential function view of (2.2). Denoting by $\lambda_1 \geq \dots \geq \lambda_n$ the eigenvalues of $\mathbf{W}$, this condition holds if and only if $1 - \|\mathbf{W}\| > 0$, or equivalently, $\phi < \frac{1}{\lambda_1}$.

Here we used that $\mathbf{W}$ is a Frobenius type matrix, therefore λ_1 is the maximum absolute value eigenvalue of $\mathbf{W}$ with eigenvector of nonnegative coordinates. Since $\mathrm{tr}\,(\mathbf{W}) = 0$, $\lambda_n = \lambda_{min}(\mathbf{W}) < 0$ and $|\lambda_n| < \lambda_1$. In this case, the following expansion works:

$$(\mathbf{I} - \phi\mathbf{W})^{-1} = \sum_{k=0}^{\infty} \phi^k \mathbf{W}^k = \mathbf{I} + \phi\mathbf{W} + \phi^2\mathbf{W}^2 + \dots .$$

Consequently, when $\alpha > 0$, then

$$\mathbf{x}^* = \alpha(\sum_{k=0}^{\infty} \phi^k \mathbf{W}^k)\mathbf{1}.$$

Note that the ith coordinate of $\mathbf{W1}$ is d_i, whereas the ith coordinate of $\mathbf{W}^k\mathbf{1}$, denoted by $d_i(k,\mathbf{W})$, is the sum of the positive edge-weights of walks of length k emanating from vertex i; in particular, $d_i(1,\mathbf{W}) = d_i$. Hence, all the coordinates of $\mathbf{x}^*$ are positive:

$$x_i^* = \alpha\left(1 + \sum_{k=1}^{\infty} \phi^k d_i(k,\mathbf{W})\right), \quad i = 1,\dots,n.$$

When $\mathbf{W}$ is the usual 0-1 adjacency matrix $\mathbf{A}$ of an unweighted graph, then $d_i(k,\mathbf{A})$ is the number of walks of length k emanating from i, and $1+\sum_{k=1}^{\infty}\phi^k d_i(k,\mathbf{A})$ is called the *Katz–Bonacich centrality* of vertex i. Therefore, $x_i^* \geq \alpha$, and equality holds if and only if $\phi = 0$. Observe that now d_i is the usual degree of vertex i, and as a consequence of the Frobenius theory, $d_{min} \leq \lambda_1 \leq d_{max}$, therefore $\lambda_1 \geq 1$ and $\phi < 1$. The closer ϕ to 0, the more rapidly ϕ^k decreases, and the shorter walks dominate this centrality.

If $\alpha = 0$ (no individual benefit, the payoff is only due to collaboration with others), then $x_i^* = \phi\sum_{j=1}^n w_{ij}x_j^*$, and the payoff is maximal when ϕ is the largest eigenvalue λ_1 of $\mathbf{W}$ and $\mathbf{x}^*$ is the corresponding eigenvector (with nonnegative coordinates, due to the Frobenius theory).

An equivalent way of reasoning is via *potential function* (the sum of the utilities corrected by a term which takes into account the network extremalities exerted by each player) as follows:

$$P(\mathbf{x}) = \sum_{i=1}^n u_i(\mathbf{x}) - \frac{\phi}{2}\sum_{i=1}^n\sum_{j=1}^n w_{ij}x_ix_j = \alpha\mathbf{x}^T\mathbf{1} - \frac{1}{2}\mathbf{x}^T(\mathbf{I}-\phi\mathbf{W})\mathbf{x}. \tag{2.2}$$

It is easy to verify that $\frac{\partial u_i(\mathbf{x})}{\partial x_i} = \frac{\partial P(\mathbf{x})}{\partial x_i}$ for $i = 1,\dots,n$. The above $P(\mathbf{x})$ has a unique interior maximum if P is strictly concave, i.e., $-(\mathbf{I}-\phi\mathbf{W})$ is negative definite. Equivalently, $\mathbf{I}-\phi\mathbf{W}$ is positive definite, for which fact a necessary and sufficient condition is that $\phi\lambda_1 < 1$. After differentiating P with respect to $\mathbf{x}$, we get back (2.1).

2.3 Game of strategic substitutes

This type of an interaction, defined in [1], is computationally less tractable, but indicates real competition between the agents, where agents want to use their neighbors' benefit instead of their own actions; in particular, free-riders. We adapt the setup of [9] to edge-weighted graphs, with the strategies $x_i \geq 0$ $(i = 1,\dots,n)$ and given positive parameters α,δ. In view of this model, the payoff (utility) of player i is

$$u_i(\mathbf{x}) = \alpha x_i - \frac{1}{2}x_i^2 - \delta\sum_{j=1}^n w_{ij}x_ix_j,$$

where the first term is the benefit of agent i using strategy x_i, the second is the cost of agent i, and the last term is his/her utility (under strategic substitute in efforts), i.e., the payoff due to the competition with the neighbors. Here efforts are decreased by the actions of the neighbors; for example, one do not want to borrow a book if their friends have it, or farmers do not want to plant the same crop as their neighbors do.

Agents again want to maximize their payoffs $u_i(\mathbf{x})$ with respect to x_i at the same time ($i = 1, \dots, n$). This is equivalent to maximizing the potential function:

$$P(\mathbf{x}) = \sum_{i=1}^{n} u_i(\mathbf{x}) + \frac{\delta}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} x_i x_j = \alpha \mathbf{x}^T \mathbf{1} - \frac{1}{2} \mathbf{x}^T (\mathbf{I} + \delta \mathbf{W}) \mathbf{x}.$$

Via differentiation, we get

$$\frac{\partial P(\mathbf{x})}{\partial x_i} = \alpha - x_i - \delta \sum_{j=1}^{n} a_{ij} x_j = 0, \quad i = 1, \dots, n.$$

This yields the system of equations

$$\mathbf{x}^* = \alpha \mathbf{1} - \delta \mathbf{W} \mathbf{x}^* \quad \text{if} \quad \mathbf{x}^* \geq \mathbf{0}. \tag{2.3}$$

P has a unique interior maximum if it is strictly concave, i.e., $-(\mathbf{I}+\delta\mathbf{W})$ (the Hessian of P) is negative definite. Equivalently, $\mathbf{I} + \delta \mathbf{W}$ is positive definite, for which fact a necessary and sufficient condition is that $\delta < \frac{-1}{\lambda_n} = \frac{1}{|\lambda_n|}$. However, we have to ensure that the coordinates of the optimizing $\mathbf{x}^*$ are nonnegative. Hence we get the quadratic programming task:

$$\begin{aligned} &\text{maximize} \quad P(\mathbf{x}) = \alpha \mathbf{x}^T \mathbf{1} - \frac{1}{2} \mathbf{x}^T (\mathbf{I} + \delta \mathbf{W}) \mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \geq 0, \end{aligned}$$

where $\mathbf{x} \geq \mathbf{0}$ means that $x_i \geq 0$, $i = 1, \dots, n$. In accord with [2] and Lemma 1 of [9]: $\mathbf{x}$ is a Nash equilibrium of the substitute game if and only if $\mathbf{x}$ satisfies the following Kuhn–Tucker conditions:

$$\frac{\partial P}{\partial x_i} = 0 \quad \text{and} \quad x_i > 0 \qquad \text{or} \qquad \frac{\partial P}{\partial x_i} \leq 0 \quad \text{and} \quad x_i = 0.$$

By Proposition 1 of [9], in the Nash equilibrium, there will be active agents with $x_i > 0$ ($i \in U$), and inactive ones with $x_i = 0$ ($i \in \overline{U}$); such an $\mathbf{x}$ is called *corner* solution with *support* U. Then the above conditions are equivalent to

$$(\mathbf{I}_U + \delta \mathbf{W}_U) \mathbf{x}_U = \alpha \mathbf{1} \quad \text{and} \quad \delta \mathbf{W}_{\overline{U},U} \mathbf{x}_U \geq \alpha \mathbf{1},$$

where the set in the lower index indicates the corresponding segment of the vector or matrix. The authors of [9] recommend maximizing over all subsets U, but it is computationally intractable. In Section 3 we will show how corner solutions are obtained, at least approximately, by using iterative algorithms.

In [9], it is also shown how partial transformations between substitutes and complements can be applied when δ is 'small'. Based on this, they distinguish between different types of solutions according to the range of δ. Actually, local substitutes can be changed into global substitutes and local complements in the following way; we adapt their reasoning to an edge-weighted graph in the case when $0 \le w_{ij} \le 1$ ($i \neq j$). Let $\overline{G} = (V, \overline{\mathbf{W}})$ denote the *complementary graph* of $G = (V, \mathbf{W})$ with edge-weights $\bar{w}_{ij} = 1 - w_{ij}$ for $i \neq j$ and $\bar{w}_{ii} = 0$ for $i = 1, \dots, n$. If $\mathbf{C}$ is the adjacency matrix of the complete graph K_n, i.e., $\mathbf{C} = \mathbf{1}\mathbf{1}^T - \mathbf{I}$, then $\overline{\mathbf{W}} = \mathbf{C} - \mathbf{W}$. Therefore,

$$
\begin{aligned}
u_i(\mathbf{x}) &= \alpha x_i - \frac{1}{2}x_i^2 - \delta \sum_{j=1}^{n} [1 - (1 - w_{ij})] x_i x_j \\
&= \alpha x_i - \frac{1}{2}x_i^2 - \delta \sum_{j=1}^{n} x_i x_j + \delta \sum_{j=1}^{n} (1 - w_{ij}) x_i x_j \\
&= \alpha x_i - \frac{1}{2}x_i^2 - \delta \sum_{j=1}^{n} x_i x_j + \delta \sum_{j \neq i} (1 - w_{ij}) x_i x_j + \delta x_i^2 \\
&= \alpha x_i - \frac{1}{2}x_i^2 - \delta \sum_{j=1}^{n} x_i x_j + \delta \sum_{j=1}^{n} \bar{w}_{ij} x_i x_j + \delta x_i^2 \\
&= \alpha x_i - \frac{1}{2}(1 - 2\delta) x_i^2 - \delta \sum_{j=1}^{n} x_i x_j + \delta \sum_{j=1}^{n} \bar{w}_{ij} x_i x_j ,
\end{aligned}
\tag{2.4}
$$

which is a game of global substitutes and local complements investigated by [1]. Here the complementarities are realized via $\overline{G}$.

In view of Theorem 1 of [1], there is a unique equilibrium if $1 - \delta > \delta \lambda_{max}(\overline{\mathbf{W}})$. Hence,

$$
\delta < \frac{1}{1 + \lambda_{max}(\overline{\mathbf{W}})}. \tag{2.5}
$$

For finding the unique equilibrium $\mathbf{x}^*$, the constant of the Katz-Bonacich centrality is $\lambda^* = \frac{\delta}{1-\delta}$. Now, let us solve (2.3), i.e., $\alpha\mathbf{1} - \mathbf{x} - \delta\mathbf{W}\mathbf{x} = \mathbf{0}$. Making use of the previous transformations,

$$
\alpha\mathbf{1} - \mathbf{x} - \delta\mathbf{C}\mathbf{x} + \delta(\mathbf{C} - \mathbf{W})\mathbf{x} = \alpha\mathbf{1} - \delta\mathbf{I}\mathbf{x} - (1 - \delta)\mathbf{x} - \delta\mathbf{C}\mathbf{x} + \delta\overline{\mathbf{W}}\mathbf{x} = \mathbf{0}
$$

and

$$
\alpha\mathbf{1} - \delta(\mathbf{I} + \mathbf{C})\mathbf{x} - (1 - \delta)(\mathbf{I} - \frac{\delta}{1 - \delta}\overline{\mathbf{W}})\mathbf{x} = \mathbf{0}.
$$

We will use that

$$(\mathbf{I}+\mathbf{C})\mathbf{x} = (\mathbf{1}\mathbf{1}^T)\mathbf{x} = (\mathbf{1}^T\mathbf{x})\mathbf{1} = x\mathbf{1},$$

where $x = \mathbf{1}^T\mathbf{x} = \sum_{i=1}^n x_i$. Therefore,

$$(\alpha - \delta x)\mathbf{1} = (1-\delta)(\mathbf{I} - \frac{\delta}{1-\delta}\overline{\mathbf{W}})\mathbf{x},$$

consequently,

$$\mathbf{x} = \frac{\alpha - \delta x}{1-\delta}(\mathbf{I} - \frac{\delta}{1-\delta}\overline{\mathbf{W}})^{-1}\mathbf{1} = \frac{\alpha-\delta x}{1-\delta}\mathbf{y}.$$

The inverse exists under (2.5), and can be expanded like the Katz-Bonacich centrality. However, the right hand side also depends on $\mathbf{x}$ through x. To get rid of this dependence, we introduce $\mathbf{y}$ and $y = \sum_{i=1}^n y_i$. Summing up the coordinates, $x = \frac{\alpha-\delta x}{1-\delta} y$, consequently, $x = \frac{\alpha y}{1-\delta+\delta y}$. This implies that

$$\mathbf{x} = \frac{\alpha - \frac{\delta\alpha y}{1-\delta+\delta y}}{1-\delta}\mathbf{y} = \frac{\alpha}{1-\delta+\delta y}\mathbf{y},$$

where

$$\mathbf{y} = (\mathbf{I} - \frac{\delta}{1-\delta}\overline{\mathbf{W}})^{-1}\mathbf{1} = \left[\sum_{k=0}^{\infty}\left(\frac{\delta}{1-\delta}\right)^k \overline{\mathbf{W}}^k\right]\mathbf{1}.$$

Therefore,

$$x_i^* = \frac{\alpha}{1-\delta+\delta y}\left[1 + \sum_{k=1}^{\infty}\left(\frac{\delta}{1-\delta}\right)^k d_i(k, \overline{\mathbf{W}})\right],$$

where $d_i(k, \overline{\mathbf{W}})$ is the sum of the positive edge-weight of walks of length k emanating from vertex i of $\overline{G}$. Since $\frac{\delta}{1-\delta} < 1$ (it decreases with δ), it suffices to consider the first terms. Consequently, x_i^* is 'large' if i has 'strong' connections in the complement graph, or equivalently, 'weak' connections in the original graph. Hence, it seems reasonable, that a set close to the maximal independent one carries the leading strategies.

Summarizing, the following cases of [9] apply in the edge-weighted setup too:

- If $\delta < \frac{1}{1+\lambda_{max}(\overline{\mathbf{W}})}$, then a unique inner equilibrium exists $x_i > 0$ $(i = 1, \dots, n)$.
- If $\frac{1}{1+\lambda_{max}(\overline{\mathbf{W}})} \le \delta < -\frac{1}{\lambda_{min}(\mathbf{W})}$, then a unique equilibrium exits which is a corner or inner point.
- If $-\frac{1}{\lambda_{min}(\mathbf{W})} \le \delta < 1$, then there are multiple equilibria among those there are corners. In this case, only corner equilibria can be stable.

If $\delta = 1$ (see [7, 8], the stable equilibrium is corner: $\mathbf{x}_U = \mathbf{1}$, where U is a maximal independent set of G. Note that the maximal independent sets of G are the maximal cliques of $\overline{G}$, and to find them we recommend algorithms in Section 3.

We remark that the lower range of δ can be made wider and the middle range $\frac{1}{1+\lambda_{max}(\overline{\mathbf{A}})} \le \delta < -\frac{1}{\lambda_{min}(\mathbf{A})}$ narrower by using results of [10, 17], when we have an unweighted graph $G = (V, \mathbf{A})$ at the beginning. In view of these, we are able to find an edge-weighted graph $(V, \mathbf{W})$ with the same skeleton as G, i.e., $w_{ij} = 0$ whenever $a_{ij} = 0$, for which $\lambda_{min}(\mathbf{W})$ is the largest possible. Likewise, for the complementary graph $\overline{G} = (V, \overline{\mathbf{A}})$ we are also able to find an edge-weighted graph $(V, \overline{\mathbf{W}})$, with the same skeleton as $\overline{G}$, for which $\lambda_{max}(\overline{\mathbf{W}})$ is the smallest possible. To find the optimal edge-weights, the authors of [10, 11, 17] suggest theory and algorithms. Roughly speaking, we have to decompose the underlying graph into odd cycles and balanced bipartite graphs, and assign symmetric evaluations to their vertices, which in turn give the optimal evaluations of the edges.

3 Optimizing over the unit simplex

3.1 Maximal cliques and interactions

First, let us consider the simplest case of an edge-weighted graph $G = (V, \mathbf{W})$ when the agents have only mutual benefits and there are complementarities between them. Then the utility of agent i is

$$u_i(\mathbf{x}) = b \sum_{j=1}^{n} w_{ij} x_i x_j$$

with positive normalizing constant b, and we maximize it with respect to x_i, the strategy of agent i, for $i = 1, \dots, n$ over the simplex

$$S = \{x_i \ge 0\, (i = 1, \dots, n), \quad \sum_{i=1}^{n} x_i = 1\} = \{\mathbf{x} \ge \mathbf{0}, \quad \mathbf{x}^T \mathbf{1} = 1\}.$$

This is equivalent to the following quadratic programming task:

$$\begin{aligned} &\text{maximize} \quad P(\mathbf{x}) = \frac{1}{2} b \mathbf{x}^T \mathbf{W} \mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \in S. \end{aligned} \tag{3.1}$$

Apart from the constant $b > 0$, the quadratic form $\mathbf{x}^T \mathbf{W} \mathbf{x}$ maximizes the cohesiveness of a cluster of vertices with fuzzy membership vector $\mathbf{x} \ge 0$ under the simplex constraint. With a 'small' value of the coordinate x_i, vertex i is weakly, while with a 'large' value, it is strongly associated with the cluster. Under cluster we understand internal homogeneity and external inhomogeneity of the vertices included in it.

Motzkin and Strauss were the first to consider this quadratic programming task for simple graphs as the continuous relaxation of the *maximal clique* problem. A clique $C \subset V$ (complete subgraph) of the simple graph $G = (V, \mathbf{A})$ is maximal if no strict superset of C is a clique. A maximal clique C is *strictly maximal* if no vertex i external to C has the property that that the enlarged set $C \cup \{i\}$ contains a clique of the size $|C|$. Maximal cliques can be several (even overlapping), and to find all of them is NP-hard. A *maximum clique* is a maximal clique with largest cardinality. The characteristic vector of a vertex-subset $U \subset V$ is denoted $\mathbf{x}^U$ and is defined with the following coordinates: $x_i^U = \frac{1}{|U|}$ if $i \in U$ and 0 otherwise.

Theorem 3.1 (Motzkin–Strauss theorem as formulated in [6]). *Let $G = (V, \mathbf{A})$ be a simple graph and $C \subset V$. Then $(\mathbf{x}^C)^T \mathbf{A} (\mathbf{x}^C) = 1 - \frac{1}{|C|}$ if and only if C is a clique. Moreover,*

- *$\mathbf{x}^C$ is a strict local maximizer of $\mathbf{x}^T \mathbf{A} \mathbf{x}$ over S if and only if C is a strictly maximal clique.*
- *$\mathbf{x}^C$ is a global maximizer of $\mathbf{x}^T \mathbf{A} \mathbf{x}$ over S if and only if C is a maximum clique.*

In case of an unweighted graph G, Motzkin and Straus [13] further generalized the maximization problem to what they called *non-square-free quadratic forms*. Their theorem solves the problem of maximizing the utility function

$$u_i(\mathbf{x}) = d_i x_i^2 + \sum_{j \sim i} x_j^2 + b x_i \sum_{j \sim i} x_j$$

with respect to x_i $(i = 1, \dots, n)$ over S. Here the first term is the benefit of the agent i proportional to his/her number of ties (d_i is the degree of vertex i), the second term is the sum of the benefits of the neighbors, while the last term is the mutual benefit due to collaboration multiplied with the constant $b > 0$. This model may not be applicable in economy, but in cultural collaborations and co-authorships, where personal costs are not counted and the agents are glad with the success of their neighbors, it indeed has rational. Since

$$u_i(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (x_i^2 + x_j^2 + b x_i x_j),$$

in the potential function context this is equivalent to

$$\begin{aligned} &\text{maximize} \quad P(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T (\boldsymbol{D} + \frac{b}{2} \mathbf{A}) \mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \in S, \end{aligned} \tag{3.2}$$

where $\boldsymbol{D}$ is the diagonal degree-matrix. The solution is given in [13], and depending on the relation of d_{max} and $\frac{b}{2}$, local maxima are again related to characteristic vectors supported by maximal cliques or vertices having maximal degree.

Theorem 3.2 (Theorem 4 of [13]). *Let $G = (V, \mathbf{A})$ be unweighted graph, and let d_{max} denote its maximal vertex degree. Then a strict local maximum of (3.2) is the following:*

- *If $d_{max} > \frac{b}{2}$ then $\max_S P(\mathbf{x}) = d_{max}$, and the maximum is attained at an $\mathbf{x}$ which is the characteristic vector of a vertex of degree d_{max}.*
- *If $d_{max} = \frac{b}{2}$, then $\max_S P(\mathbf{x}) = d_{max}$ and the maximum is attained at the weighted characteristic vector of a complete subgraph, all of whose vertices have degree d_{max}.*
- *If $d_{max} < \frac{b}{2}$, then $\max_S P(\mathbf{x}) = \frac{b}{2} - \frac{c}{2}$ with $\frac{1}{c} = \max_{G'} \sum_{G'} (b - 2d_i)^{-1}$, where G' ranges over the cliques of G; the maximum is attained at an $\mathbf{x}$ with coordinates $x_i = \frac{c}{b-2d_i}$ for $i \in G'$ and $x_j = 0$ for $j \notin G'$.*

3.2 Dominant sets and weighted characteristic vectors

Now let $G = (V, \mathbf{W})$ be an edge-weighted graph. We will use the notion of a dominant set as introduced by Pavan and Pelillo [16] as follows. Let $U \subset V$ and $j \notin U$. Then

$$\varphi_U(i, j) = w_{ij} - \frac{1}{|U|} \sum_{l \in S} w_{il}, \quad i \in U$$

is the relative similarity between vertices i and j with respect to the average similarity between vertex i and its neighbors in U, where the second term is the average weighted degree of i with respect to vertices of U. Note that $\varphi_U(i, j)$ is positive if the connection between vertices i and j is stronger than the connection between vertex i and its neighbors in U, and it is negative, otherwise. Using their relative similarity, the weight of vertex i with respect to U is defined by the following recursive formula:

$$\mathbf{w}_U(i) = \begin{cases} 1, & \text{if } |U| = 1 \\ \sum_{l \in U \setminus \{i\}} \varphi_{U \setminus \{i\}}(l, i) \mathbf{w}_{U \setminus \{i\}}(l), & \text{otherwise.} \end{cases}$$

The total weight of U is $W(U) = \sum_{i \in U} \mathbf{w}_U(i)$. The function $\mathbf{w}_U(i)$ measures the relative similarity between vertex i and the vertices of $U \setminus \{i\}$ with respect to the overall similarity among the vertices in $U \setminus \{i\}$.

Definition 3.3. If $W(T) > 0$ for any nonempty $T \subseteq U$, $U \subseteq V$, then U is a dominant set if

- $\mathbf{w}_U(i) > 0$, for all $i \in U$,
- $\mathbf{w}_{U\cup\{i\}}(i) < 0$, for all $i \notin U$.

These two conditions correspond to the main properties of a cluster: internal homogeneity and external inhomogeneity. The first condition ensures that vertices in U are strongly connected to each other, i.e., U induces a strongly connected subgraph, while the second condition ensures that the set U induces the most strongly connected subset in G. This definition shows that in a dominant set, the overall similarity among its vertices is higher than the similarity between its vertices and the rest of the vertices in V. Note that in an unweighted graph (with 0-1 weights) dominant sets correspond to the strictly maximal cliques. The quadratic programming task

$$\begin{aligned} &\text{maximize} \quad P(\mathbf{x}) = \mathbf{x}^T\mathbf{W}\mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \in S, \end{aligned} \tag{3.3}$$

is the generalization of the problem (3.1) and it favors pairs of vertices with similar coordinates in $\mathbf{x}$ that also have strong connection in $\mathbf{W}$. Pavan and Pelillo [16] characterized the strict local maxima of the above task by means of weighted characteristic vectors.

Definition 3.4. The weighted characteristic vector of a set U, also denoted by $\mathbf{x}^U$, has the following coordinates:

$$x_i^U = \begin{cases} \frac{\mathbf{w}_U(i)}{W(U)}, & \text{if } i \in U \\ 0, & \text{otherwise.} \end{cases}$$

Note that the weighted characteristic vector satisfies the simplex constrains, and it also corresponds to a corner solution in Section 2.3.

Theorem 3.5 (Theorem 1 of [16])**.** *Let $G = (V, \mathbf{W})$ be an edge-weighted graph.*

- *If U is a dominant set of G, then its weighted characteristic vector $\mathbf{x}^U$ is a strict local solution of the program (3.3).*
- *Conversely, if $\mathbf{x}^*$ is a strict local solution of the program (3.3), then its support $\sigma = \{i : x_i^* \neq 0\}$ is a dominant set, provided that $\mathbf{w}_{\sigma\cup\{i\}}(i) \neq 0$ for all $i \notin \sigma$.*

In [15, 16], the authors recommend the so-called *replicator dynamics* to solve the problem (3.3). Namely, they used the following iteration:

$$x_i(t+1) = x_i(t) \frac{(\mathbf{W}\mathbf{x}(t))_i}{\mathbf{x}(t)^T \mathbf{W}\mathbf{x}(t)} \tag{3.4}$$

for $i = 1, \dots, n$ and $t = 0, 1, 2, \dots,$ until convergence. The simplex S is invariant under the above dynamics, which means that every trajectory starting in S will remain in S for the eternity. Further, if $\mathbf{W}$ is symmetric, the objective function is strictly increasing along any nonconstant trajectory of (3.4), and its asymptotically stable points are in one-to-one correspondence to the strict local solutions of (3.3).

To avoid spurious solutions (that are not characteristic vectors), in the unweighted case (0-1 weights) Bomze et al. [6] suggested the following regularization of (3.1) with introducing a positive parameter α:

$$\begin{aligned} &\text{maximize} \quad \mathbf{x}^T(\mathbf{A} + \alpha \mathbf{I})\mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \in S. \end{aligned} \tag{3.5}$$

They proved the following.

Theorem 3.6 (Theorem 10 of [6]). *Let $G = (V, \mathbf{A})$ be an unweighted graph and $0 < \alpha < 1$. Then*

- *the only strict local maximizers of $\mathbf{x}^T(\mathbf{A}+\alpha\mathbf{I})\mathbf{x}$ over S (i.e., the only attracting stationary points under the replicator dynamics with $\mathbf{A} + \alpha\mathbf{I}$ instead of $\mathbf{A}$) are characteristic vectors $\mathbf{x}^C$ where C is a maximal clique of G;*
- *conversely, if C is a maximal clique of G, then $\mathbf{x}^C$ represents a strict local maximizer.*

Therefore, when selecting an $\alpha \in (0, 1)$, e.g., $\alpha = \frac{1}{2}$, all local maximizers of (3.5) are strict and are in one-to-one correspondence with the characteristic vectors of the maximal cliques of the unweighted graph $G = (V, \mathbf{A})$.

It is an open question, what kind of regularization is useful when we have an edge-weighted graph $G = (V, \mathbf{W})$. Since the argmax $\mathbf{x}^T\mathbf{W}\mathbf{x}$ is invariant under scaling the entries of $\mathbf{W}$, we may assume that $0 \le w_{ij} \le 1$ $(i \ne j)$. We conjecture that the regularization with $\alpha \in (0, 1)$ will have the same effect. Alternatively, without normalizing $\mathbf{W}$, we could run the dynamics for $\mathbf{W} + \alpha\mathbf{I}$, where $0 < \alpha < \max_{i \ne j} w_{ij}$.

3.3 Interactions and dominant sets

First, let us consider the simplest case when the agents have individual costs and mutual benefit based on complementaries between them. The connections between the agents is described by the edge-weighted graph $G = (V, \mathbf{W})$. The utility of agent i is

$$u_i(\mathbf{x}) = \beta x_i \sum_{j\sim i} x_j - \alpha x_i^2 = \beta \sum_{j=1}^{n} w_{ij} x_i x_j - \alpha x_i^2 \tag{3.6}$$

with positive constants α and β, balancing between the benefit of agent i due to collaborations and its individual quadratic cost; further, we maximize it with respect to x_i for $i = 1, \dots, n$ over the simplex S.

In potential function view, (3.6) is equivalent to the following quadratic programming task:

$$\begin{aligned} &\text{maximize} \quad P(\mathbf{x}) = \frac{1}{2}\beta \mathbf{x}^T \mathbf{W}\mathbf{x} - \frac{1}{2}\alpha \mathbf{x}^T \mathbf{I}\mathbf{x} = \frac{1}{2}\mathbf{x}^T(\beta\mathbf{W} - \alpha\mathbf{I})\mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \in S. \end{aligned} \tag{3.7}$$

Using the ideas of [15], the solutions of (3.7) remain the same if the matrix $\beta\mathbf{W} - \alpha\mathbf{I}$ is replaced with $\beta\mathbf{W} - \alpha\mathbf{I} + \kappa\mathbf{1}\mathbf{1}^T$, where κ is an arbitrary real number. Indeed, $\kappa\mathbf{x}^T\mathbf{1}\mathbf{1}^T\mathbf{x} = \kappa(\mathbf{x}^T\mathbf{1})^2 = \kappa$, since $\mathbf{x}^T\mathbf{1} = 1$ due to $\mathbf{x} \in S$. In particular, if $\kappa = \alpha$, the resulting matrix has nonnegative entries and zero diagonal. Therefore, Theorem 3.5 is applicable to it, and implies that the strict local maxima of (3.7) are weighted characteristic vectors of dominant sets for the scaled edge-weight matrix $\beta\mathbf{W} + \alpha(\mathbf{1}\mathbf{1}^T - \mathbf{I})$ having zero diagonal and off-diagonal entries equal to $\beta w_{ij} + \alpha \geq 0$ $(i \neq j)$. Let us denote by G' this new edge-weighted graph: $G' = (V, \beta\mathbf{W} + \alpha(\mathbf{1}\mathbf{1}^T - \mathbf{I}))$.

In [15, 16], the authors adapted the replicator dynamics (3.4) to maximize (3.7) over S. Namely, they recommended the following iteration:

$$x_i(t+1) = x_i(t)\frac{(\beta\mathbf{W}\mathbf{x}(t))_i - \alpha x_i(t)}{\mathbf{x}(t)^T(\beta\mathbf{W} - \alpha\mathbf{I})\mathbf{x}(t)} \tag{3.8}$$

for $i = 1, \dots, n$ and $t = 0, 1, 2, \dots$, until convergence.

However, α could basically change the scale that would result in excluding dominant sets under a certain size. When α is large, namely $\alpha > \beta\lambda_{max}(\mathbf{W})$, then the regularization term dominates, and the only solution is an $\mathbf{x}$ having all positive coordinates, and hence, being the weighted characteristic vector of the whole V. If α gets smaller, but $\alpha > \beta\lambda_{max}(\mathbf{W}_U)$, where $\mathbf{W}_U$ is the edge-weight matrix of the induced subgraph of G on the vertex-set $U \subset V$, then there is no maximizing $\mathbf{x}$ with support which is the subset or equal to U. Therefore, if one wants to avoid too small clusters, we select an α according to this rule. Starting with $\alpha = \beta(n-1) \geq \beta\lambda_{max}(\mathbf{W})$, we can decrease α one by one to obtain smaller and smaller clusters, which support the weighted characteristic vector of the solution. However, if $\alpha > \beta(m-1)$, then we exclude characteristic vectors of dominant sets with $|U| \leq m$. Nonetheless, if α

is very small, the effect of regularization becomes negligible and dominant sets of $G = (V, \beta\mathbf{W})$, or equivalently, those of $G = (V, \mathbf{W})$ will enter into the solution.

Summarizing, the constants α and β are built into the edge-weight matrix of G, hence reshaping its structure, and suppressing the ties w_{ij}'s if α is large relative to β. The smaller α, the smaller dominant sets of agents will pursue a non-zero strategy (with the coordinates of the support of their weighted characteristic vectors). This means that if the individual costs are large compared to the mutual benefit, then larger sets of agents can collaborate fruitfully. On the contrary, when the individual costs are small compared to the mutual benefits, then the effect of the original edge-weights dominates, and smaller dominant sets – close to the ones of the original graph – of agents maximize their payoffs at the same time. However, in this case, a larger number of agents is rendered to have zero strategy.

We illustrate this process on a so-called generalized random graph.

Definition 3.7. Let n be a natural number and $k \le n$ be a positive integer. The graph $G_n(\mathbf{P}, \mathcal{P}_k)$ is a generalized random graph with probability matrix $\mathbf{P}$ and proper k-partition $\mathcal{P}_k = (V_1, \dots, V_k)$ of the vertices if it satisfies the following. The vertex set is V, $|V| = n$; the $k \times k$ symmetric matrix $\mathbf{P}$ is such that its entries satisfy $0 \le p_{ij} \le 1$ $(1 \le i \le j \le k)$. Then vertices of V_i and V_j are connected independently, with probability p_{ij}, $1 \le i \le j \le k$.

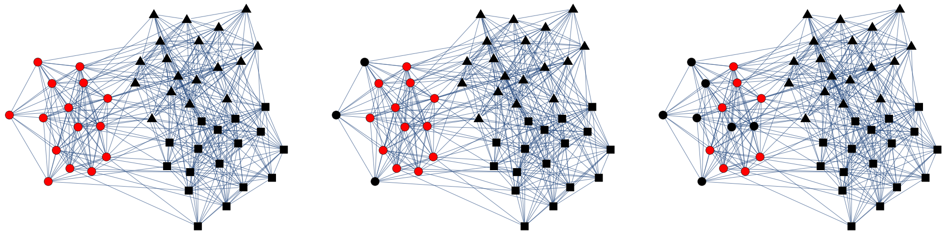
With the probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.1 & 0.15 \\ 0.1 & 0.75 & 0.2 \\ 0.15 & 0.2 & 0.7 \end{pmatrix}$$

a random graph on 50 vertices was generated, where the vertices formed three loosely connected clusters; particularly, Cluster 1 (V_1)is loosely connected to Cluster 2 (V_2) and Cluster 3 (V_3). Depending on the initialization, we obtained indicator vectors of subsets of V_1, V_2, or V_3. The support of them is indicated by red points in Figures 1,2,3. With $\beta = 1$ and decreasing values of α, smaller and smaller supports appeared, but they were concentrated on one of the clusters. The weighted characteristic vectors supported on parts of the first cluster appeared soon, whereas those supported on parts of the second and third clusters were separated later. With $\alpha = -0.5$, the result of Theorem 3.6 is applicable, and we indeed obtain the support of a strongly maximal clique within one of the clusters.

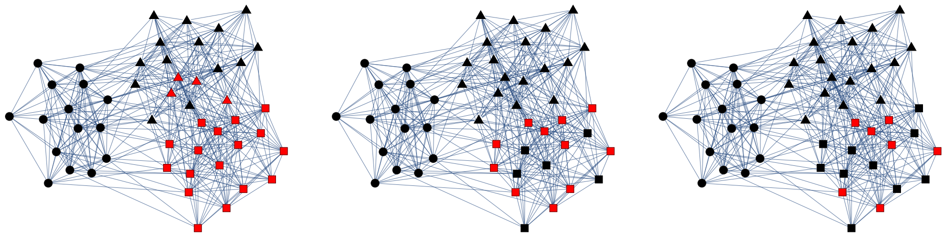
4 Optimizing over spheres and ellipsoids

From now on, we consider *multiple strategies*. The k-dimensional strategies of the agents can be thought of as intensities of buying/selling k different stocks or borro-



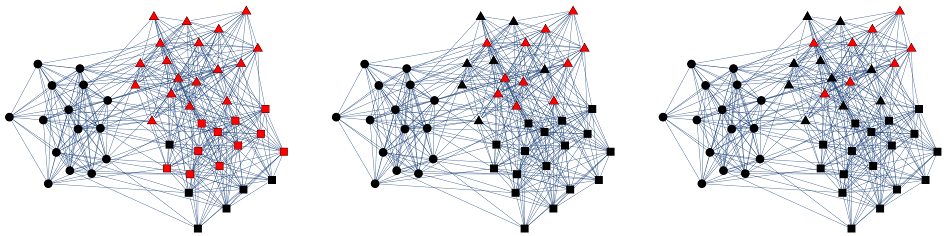
$\alpha \in (3.5, 6.5)$ $\alpha = 1$ $\alpha = -0.5$

Figure 1: Dominant sets with weighted characteristic vectors concentrated on the first cluster. Vertices of the three clusters are denoted by $\text{O}, \square, \triangle$ and red dots indicate the support of the weighted characteristic vector obtained by the dynamics with the actual values of α.



$\alpha = 6.5$ $\alpha \in (1.5, 2)$ $\alpha = -0.5$

Figure 2: Dominant sets with weighted characteristic vectors concentrated on the second cluster. Vertices of the three clusters are denoted by $\text{O}, \square, \triangle$ and red dots indicate the support of the weighted characteristic vector obtained by the dynamics with the actual values of α.



$\alpha = 6.5$ $\alpha = 1.5$ $\alpha \in (-0.5, 0.5)$

Figure 3: Dominant sets with weighted characteristic vectors concentrated on the third cluster. Vertices of the three clusters are denoted by $\text{O}, \square, \triangle$ and red dots indicate the support of the weighted characteristic vector obtained by the dynamics with the actual values of α.

wing/lending k different goods (they may have negative coordinates).

Now the quadratic objective function of Section 3 or its multidimensional extension will be maximized with respect to quadratic constraints. Here we have exact solutions: the maxima are given in terms of the bottom or top eigenvalues of the transformed edge-weight matrix, whereas the optimal multiple strategies are derived by means of the corresponding eigenvectors. The two extremes, corresponding to strategic complements or substitutes are unified into a multiway clustering problem, where we are looking for groups of agents following similar strategies with respect to the other groups, and in this case, strategies can be assigned to the agents, depending on their group memberships.

We saw that in the classical setup of strategic complements (see Section 2.2) when the parameter δ is small ($\delta < \frac{1}{1+\lambda_{max}(\bar{\mathbf{G}})}$), a unique inner equilibrium exists ($\forall x_i > 0$), and it can be found by matrix inversion, also using the Katz–Bonacich centrality. However, in the case of strategic substitutes (see Section 2.3), for larger δ's corner equilibria appear, and these are the only stable equilibria. To find corner equilibria, in [9] the authors define an algorithm which examines all subsets of vertices for possible corner solutions. This is computationally not tractable if the number of vertices is very large, since it is NP-complete. Instead, we may approximate corner equilibria by spectral clustering tools of [4] in polynomial time.

4.1 When there are complementarities between the agents

The utility function of agent i is defined by

$$u_i(\mathbf{X}) = \sum_{\ell=1}^{k} \alpha s_{i\ell}^2 - \frac{1}{2}\sum_{\ell=1}^{k} s_{i\ell}^2 + \phi \sum_{\ell=1}^{k}\sum_{j=1}^{n} w_{ij} s_{i\ell} s_{j\ell} \tag{4.1}$$

where α and ϕ are given positive constants. The first term is the benefit of agent i using strategy x_i, the second is the cost of agent i, and the last term is the utility (under strategic complementarity in efforts), i.e., the payoff due to his/her collaboration with the neighbors. The k-dimensional strategies $\mathbf{s}_1, \dots, \mathbf{s}_n \in \mathbb{R}^k$ of the agents are collected as row vectors of the $n \times k$ matrix $\mathbf{X}$. The coordinate $s_{i\ell}$ of $\mathbf{s}_i$ denotes the strategy of agent i towards the ℓ-th subject. The constant α now scales the quadratic gain of the agents. We assume that $0 < \alpha \leq \frac{1}{2}$, so the gain would not exceed the costs for solitary agents; further, $\phi > 0$ is a constant that serves to regulate the effect of complementarities.

The simultaneous maximization of $u_i(\mathbf{X})$'s with respect to $\mathbf{s}_1, \dots, \mathbf{s}_n$ subject to $\sum_{i=1}^{n} \mathbf{s}_i \mathbf{s}_i^T = \mathbf{X}^T\mathbf{X} = \mathbf{I}_k$ is equivalent to maximizing the following potential function under the same constraint:

$$P(\mathbf{X}) = \sum_{i=1}^{n} u_i(\mathbf{X}) - \frac{\phi}{2} \sum_{i=1}^{n} \sum_{\ell=1}^{k} \sum_{j=1}^{n} w_{ij} s_{i\ell} s_{j\ell}$$
$$= \frac{1}{2} \sum_{\ell=1}^{k} \mathbf{x}_\ell^T [(2\alpha - 1)\mathbf{I} + \phi \mathbf{W}] \mathbf{x}_\ell = \frac{1}{2} \mathrm{tr}\, \mathbf{X}^T [(2\alpha - 1)\mathbf{I} + \phi \mathbf{W}] \mathbf{X},$$

where $\mathbf{x}_1, \dots, \mathbf{x}_k$ denote the column vectors of the suborthogonal matrix $\mathbf{X}$. The maximum of $P(\mathbf{X})$ subject to $\mathbf{X}^T \mathbf{X} = \mathbf{I}_k$ is taken on with the $\mathbf{X}$ that maximizes

$$\mathrm{tr}\, \mathbf{X}^T [(2\alpha - 1)\mathbf{I} + \phi \mathbf{W}] \mathbf{X}$$

on the constraint $\mathbf{X}^T \mathbf{X} = \mathbf{I}_k$. Irrespective of the definiteness of the matrix in brackets, the maximum is attained by an $\mathbf{X}^*$ which contains pairwise orthogonal, unit-norm eigenvectors, corresponding to the k largest eigenvalues of $(2\alpha - 1)\mathbf{I} + \phi \mathbf{W}$ in its columns, and the maximum is $\sum_{l=1}^{k} (2\alpha - 1 + \phi \lambda_l)$, where $\lambda_1 \geq \dots \geq \lambda_n$ are the eigenvalues of $\mathbf{W}$, and it is attained by the corresponding eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ as columns of $\mathbf{X}^*$. These may contain negative coordinates, but they can be approximated by stepwise constant vectors of mainly nonnegative coordinates if the following condition is met: the subspace of these partition-vectors is close to the subspace spanned by $\mathbf{u}_1, \dots, \mathbf{u}_k$. This is the case if there is a gap between λ_k and λ_{k+1}. In this case, the squared distance between these two subspaces is the k-variance of the clusters S_k^2 (see [4]), which is the minimum of the objective function of the k-means algorithm. Hence, the clusters of agents following similar strategies are obtained by applying the k-means algorithm to the optimum strategy vectors, row vectors of the optimum $\mathbf{X}$. Note, that the representatives can as well be rotated so that the column vectors of the matrix $\mathbf{X}^*$ are near to characteristic vectors of the optimizing vertex clusters, giving the same representation, but resulting in near zero or positive strategies. In this way, a k-partition of the vertices is obtained, so that each cluster of the partition is specialized to a strategy out of the k ones. Members of the same cluster pursue the same strategy with the same (positive) intensity , and the others do almost nothing. There are different groups responsible for different strategies (it is possible, since the number of clusters is equal to the number of strategies). In view of [12], when there is a remarkable gap between λ_k and λ_{k+1} these clusters are loosely connected, but themselves define dense subgraphs. Consequently, neighbors, or agents with strong connections will follow similar strategies in all the k respects. In Tables 1,2,3, one rotated eigenvector is concentrated on one cluster, and after suitable normalization it shows good agreement with the weighted characteristic vector obtained in Section 3.3, in terms of the MSE.

Ev1	0.034	0.058	0.044	0.025	0.058	0.006	0.055	0.041	0.003	0.066	-0.007	0.005	0.019	0.065	0.002
Ev2	-0.022	-0.067	-0.043	0.016	-0.085	0.032	-0.065	0.009	0.035	-0.050	0.016	-0.019	-0.019	-0.086	-0.020
Ev3	0.245	0.188	0.281	0.243	0.266	0.275	0.253	0.269	0.290	0.242	0.226	0.259	0.242	0.272	0.196
Wcv	0.065	0.040	0.081	0.063	0.075	0.073	0.069	0.077	0.081	0.065	0.054	0.069	0.060	0.079	0.040

Table 1: Coordinates of the three rotated leading eigenvectors corresponding to the first cluster. The third one (Ev3) is concentrated on Cluster 1, and the MSE between its normalized version and the weighted characteristic vector (Wcv) of this cluster (its non-zero coordinates are in the last row) is 0.0674341.

Ev1	0.263	0.262	0.288	0.190	0.178	0.226	0.201	0.230	0.248
Ev2	-0.105	0.008	-0.111	0.021	-0.087	-0.006	-0.004	0.025	-0.065
Ev3	-0.04	-0.048	-0.033	-0.040	-0.0008	-0.03	-0.016	-0.064	-0.052
Wcv	0.082	0.100	0.103	0.027	0.023	0.032	0.073	0.066	0.068

Ev1	0.232	0.214	0.216	0.295	0.22	0.191	0.21	0.234
Ev2	-0.008	-0.013	0.005	-0.039	-0.047	-0.014	-0.047	-0.039
Ev3	-0.023	-0.013	-0.031	-0.069	0.005	0.022	-0.053	-0.068
Wcv	0.065	0.064	0.106	0.0216	0.046	0.031	0	0.084

Table 2: Coordinates of the three rotated leading eigenvectors corresponding to Cluster 2. The first one (Ev1) is concentrated on Cluster 2, and the MSE between its normalized version and the weighted characteristic vector (Wcv) of this cluster (its non-zero coordinates are in the last row, except the last coordinate, instead of which we have a non-zero coordinate corresponding to a vertex of Cluster 3) is 0.135404.

Ev1	0.066	0.019	0.057	-0.008	0.015	0.037	0.076	0.050	-0.006
Ev2	0.224	0.270	0.205	0.260	0.213	0.268	0.270	0.187	0.299
Ev3	0.045	-0.020	0.011	-0.016	0.026	-0.004	0.006	0.081	0.011
Wcv	0.061	0.079	0.051	0.072	0.032	0.085	0.089	0.036	0.093

Ev1	0.058	0.053	0.139	0.076	0.056	0.077	0.024	0.024	0.003
Ev2	0.190	0.150	0.120	0.122	0.253	0.135	0.172	0.296	0.290
Ev3	0.003	0.029	-0.008	0.009	0.034	0.044	0.015	0.006	0.070
Wcv	0.057	0.011	0.038	0.080	0.023	0.018	0.087	0	0.080

Table 3: Coordinates of the three rotated leading eigenvectors corresponding to Cluster 3. The second one (Ev2) is concentrated on Cluster 3, and the MSE between its normalized version and the weighted characteristic vector (Wcv) of this cluster (its non-zero coordinates are in the last row, except the last coordinate, instead of which we have a non-zero coordinate corresponding to a vertex of Cluster 2) is 0.129198.

In the case of $k = 1$ we optimize over the sphere $\|\mathbf{x}\|=1$, and the above maximum is $2\alpha - 1 + \phi\lambda_1$, which is positive if and only if $\phi > \frac{1-2\alpha}{\lambda_1}$, in view of $\lambda_1 > 0$ (since $\mathbf{W}$ is a Frobenius-type matrix). Because of $0 < \alpha < \frac{1}{2}$, this gives a positive lower bound for ϕ. Consequently, the above maximum is positive.

When $k > 1$ is such that $\lambda_1 \geq \cdots \geq \lambda_k > 0$, then $\sum_{\ell=1}^{k}(2\alpha - 1 + \phi\lambda_\ell) > 0$ holds if $\phi > \frac{k(1-2\alpha)}{\sum_{\ell=1}^{k} \lambda_\ell}$. Therefore, the number of strategies cannot exceed the number of positive eigenvalues of $\mathbf{W}$ to get a positive optimum. However, when the size of G is large, it suffices to select a k such that $\lambda_k > 0$ and it is much 'larger' that λ_{k+1}.

The utility function can be further generalized to

$$u_i(\mathbf{X}) = \sum_{\ell=1}^{k} \alpha s_{i\ell}^2 - \frac{1}{2}\sum_{\ell=1}^{k} s_{i\ell}^2 + \sum_{\ell=1}^{k} \phi_\ell \sum_{j=1}^{n} a_{ij} s_{i\ell} s_{j\ell}, \tag{4.2}$$

when the potential function becomes

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=1}^{n} u_i(\mathbf{X}) - \sum_{\ell=1}^{k} \frac{\phi_\ell}{2} \sum_{j=1}^{n} w_{ij}^{(\ell)} s_{i\ell} s_{j\ell} \\ &= \frac{1}{2}\sum_{l=1}^{k} \mathbf{x}_l^T [(2\alpha-1)\mathbf{I} + \phi_\ell \mathbf{W}^{(\ell)}]\mathbf{x}_\ell, \end{aligned}$$

where $\mathbf{W}^{(\ell)}$ is the edge-weight matrix of the agents under strategy ℓ, $\ell = 1,\dots,k$ (these connections are given, and they may differ for different strategies). For maximizing the sum of the inhomogeneous quadratic forms we introduced an algorithm in [5]. In particular, when $\mathbf{W}^{(1)} = \cdots = \mathbf{W}^{(k)} = \mathbf{W}$, i.e., the matrices in the brackets commute, we select their largest eigenvalues (assuming that ϕ_ℓ's are different) with the corresponding eigenvectors.

Another possibility is to take into consideration the vertex degrees in G. Then the modified utility is

$$u_i(\mathbf{X}) = \sum_{\ell=1}^{k} \alpha d_i s_{i\ell}^2 - \frac{1}{2}\sum_{\ell=1}^{k} d_i s_{i\ell}^2 + \phi \sum_{\ell=1}^{k}\sum_{j=1}^{n} a_{ij} s_{i\ell} s_{j\ell}. \tag{4.3}$$

The simultaneous maximization of $u_i(\mathbf{X})$'s ($i = 1,\dots,k$) subject to $\mathbf{X}^T \boldsymbol{D}\mathbf{X} = \mathbf{I}_k$ is equivalent to maximizing the following potential function under the same constraint:

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=1}^{n} u_i(\mathbf{X}) - \frac{\phi}{2}\sum_{\ell=1}^{k}\sum_{j=1}^{n} a_{ij} s_{i\ell} s_{j\ell} \\ &= \frac{1}{2}\sum_{\ell=1}^{k} (\boldsymbol{D}^{1/2}\mathbf{x}_\ell)^T [(2\alpha-1)\mathbf{I} + \phi \mathbf{W}_D](\boldsymbol{D}^{1/2}\mathbf{x}_\ell) \\ &= \frac{1}{2}\mathrm{tr}\,(\boldsymbol{D}^{1/2}\mathbf{X})^T [(2\alpha-1)\mathbf{I} + \phi \mathbf{W}_D](\boldsymbol{D}^{1/2}\mathbf{X}). \end{aligned}$$

Its maximum subject to $\mathbf{X}^T \boldsymbol{D}\mathbf{X} = \mathbf{I}_k$ (ellipsoid) is taken on with the $\mathbf{X}$ that maximizes

$$\mathrm{tr}\,(\boldsymbol{D}^{1/2}\mathbf{X})^T [(2\alpha-1)\mathbf{I} + \phi \mathbf{W}_D](\boldsymbol{D}^{1/2}\mathbf{X})$$

on the constraint $\mathbf{X}^T \boldsymbol{D}\mathbf{X} = \mathbf{I}_k$. Irrespective whether the matrix in brackets is positive semidefinite, it is attained by an $\boldsymbol{D}^{-1/2}\mathbf{X}^*$, where the columns of $\mathbf{X}^*$ are pairwise orthogonal, unit-norm eigenvectors, corresponding to the k largest eigenvalues of $(2\alpha - 1)\mathbf{I} + \phi\mathbf{W}_D$ (see Section 2), and the maximum is $\sum_{l=1}^{k}(2\alpha - 1 + \phi\lambda'_l)$, where $\lambda'_1 \geq \cdots \geq \lambda'_n$ are the eigenvalues of $\mathbf{W}_D$, and it is attained by the corresponding eigenvectors $\mathbf{u}'_1, \dots, \mathbf{u}'_k$ as columns of $\mathbf{X}^*$. Since the eigenvalues of $\mathbf{W}_D$ are in the [-1,1] interval and $0 \leq 2\alpha - 1 \leq 1$, the eigenvalues of $(2\alpha - 1)\mathbf{I} + \phi\mathbf{W}_D$ are in the $[-\phi - 1, \phi]$ interval.

4.2 When there are substitutes between the agents

The utility function of agent i is now defined by

$$u_i(\mathbf{X}) = \sum_{\ell=1}^{k} \alpha s_{i\ell}^2 - \frac{1}{2}\sum_{\ell=1}^{k} s_{i\ell}^2 - \delta \sum_{\ell=1}^{k}\sum_{j=1}^{n} a_{ij} s_{i\ell} s_{j\ell} \tag{4.4}$$

with constants $0 < \alpha \leq \frac{1}{2}$ and $\delta > 0$ to regulate the effect of substitutes. The simultaneous maximization of $u_i(\mathbf{X})$'s subject to $\sum_{i=1}^{n} \mathbf{s}_i \mathbf{s}_i^T = \mathbf{X}^T\mathbf{X} = \mathbf{I}_k$ is equivalent to maximizing the following potential function under the same constraint:

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=1}^{n} u_i(\mathbf{X}) + \frac{\delta}{2}\sum_{i=1}^{n}\sum_{\ell=1}^{k}\sum_{j=1}^{n} a_{ij} s_{i\ell} s_{j\ell} \\ &= -\frac{1}{2}\sum_{\ell=1}^{k} \mathbf{x}_\ell^T [(1-2\alpha)\mathbf{I} + \delta\mathbf{W}]\mathbf{x}_\ell \\ &= -\frac{1}{2}\mathrm{tr}\, \mathbf{X}^T[(1-2\alpha)\mathbf{I} + \delta\mathbf{W}]\mathbf{X}. \end{aligned}$$

Its maximum subject to $\mathbf{X}^T\mathbf{X} = \mathbf{I}_k$ is taken on with the same $\mathbf{X}$ that gives the minimum of

$$\mathrm{tr}\, \mathbf{X}^T[(1-2\alpha)\mathbf{I} + \delta\mathbf{W}]\mathbf{X}$$

on the same constraint. Irrespective of the definiteness of the matrix in brackets, the minimum is attained at an $\mathbf{X}^*$ which contains pairwise orthogonal, unit-norm eigenvectors, corresponding to the k smallest eigenvalues of $(1-2\alpha)\mathbf{I} + \delta\mathbf{W}$ in its columns, and the minimum is $\sum_{\ell=1}^{k}(1 - 2\alpha + \delta\lambda_{n-\ell+1})$, where $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\mathbf{G}$, and it is attained by the corresponding eigenvectors $\mathbf{u}_n, \dots, \mathbf{u}_{n-k+1}$ as columns of $\mathbf{X}$. These may contain negative coordinates, but they can be approximated by stepwise constant vectors of nonnegative coordinates. The subspace of these partition-vectors is close to the subspace spanned by $\mathbf{u}_n, \dots, \mathbf{u}_{n-k+1}$ if there is a gap between λ_{n-k+1} and λ_{n-k}. In this case, the clusters of agents following similar strategies are obtained by applying the k-means algorithm to the optimum strategy vectors, row vectors of the optimum $\mathbf{X}$.

In the case of $k = 1$, this minimum is $1 - 2\alpha + \delta\lambda_1$, which is negative if and only if $\delta > \frac{2\alpha-1}{\lambda_n}$, in view of $\lambda_n < 0$ and $0 < \alpha < \frac{1}{2}$. It means that the above maximum is positive.

The inequality $\delta > \frac{2\alpha-1}{\lambda_n}$ can be restricted to the range of δ where corner equilibria are stable. The corresponding 2-partition of the vertices is obtained by the k-means algorithm applied for the coordinates of $\mathbf{u}_1$. In the $k > 1$ case the same holds with applying the k-means algorithm with the optimal $\mathbf{s}_1^*, \dots, \mathbf{s}_n^*$ as row vectors of the $n \times k$ matrix $\mathbf{X}^*$.

When $k > 1$ is such that $\lambda_n \le \dots \le \lambda_{n-k+1} < 0$, then $\sum_{\ell=1}^{k}(1-2\alpha+\delta\lambda_{n-\ell+1}) < 0$ holds if $\delta > \frac{k(2\alpha-1)}{\sum_{\ell=1}^{k}\lambda_{n-\ell+1}}$. Therefore, the number of strategies cannot exceed the number of negative eigenvalues of G to get a positive optimum. However, when the size of G is large, it suffices to select a k such that $\lambda_{n-k+1} < 0$ and it is much 'smaller' that λ_{n-k}.

The potential function can be also generalized to

$$P(\mathbf{X}) = \sum_{i=1}^{n} u_i(\mathbf{X}) - \sum_{\ell=1}^{k} \frac{\delta_\ell}{2} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} s_{i\ell} s_{j\ell}$$
$$= \frac{1}{2}\sum_{\ell=1}^{k} \mathbf{x}_\ell^T [(2\alpha - 1)\mathbf{I} + \delta_l \mathbf{G}^{(l)}]\mathbf{x}_\ell$$

as in Section 4.1.

Since the eigenvectors not always have positive coordinates, we approximate them by partition vectors. In this way, clusters of agents, following similar strategy are found. In the substitute case, these clusters have sparse within- and dense between-cluster connections.

When the similarity matrix depends on the actual strategy, the potential function can be further generalized to

$$P(\mathbf{X}) = \sum_{i=1}^{n} u_i(\mathbf{X}) + \frac{\delta}{2} \sum_{l=1}^{k}\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}^{(\ell)} s_{i\ell} s_{j\ell}$$
$$= \frac{1}{2}\sum_{\ell=1}^{k} \mathbf{x}_\ell^T [(2\alpha - 1)\mathbf{I} + \delta \mathbf{G}^{(l)}]\mathbf{x}_\ell,$$

where $W^{(\ell)}$ is the connection matrix of the agents under strategy ℓ ($\ell = 1, \dots, k$), and $\mathbf{x}_1, \dots, \mathbf{x}_k$ form an orthonormal set. The solution is given by the compromise vectors of the symmetric matrices in brackets. This generalization corresponds to the real-life situation when the agents have different connections with respect to different strategies (e.g., for buying different kinds of stocks or planting different kinds of crops).

5 Discussion

When maximizing the mutual utility of agents in a network of interactions, we consider edge-weighted graphs describing pairwise relations of the agents. We show how the graph structure determines the optimal strategies with respect to quadratic objective functions maximized on linear or quadratic constraints. Under simplex constraints, dominant sets of an edge-weighted graph will give the solution, where the model parameters are built into the edge-weights. Under quadratic constraints, the spectrum of the unnormalized or normalized edge-weight matrix decides which strategy to follow. Large positive eigenvalues favor complementary strategies in as many respect as the number of the structural positive eigenvalues; while negative eigenvalues of large absolute value favor substitute strategies in as many respect as the number of the structural negative eigenvalues. This is also supported by social network studies, see, e.g., [3, 14]. Note that an eigenvector-based feature organization is also discussed in [18].

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References

[1] Ballester, C., Calvo-Armengol, A. and Zenou, Y., Who's who in networks. Wanted: the key player, International Library of Critical Writings in Economics, **273** (2013), 498-512.

[2] Bazaraa, M. S. and Shetty, C. M., Nonlinear Programming, Theory and Algorithms. Wiley, 1979.

[3] Bolla,M., Penalized versions of the Newman–Girvan modularity and their relation to multiway cuts and k-means clustering, *Physical Review E* **84** (2011), 016108.

[4] Bolla,M., *Spectral Clustering and Biclustering. Learning Large Graphs and Contingency Tables.* Wiley (2013).

[5] Bolla, M., Michaletzky, Gy., Tusnády, G., Ziermann, M., Extrema of sums of Heterogeneous Quadratic Forms, *Linear Algebra and its Applications* **269** (1998), 331–365.

[6] Bomze, I. M., Budinich, M., Pelillo, M., Annealed replication: a new heuristic for the maximum clique problem, *Discrete Applied Mathematics* **121** (2002), 27-49.

[7] Y. Bramoullé, R. E. Kranton, Public goods in networks, *Journal of Economic Theory* **135** (2007), 478-494.

[8] Y. Bramoullé, R. E. Kranton, Risk-sharing networks, *Journal of Economic Behavior and Organization* **64** (2007), 275-294.

[9] Bramoulle, Y., Bramoulle, Y., D'Amours, M., R. Kranton, Strategic interaction and networks, American Economic Review, **104** (2014), 898-930 .

[10] Fiedler, M., An extremal problem for the spectral radius of a graph, *Discrete Mathematics* **108** (1992), 149–158.

[11] Göring, F., Helmberg, C. and Reiss, S., Graph realizations asociated with minimizing the maximum eigenvalue of the Laplacian, *Mathematical Programming* **131** (2012), 95–111.

[12] Lee, J. R., Gharan, S. O. and Trevisan, L., Multi-way spectral partitioning and higher-order Cheeger inequalities. In Proc. 44th Annual ACM Symposium on the Theory of Computing, New York, 1117–1130, 2012.

[13] Motzkin, T. S., Straus, E. G., Maxima for graphs and a new proof of a theorem of Turán, *Canad. J. Math.* **17** (1965), 533–540.

[14] Newman MEJ, *Networks, An Introduction.* Oxford University Press (2010).

[15] Pavan, M., Pelillo, M., Dominant sets and hierarchical clustering. In Proc. of the 9th IEEE International Conference on Computer Vision (ICCV 2003), **2** (2003), 362-369.

[16] Pavan, M., Pelillo, M., Dominant sets and pairwise clustering, *IEEE Trans. Pattern Anal. Machine Intell.* **29**(1) (2007), 167–172.

[17] Poljak, S., Minimum spectral radius of a weighted graph, *Linear Algebra and its Applications* **171** (1992), 53–63.

[18] Sarkar, S. and Boyer, K. L., Quantitative measures of change based on future. Organization: eigenvalues and eigenvectors. *Computer Vision and Image Understanding* **71** (1998), 110-136.

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