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Comaximal Submodule Graphs of Unitary Modules

Elham Mehdi-Nezhad and Amir M. Rahimi

Abstract: In this paper, a new kind of graph on a unitary module A over a commutative ring R with identity, namely the co-maximal submodule graph is defined and studied as a natural generalization of the comaximal ideal graph of a commutative ring R, denoted by $\mathbb{C}(R)$. We use $\mathbb{C}(A)$ to denote this graph, with its vertices the proper submodules of A which are not contained in the Jacobson radical of A, and two vertices B_1 and B_2 are adjacent if and only if $B_1 + B_2 = A$. We show some properties of this graph and compare some of the results of $\mathbb{C}(A)$ and $\mathbb{C}(R)$. For example, this graph is a simple, connected graph with diameter less than or equal to three, and both the clique number and the chromatic number of the graph are equal to the number of maximal submodules of the module A. It is shown that $\mathbb{C}(R)$ is isomorphic to a subgraph of $\mathbb{C}(A)$ when A is a finitely generated cancellation (in particular, a finitely generated free) R-module. We also discuss the conditions under which A is a finite direct sum of simple modules, $\mathbb{C}(A)$ is isomorphic to a finite Boolean graph, and $\mathbb{C}(A)$ and $\mathbb{C}(R)$ are isomorphic graphs.

Keywords: Co-maximal (submodule, ideal) graph, (finitely generated, cancellation, content) module, number of maximal submodules, injector ideal, connectedness and diameter, clique number, chromatic number, weakly perfect, Boolean graph.

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Introduction

The main purpose of this paper is to study the *comaximal submodule graph* of a (finitely generated) *unitary module* A, denoted by $\mathbb{C}(A)$, as a natural extension of the *comaximal ideal graph* of a commutative ring R, denoted by $\mathbb{C}(R)$ [41], by replacing proper ideals of R (not contained in the Jacobson radical of R) with proper submodules of A (not contained in the *Jacobson radical* of A) as the vertex set of the graph $\mathbb{C}(A)$ and edges are defined by *comaximal submodules* of A (Definition 0.1). In [41], Ye and Wu extended the notion of the *comaximal graph* of a commutative ring R (by Sharma and Bhatwadekar [38]) to the comaximal ideal graph of the ring R by replacing *principal ideals* of R with proper ideals of R that are not contained in $J(R)$ (the Jacobson radical of R) as the vertex set of their graph $\mathbb{C}(R)$, where edges are defined by *comaximal ideals* of R. For some recent works on comaximal ideal graphs of commutative rings, see [1, 8, 19, 42, 43] with more algebraic and graph-theoretic properties of R and $\mathbb{C}(R)$ such as *planarity, classification of diameter, and graph perfection* for rings with (infinitely) many maximal ideals. In this work, besides applying direct arguments on the module A for characterizing $\mathbb{C}(A)$, we will use the results and techniques that are related to $\mathbb{C}(R)$ by making a bridge between $\mathbb{C}(R)$ and $\mathbb{C}(A)$ by applying the intrinsic connection between the ring R and its associated module, which is natural in commutative algebra. Notice that in the next section, we provide all necessary results and definitions on modules and graphs that are required in this work, for the sake of completeness, and use them throughout the paper in the sequel.

Throughout this paper, unless otherwise indicated, all rings are assumed to be commutative with identity and modules are unitary. For a ring R [resp. an R-module A], we use (R) [resp. (A)] to denote the set of all *maximal ideals* of R [resp. maximal submodules of A]. A ring R [resp. module A] is said to be *local* if it has a unique maximal ideal [resp. unique maximal submodule (see Example 2.1)] (i.e., $|(R)| = 1$ [resp. $|(A)| = 1$]). Some authors, equivalently, (as in [26]) use "quasi-local" to mean a ring with a unique maximal ideal. We use $J(R)$ and $U(R)$ [resp. $J(A)$ and $U(A)$] to denote the Jacobson radical of R (i.e. the intersection of all maximal ideals of R) [resp. the Jacobson radical of A (i.e. the intersection of all maximal submodules of A)] and the set of all the *invertible elements* of R [resp. the set of all the *units* of A (Definition 1.8)], respectively. As usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$ (i.e., $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$), respectively. References for *graph theory* are [15], [16], [18], and [21]; for *commutative ring theory* and modules, see [4], [11], [7], [23], and [37].

Definition 0.1. The co-maximal submodule graph of a (unitary) module A over a ring R, denoted by $\mathbb{C}(A)$, is a graph whose vertices are the proper submodules of A which are not contained in the Jacobson radical of A, and two vertices B_1 and B_2 are adjacent if and only if $B_1 + B_2 = A$.

• Notice that, besides using many results from module theory (which are stated in Section 1 of this paper as background), the general pattern of the proofs related to characterization of $\mathbb{C}(A)$, respectively,

($*$) are exactly parallel to the ring case without any extra (major) assumptions on the module A or the ring R;

(∗) are (somewhat) parallel to the ring case by assuming that A is a (*finitely generated, cancellation, content [in particular, free, projective]*) module and the *injector ideals* (Definition 1.1) of any two maximal submodules of A are not contained in each other.

The rest of this section is devoted on a *brief historical note* on some graphs associated to some algebraic structures and concluded with a *description* on the organization of the other sections. The area of research on assigning a graph to an algebra (algebraic structure) has been very active (specially) since last two decades and There are many papers which apply combinatorial methods (using graph-theoretic properties and parameters such as *planarity, clique number, chromatic number, independence number, domination number*, and so on) to obtain algebraic results and vice versa, for instance, there are many papers on this interdisciplinary subject and for a short list of them, see for example the reference of [31].

• The co-maximal ideal graph of a ring R, denoted by $\mathbb{C}(R)$, is a graph whose vertices are the proper ideals of R which are not contained in the Jacobson radical of R, and two vertices I_1 and I_2 are adjacent if and only if $I_1 + I_2 = R$. This graph was first introduced and studied in [41]. They studied the *diameter, girth, and bipartiteness* of $\mathbb{C}(R)$ and showed that $\mathbb{C}(R)$ is a simple, connected graph with diameter less than or equal to three, and both the clique number and the chromatic number of the graph are equal to the number of maximal ideals of the ring R. They also studied the conditions under which $\mathbb{C}(R)$ is a *finite Boolean graph* if and only if R is a finite direct product of fields.

In [38], Sharma and Bhatwadekar define a graph G on a commutative ring R with vertices as elements of R, where two distinct vertices a and b are adjacent if and only if $Ra+Rb = R$. They showed that $\chi(G)$ (the chromatic number of G) is finite if and only if R is a finite ring. In this case $\chi(G) = \omega(G)$(the clique number of G) $= t+l$, where t and l, respectively, denote the number of maximal ideals and the number of units of R (see Theorem 2.3 in [38]). Further, in [26], Maimani et al. studied the graph structure defined by Sharma and Bhatwadekar and called it "comaximal graph of commutative rings". In their work, they mostly focused on the graph-theoretic and related ring-theoretic properties of the subgraph generated by nonunit elements of R (see Definition 1.20).

The *zero-divisor graph of a commutative ring* R, denoted $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero-divisors of R with two distinct vertices x and y joined by an edge if and only if $xy = 0$. Thus $\Gamma(R)$ is the *empty graph* if and only if R is an *integral domain.*

In [12] (1988), Beck introduced the concept of a *zero-divisor graph* of a commutative ring, but this work was mostly concerned with *colorings* of rings. The above definition first appeared in the work of Anderson and Livingston [5] (1999), which contains several fundamental results concerning $\Gamma(R)$. This definition, unlike the earlier work of Anderson and Naseer [6] and Beck [12], does not take zero to be a vertex of $\Gamma(R)$.

In [35], Redmond introduced the notion of an *ideal-based zero-divisor graph* of a commutative ring and his work continued and developed further in [27]. Let I be a proper ideal of R. The *zero-divisor graph* of R with respect to I, denoted by $\Gamma_I(R)$, is the graph whose vertex set is the set $\{x \in R \setminus I | \ xy \in I \text{ for some } y \in R \setminus I\}$ with distinct vertices x and y adjacent if and only if $xy \in I$. Thus, if $I = 0$ then $\Gamma_I(R) = \Gamma(R)$, and I is a *nonzero prime ideal* of R if and only if $\Gamma_I(R) = \emptyset$. In both papers [35] and [27], the authors explored the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$.

Moreover, the concept of a zero-divisor graph of a commutative ring has been generalized to a *k-zero-divisor hypergraph* by Eslahchi and Rahimi (the second author of this paper) in [20]. Besides providing many examples and introducing the notion of a k-zero-divisor, k-prime ideal and k-integral domain, there is a discussion on some of the properties and parameters of this hypergraph such as connectedness, diameter, and girth. Moreover, for other properties of this hypergraph, see [36] and [39]. Also, Rahimi in [34], in connection to the *Smarandache zero divisors* of a commutative ring, introduced the notion of a *Smarandache vertex* (S-vertex for short) of a (simple) graph (which is independent of any algebraic structure) to study the S-zero divisors of a commutative ring via its associated zero-divisor graph. Consequently, by this generalization, study of S-vertices of any simple graph can be done directly in a pure graph-theoretic sense, and specially, discussing the S-vertices of any graph associated to an algebra (algebraic structure) is possible and can lead to the study of the interplay between some graph-theoretic properties and algebraic properties of the related algebra. For instance, S. Visweswaran and Hiren D. Patel in [40] studied the S-vertices of *the complement of the annihilating-ideal graph* in connection to some ring-theoretic properties in Sections 2 (Lemma 2.5), 4 (Lemma 4.2(v)), and 5 (Proposition 5.1(iv)) of their paper.

• The concept of the *annihilating-ideal graph* of a commutative ring, denoted by $\mathbb{AG}(R)$, was first introduced by Behboodi and Rakeei in [13] and [14] (for other types of the annihilator-graph of a commutative ring R, denoted by $\mathbb{AG}(R)$ as well, see [9]). Actually, $\mathbb{AG}(R)$ is the *zero-divisor graph* of the *multiplicative semigroup* of the ideals of R (see [17]). Also in [2], Aliniaeifard, Behboodi and authors of this paper have extended and studied this notion to a more general setting as *the annihilating-ideal graph of a commutative ring R with respect to an ideal I of R*, denoted by $\mathbb{AG}_I(R)$, by replacing (nonzero) ideals whose product is zero with ideals ($\nsubseteq I$) whose product lies in I. Thus, $\mathbb{AG}_I(R) = \mathbb{AG}(R)$ for $I = (0)$. Clearly, I is a *prime ideal* of R if and only if $\mathbb{AG}_I(R) = \emptyset$. Notice that $\mathbb{AG}_I(R)$ can be regarded as an *ideal-based zero-divisor graph* of the semiring of the ideals of R and can be denoted as $\Gamma_{C_I}(\mathbb{I}(R))$, where $\mathbb{I}(R)$ is the semiring of the ideals of R and C_I is the set of all ideals of R that are contained in I.

• Furthermore, we now mention a generalization of "an annihilating-ideal graph with respect to an ideal" [2] from two different directions as follows. The authors of this paper, in [31], extended the notion of an annihilating-ideal graph with respect to an ideal to *the annihilation graphs of commutator posets and lattices with respect to an element*, which is a generalization of [2] in a very broad sense. That is, besides defining and discussing the mentioned graph on a *commutator poset* as a general model, we provide examples by applying the commutator theory to define our graph on the substructures (as vertices) of any algebra and define the edge between any two substructures to be the commutator of them that satisfies a special property. Also, in [30], from another direction, we generalized the graph in [2] (see also the last sentence in the previous paragraph) by extending the work of Redmond in [35] to *an ideal-based zero-divisor graph of a commutative semiring R*, denoted by $\Gamma_I(R)$, where I is an ideal of R, and (in contrast to the ring case [35, Theorem 3.2] and [35, Proposition 3.5]) by an example showed that $\Gamma_I(R)$ has a *cut-point* and more than one *bridge* ([30, Example 3.1]). See also [32], which is an ideal-based version of [31], namely "The annihilation graphs of commutator posets and lattices with respect to an ideal".

• The organization of this paper is as follows: In Section 1, we recall some basic properties and definitions of modules and graphs, respectively, and use them (implicitly) in the sequel. Section 2 is devoted on some fundamental properties of the graph $\mathbb{C}(A)$. In this section, besides some simple examples and trivial results, we will discuss some basic properties of the graph $\mathbb{C}(A)$ such as the *diameter* and *core* of $\mathbb{C}(A)$ (Theorems 2.3 and 2.6). Finally, we close the section by showing that $\mathbb{C}(R)$ is isomorphic to a subgraph of $\mathbb{C}(A)$ whenever A is a finitely generated *cancellation* R-module (Theorem 2.7). In Section 3, we discuss the *clique number* and the *chromatic number* of $\mathbb{C}(A)$. In this section, we will prove that the graph $\mathbb{C}(A)$ has the property that its clique number and chromatic number are equal (Theorem 3.1). Moreover, we show that a *content [in particular, free, or more generally, projective] module* A over a ring R is a finite direct sum of *simple modules* and $\mathbb{C}(A)$ is a *finite Boolean graph* whenever $\mathbb{C}(R)$ is a finite Boolean graph (Theorem 3.4). Finally, we will show that for a module A over a ring R with n maximal submodules, $\mathbb{C}(A)$ can be *retracted* to the n-Boolean graph (Corollary 3.12 and see also Proposition 3.11). In Section 4, we study the graph $\mathbb{C}(A)$ of modules A with exactly two maximal submodules. We will show that in such a situation, $\mathbb{C}(A)$ is a *complete bipartite graph* (Lemma 4.1) and consequently, $\mathrm{gr}(\mathbb{C}(A))$ is 4 or ∞ (Corollary 4.3) provided that A is finitely generated. Moreover, For a finitely generated module A, $\mathbb{C}(A)$ is a (complete) bipartite graph if and only if A has exactly two maximal submodules (Theorem 4.5). Theorem 4.7 gives a necessary and sufficient condition for $\mathrm{diam}(\mathbb{C}(A)) = 1$ when A is a finitely generated module. Finally, in Theorems 4.9 and 4.10, we study the conditions under which $\mathbb{C}(A)$ is a *complete graph* if and only if A is a direct sum of two simple R-modules.

1 Background on Modules and Graphs

In this section we recall some basic definitions and properties of (unitary) modules and (simple) graphs, respectively, and will use them (implicitly) in the sequel. The ideal generated by a subset Y of a ring R will be denoted by (Y), while the submodule generated by a subset X of a module A will be denoted by $\langle X\rangle$. We also write $B \leq A$ if B is a submodule of A, and if B is proper, by $B < A$.
Let I be an ideal of a ring R, A an R-module, and S a nonempty subset of A. Then $IS = \{\sum_{i=1}^{n} r_i a_i \mid r \in I; a \in S; n \text{ a positive integer}\}$ is a submodule of A. Similarly if $a \in A$, then $Ia = \{ra \mid r \in I\}$ is a submodule of A. A nonzero unitary R-module A is simple if its only sub- modules are 0 and A. Thus, every simple R-module is cyclic and every R-module endomorphism of a simple module A is either the zero map or an isomorphism. Note that a submodule is itself a module. Also a submodule of a unitary module over a ring with identity is necessarily unitary. In addition, when there is no confusion in the context, 0_R, 0_A, $0 \in \mathbb{Z}$ and the trivial module $\{0\}$ will all be denoted 0.

Definition 1.1. Let B be a submodule of an R-module A. Then the ideal $S_B = \{r \in R \mid rA \subseteq B\}$ of R is called the injector ideal of B. Note that if the commutative ring R has an identity, then for any submodule (ideal) B of the R-module R, S_B is precisely B.

Remark 1.2. Except in Proposition 1.19, if B is a submodule of an R-module A, we denote the annihilator ideal of A/B by S_B instead of $(B : A)$ $(= \{r \in R \mid rA \subseteq B\})$

and note that if the commutative ring R has an identity, then for any submodule (ideal) B of the R-module R, $(B : A)$ is precisely B.

Definition 1.3. Let A be an R-module. A proper submodule M of A is said to be maximal provided that for $N \leq A$ with $M \subseteq N \subseteq A$, then either $M = N$ or $N = A$.

Proposition 1.4. *Let $M < A$, where A is an R-module. Then M is maximal if and only if for each $x \in A \setminus M$, $\langle x, M \rangle = A$.*

Proof. the proof follows directly from the definition. □

The next result shows the existence of a maximal submodule in a module which is similar to the case of rings with identity (Theorem 2.18 of Chapter 3 in [22]).

Proposition 1.5. *Let A be a nonzero finitely generated R-module. Then every proper submodule of A is contained in a maximal submodule of A.*

Proof. See Theorem 2.8 in [4] or Corollary 2.1.15 of [11]. □

Definition 1.6. Let A be an R-module. The Jacobson radical of A (denoted $J(A)$) is the intersection of all maximal submodules of A. If no maximal submodules exist, then we set $J(A) = A$. Similarly the Jacobson radical of R will be denoted by $J(R)$.

The following corollary gives a necessary and sufficient condition for a finitely generated module to be trivial.

Corollary 1.7. *Let A be a finitely generated R-module. Then $J(A) = A$ if and only if $A = 0$.*

Proof. If $J(A) = A$, A has no maximal submodules. By Proposition 1.5, $\langle 0 \rangle$ is contained in some maximal submodule unless $\langle 0 \rangle = A$. Conversely, if $A = \langle 0 \rangle$, then clearly $J(A) = A$. □

Definition 1.8. An element u of an R-module A is said to be a unit provided that u is not contained in any maximal submodule of A.

Proposition 1.9. *Let A be a finitely generated R-module. Then $u \in A$ is a unit if and only if $\langle u \rangle = A$.*

Proof. "$\Leftarrow$" This is immediate. "$\Rightarrow$" Let u be a unit. Then $\langle u \rangle$ is contained in no maximal submodule of A. By Proposition 1.5, we have $\langle u \rangle = A$. □

As a consequence of Proposition 1.9, it follows that a finitely generated module A is *cyclic* if the set of all maximal submodules does not cover A.

Lemma 1.10. (cf. [28, Lemma 4.1] or see the lemma on page 2 of [29]) *Let A be an R-module and let $B, C \leq A$ be such that $RA + B = A$ and $S_B + S_C = R$. Then $B + C = A$.*

Proof. $A = RA + B = (S_B + S_C)A + B = S_B A + S_C A + B \subseteq B + C + B = B + C \subseteq A$. □

Remark 1.11. Clearly, in the above lemma, $RA = A$ and hence $RA + B = A$ for any submodule B of A if R is a ring with identity.

Proposition 1.12. [Chinese Remainder Theorem] (cf. [28, Theorem 4.2] or see the theorem on page 2 of [29]) *Let A be an R-module and let $B_1, \ldots, B_n \leq A$ be such that $RA + B_i = A$ for $i = 1, 2, \ldots, n$ and $S_{B_i} + S_{B_j} = R$ for $i \neq j$. If $a_1, a_2, \ldots, a_n \in A$, then there is $x \in A$ such that $x \equiv a_i (mod\ B_i)$ for $i = 1, 2, \ldots, n$. Furthermore, x is uniquely determined up to congruence modulo $\bigcap_{i=1}^{n} B_i$.*

Proof. See [28] or [29]. The proof is actually patterned after that given in [22]. □

Remark 1.13. On page 1 of [29], there is a counterexample that shows the comaximality condition of ideals (i.e. $S_B + S_C = R$) in the above proposition (Proposition 1.12) is not a superfluous assumption.

Definition 1.14. A proper submodule P of an R-module A is said to be prime provided that whenever $ra \in P$ with $r \in R$ and $a \in A \setminus P$, then $rA \subseteq P$.

Proposition 1.15. *If B is a prime submodule of an R-module A, then S_B is a prime ideal of R.*

Proof. Note that $1 \notin S_B$ (since B is a proper submodule of A by definition) and so S_B is strictly contained in R. Suppose B is a prime submodule of A and $rs \in S_B$ with $s \notin S_B$. Thus, there exists $a \in A$ such that $sa \notin B$ by definition. Now, by prime property of B, $r \in S_B$. See also [25]. □

Proposition 1.16. *If B is a maximal submodule of an R-module A, then B is prime and S_B is a maximal ideal of R.*

Proof. See Proposition 4 in [25]. □

We will use the next result in the proof of Theorem 3.4. For any element x of an R-module M, the content $c(x)$ of x is defined by $c(x) = \bigcap\{A \mid A \text{is an ideal of} R \text{such that} x \in AM\}$. M is called a *content R-module* if $x \in c(x)M$ for every $x \in M$. We remark that every projective module is a content module [33].

Proposition 1.17. (cf. [25, Theorem 5]) *Let F be a non-zero content [in particular, free, or more generally, projective] R-module. Then $J(R)F = J(F)$.*

Proof. See Theorem 5 in [25]. The "in particular part", follows from the fact that every free, or more generally, every projective module is content by [33]. For more properties of content modules, see [33] and Section 4 of [25]. □

The following result is an immediate consequence of the above proposition since every vector space is a free module over a field.

Corollary 1.18. *Let V be a vector space over a field. Then $J(V) = 0$.*

Proposition 1.19. [The Prime Avoidance Theorem] (cf. [24, Theorem 2.3]) *Let M be an R-module, $L_1, L_2, \ldots, L_n$ a finite number of submodules of M, and L a submodule of M such that $L \subseteq L_1 \cup L_2 \cdots \cup \cdots \cup L_n$. Assume that at most two of the L's are not prime, and that $(L_j : M) \nsubseteq (L_k : M)$ whenever $j \neq k$. Then $L \subseteq L_k$ for some k.*

• We close this section by recalling some definitions and notions from graph theory (for the sake of completeness) and use them throughout to keep this paper as self contained as possible.

For a graph G, by $V(G)$ and $E(G)$, we denote the set of all vertices and all edges of G, respectively. Recall that for a graph G, the *degree* of a vertex v in G is the number of edges of G incident with v. A graph G is *connected* if there is a path between any two vertices of G. The *diameter* of a connected graph G is the supremum of the distances between vertices. That is, $\mathrm{diam}(G) = \sup\{d(x, y) | x \text{ and } y \text{ are distinct vertices of } G\}$, where $d(x, y)$ is the length of a shortest path from x to y in G ($d(x, y) = \infty$ if there is no such path). The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e., each pair of distinct vertices forms an edge. The *girth* of a graph G, containing a cycle, is the smallest size of the length of the cycles of G and is denoted by $\mathrm{gr}(G)$. If G has no cycles, we define the girth of G to be infinite. An *r-partite graph* is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite graph* (2-partite graph) with parts of size m and n is denoted by $K_{m,n}$. A complete bipartite graph of the form $K_{1,n}$ is called a *star graph.* A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph.* The complete graph on n vertices is denoted K_n. For a graph G, a complete subgraph of G is called a *clique*. The *clique number*, $\omega(G)$, is the greatest integer $n \geq 1$ such that $K_n \subseteq G$ and $\omega(G)$ is infinite if $K_n \subseteq G$ for all $n \geq 1$. The *chromatic number* $\chi(G)$ of a graph G is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices have the same color. A graph G is said to be *finitely colorable* if $\chi(G)$ is finite. A graph is called *weakly perfect* if its chromatic number equals its clique number.

Definition 1.20. Let S be a nonempty set of vertices of A graph G. The *subgraph induced (= generated) by* S is the maximal subgraph of G with vertex set S and denoted by $\langle S \rangle$. That is, $\langle S \rangle$ contains precisely those edges of G joining two vertices in S.

2 Fundamental Properties of $\mathbb{C}(A)$

In this section, besides some simple examples and trivial results, we will discuss some basic properties of the graph $\mathbb{C}(A)$ such as the diameter and core of $\mathbb{C}(A)$ (Theorems 2.3 and 2.6). Finally, we close the section by showing that $\mathbb{C}(R)$ is isomorphic to a subgraph of $\mathbb{C}(A)$ whenever A is a finitely generated cancellation R-module (Theorem 2.7).

The results will show that $\mathbb{C}(A)$ has many properties similar to that of the co-maximal ideal graph $\mathbb{C}(R)$, which is defined and studied by ye and wu in [41].

We now provide a simple example of a local module before stating the next proposition. An R-module A over a ring R is called a multiplication module provided that each submodule of A is of the form IA for some ideal I of R.

Example 2.1. Clearly, a simple module A over a ring R is local with 0 its unique maximal submodule. Furthermore, a nontrivial multiplication module A over a local ring R is a local module. Suppose B is a proper submodule of A and R is a local ring with maximal ideal M. Thus, $B = IA \subseteq MA \neq A$ for some proper ideal I of R and hence MA is the unique maximal submodule of A. Note that by Nakayama's lemma, $MA = A$ implies $A = 0$ since every multiplication module over a local ring is cyclic (finitely generated) [10, Proposition 4].

Proposition 2.2. (cf. [41, Proposition 2.1]) *Let A be a module over a ring R. Then*

(a) *Let A be a nontrivial finitely generated module. Then $\mathbb{C}(A)$ is the empty graph if and only if A is a local module.*

(b) *Assume $|(A)| \geq 2$. Then for any proper submodule B such that $B \nsubseteq J(A)$, B is a vertex of $\mathbb{C}(A)$.*

Proof. (a) Clear. (b) Since $B \nsubseteq J(A)$, there is an M in (A) such that $B \nsubseteq M$. By the maximal property of M, we must have $B + M = A$ (Proposition 1.4). This gives the fact that B is a vertex. □

Throughout the rest part of this paper, all modules are assumed to be nonlocal, i.e. there are at least two maximal submodules in the module. For instance, the direct sum $A = S_1 \oplus S_2$ of two nonisomorphic simple R-modules is an R-module with exactly two maximal submodules, namely $S_1 \oplus \{0\}$ and $\{0\} \oplus S_2$. Note that a direct sum of $\mathbb{Z}_2$ and $\mathbb{Z}_2$ has three maximal submodules, i.e., the two mentioned and the diagonal $\{(0,0),(1,1)\}$.

In [41, Theorem 2.4], it is shown that for a nonlocal ring R, $\mathbb{C}(R)$ is a simple, connected graph with diameter less than or equal to 3. We now show that the co-maximal submodule graph $\mathbb{C}(A)$ has the same property.

Theorem 2.3. (cf. [41, Theorem 2.4]) *For a module A over a ring R, $\mathbb{C}(A)$ is a simple, connected graph with diameter less than or equal to three.*

Proof. Let B_1 and B_2 be any two vertices of $\mathbb{C}(A)$. If $B_1+B_2 = A$, then $\mathrm{d}(B_1, B_2) = 1$. Now we assume $B_1 + B_2 \neq A$. If there is a submodule B_3 such that $B_1 + B_3 = A$ and $B_2 + B_3 = A$, then $\mathrm{d}(B_1, B_2) = 2$. If such submodules do not exist, we can find two different maximal submodules M_1 and M_2 such that $B_1+M_1 = A$, $B_2+M_2 = A$. Since $M_1 \neq M_2$, we have $M_1 + M_2 = A$, thus $\mathrm{d}(B_1, B_2) = 3$. As the diameter of a graph is the maximum distance between any two vertices, the diameter of the graph is less than or equal to three. By the proof, we can easily see that the graph is simple and connected. □

Example 2.4. (cf. [41, Example 2.5]) Let R be a ring and $A = S \oplus S \oplus S$ the direct sum of three copies of a simple R-module S. Now it is easy to see that the diameter of $\mathbb{C}(A)$ is three. Let $a = S \oplus \{0\} \oplus \{0\}$, $b = \{0\} \oplus S \oplus S$, $c = S \oplus \{0\} \oplus S$, $d = \{0\} \oplus S \oplus \{0\}$. Clearly (a, b, c, d) is a path of length 3.

We now, similar to [41, Lemma 2.6], show that for a module A with at least three maximal submodules, the complete graph K_3 (i.e. a triangle) is an induced subgraph of $\mathbb{C}(A)$; so the girth of $\mathbb{C}(A)$ is 3.

Lemma 2.5. (cf. [41, Lemma 2.6]) *Let A be a module over a ring R and let $G = \mathbb{C}(A)$. If $|(A)| \geq 3$, then the complete graph K_3 is an induced subgraph of G. Thus G contains at least one cycle and hence its girth $gr(G) = 3$.*

Proof. If $|(A)| \geq 3$, then A has at least three different maximal submodules, say M_1, M_2, and M_3. Consider the induced subgraph on $\{M_1, M_2, M_3\}$. This graph is the complete graph K_3. As the girth of a graph with cycles is the smallest size of the length of the cycles, so $\mathrm{gr}(G) = 3$. □

• Recall that the *core* of a graph G is the subgraph induced on all vertices of cycles of G, i.e. the *union of the cycles* in G. A vertex x of G is called an *end vertex* in case the degree of x is one. In [41, Theorem 2.7], Ye and Wu proved that if $\mathbb{C}(R)$ contains a cycle, then the core of $\mathbb{C}(R)$ is always a union of triangles and squares, and a vertex in $\mathbb{C}(R)$ is either an end vertex or a vertex of the core.

We now show that the core of $\mathbb{C}(A)$ is always a union of triangles and squares for finitely generated modules.

Theorem 2.6. (cf. [41, Theorem 2.7]) *Let A be a finitely generated module over a ring R and let $G = \mathbb{C}(A)$. If G contains a cycle, then the core of G is a union of triangles and rectangles, and every vertex of G is either an end vertex or a vertex of the core.*

Proof. For the first statement, it is enough to prove that if $(B_1, B_2, \ldots, B_n, B_1)$ is a cycle in G, then each edge of this cycle is an edge of either a triangle or a rectangle. By the symmetric property of the cycle, we only have to prove that $B_1 — B_2$ is an edge of either a triangle or a rectangle. Clearly, for any cycle of size $n \geq 5$, if $B_1+B_3 = A$, or $B_2+B_{n-1} = A$, or $B_2+B_n = A$, then $B_1 — B_2$ belongs to a triangle or a rectangle. Thus we can now assume $n \geq 5$ and $B_1+B_3 \neq A$, $B_2+B_{n-1} \neq A$, and $B_2 + B_n \neq A$. In this case, either $B_1 + B_{n-1} = A$ or $B_1 + B_{n-1} \neq A$. Suppose that $B_1 + B_{n-1} \neq A$. Hence, there exists a maximal submodule M of A (Proposition 1.5) such that $M \supseteq B_1 + B_{n-1}$. One can easily see that $B_2 + M = A$ and $B_n + M = A$ since $B_1 \subseteq M$ and $B_{n-1} \subseteq M$. This gives the fact that (B_1, B_2, M, B_n, B_1) is a rectangle. Now, suppose that $B_1 + B_{n-1} = A$. In this case, the result is clear when $n = 5$. Hence, by an inductive argument, we can easily see that if the edge $B_1 — B_2$ belongs to any cycle of size $5 \leq m \leq n-1$ ($n \geq 6$), must belong to a triangle or a rectangle. So the core of $\mathbb{C}(A)$ is a union of triangles and rectangles. For the second statement, we will prove that if B is not a vertex in any cycle, then B will be an end vertex. As G contains a cycle, the vertex number of G is at least 3. Let B be a vertex of G, assume B is not a vertex in any cycle. We claim that there is only one edge adjacent to B. In fact, if this is not the case, then there is a path (B_2, B, B_1, C) (note that $\mathbb{C}(A)$ is a connected graph). Let $B_3 = B_2 + C$. If $B_3 = A$, then (B_2, B, B_1, C, B_2) would be a cycle. Otherwise, we have $B_1 + B_3 = A$ and $B + B_3 = A$, which gives another cycle (B, B_1, B_3, B). Either case leads to a contradiction. □

Finally, besides a remark related to the girth and non-planarity of $\mathbb{C}(A)$, we close this section by showing that $\mathbb{C}(R)$ is isomorphic to a subgraph of $\mathbb{C}(A)$ whenever A is a finitely generated cancellation R-module. An R-module A is a *cancellation module* if for ideals I and J of R, $IA = JA$ implies $I = J$. Examples -of cancellation modules include invertible ideals and free modules (for a detailed study of cancellation modules, see [3]).

Theorem 2.7. *Let A be a finitely generated cancellation module [in particular, a finitely generated free module] over a non-local ring R. Then $\mathbb{C}(R)$ is isomorphic to a subgraph of $\mathbb{C}(A)$.*

Proof. Clearly, if R is a local ring, then $\mathbb{C}(R)$ is the empty graph and hence is a subgraph of $\mathbb{C}(A)$. Suppose I and J are two vertices of the graph $\mathbb{C}(R)$ such that $I + J = R$. Thus $IA + JA = RA = A$. Clearly, each of IA and JA is a proper submodule of A by cancellation property of A and the choice of I and J in R. On the other hand, $J(A)$ does not contain any of IA or JA since, for example, Suppose $IA \subseteq J(A)$, then there exists a maximal submodule M of A that contains JA (Proposition 1.5) and consequently contains $A = IA + JA$, which is a contradiction. Now define a map $\varphi : V(\mathbb{C}(R)) \to V(\mathbb{C}(A))$ by $I \mapsto IA$. Clearly, this function is injective and preserves edges. □

Remark 2.8. From the above result, it is clear that $\mathrm{gr}(\mathbb{C}(A)) \leq \mathrm{gr}(\mathbb{C}(R))$ and non-planarity of $\mathbb{C}(R)$ implies non-planarity of $\mathbb{C}(A)$. For example, $\mathbb{C}(A)$ is not planar if $|(R)| \geq 5$ since $\mathbb{C}(R)$ contains an isomorphic copy of K_5 and by Kuratowski's theorem [15, Theorem 10.30], it is not planar.

3 The Clique Number and the Chromatic Number of the Graph $\mathbb{C}(A)$

In this section, we will prove that the graph $\mathbb{C}(A)$ has the property that its clique number and chromatic number are equal (Theorem 3.1). Moreover, we show that a content [in particular, free, or more generally, projective] module A over a ring R is a finite direct sum of simple modules and $\mathbb{C}(A)$ is a finite Boolean graph whenever $\mathbb{C}(R)$ is a finite Boolean graph (Theorem 3.4). Finally, we will show that for a module A over a ring R with n maximal submodules, $\mathbb{C}(A)$ can be retracted to the n-Boolean graph (Corollary 3.12 and see also Proposition 3.11).

Recall that a graph G is called weakly perfect provided $\chi(G) = \omega(G)$. The following theorem shows that $\mathbb{C}(A)$ is a weakly perfect graph.

Theorem 3.1. (cf. [41, Theorem 3.1]) *Let A be a finitely generated module over a ring R and let $G = \mathbb{C}(A)$. Suppose if $(A) = \{M_1, M_2, \ldots, M_n\}$ (i.e. $|(A)|$ is finite), then $S_{M_i} \nsubseteq S_{M_j}$ for $i \neq j$. Then the following three numbers are equal.*

(a) *The number $|(A)|$ of maximal submodules of A.*

(b) *The clique number $\omega(G)$ of the graph G.*

(c) *The chromatic number* $\chi(G)$ *of the graph* G.

Proof. If $|(A)| = \infty$, it is easy to see that the other two numbers are also infinite. Thus we may assume $|(A)| = n < \infty$ and let $(A) = \{M_1, \ldots, M_n\}$ with $S_{M_i} \nsubseteq S_{M_j}$ for $i \neq j$. Consider the induced subgraph on $\{M_1, \ldots, M_n\}$. It is the complete graph K_n, so $|(A)| \leq \omega(G)$. Since $\omega(G) \leq \chi(G)$ is a well known conclusion in graph theory, we get

$|(A)| \leq \omega(G) \leq \chi(G)$. The only thing left is to prove that $|(A)| \geq \chi(G)$. Let $V_1 = \{B \in V(G) \mid B \subseteq M_1\}$, $V_2 = \{B \in V(G) \mid B \subseteq M_2, B \notin V_1\}$, $V_3 = \{B \in V(G) \mid B \subseteq M_3, B \notin V_1 \cup V_2\}$, $V_n = \{B \in V(G) \mid B \subseteq M_n, B \notin V1 \cup \cdots \cup V_{n-1}\}$.

By Prime Avoidance Theorem (Proposition 1.19), $V_i \neq \emptyset$ and hence $M_i \in V_i$ for each i. Thus this gives an n-coloring implementation on the graph G. So $|(A)| = n \geq \chi(G)$, and thus the three numbers are equal. □

The proof of the following proposition is similar to the proof of [26, Proposition 2.3] and its comaximal ideal version stated for $\mathbb{C}(R)$ in [41, Proposition 3.2] with no proof. We now prove it for $\mathbb{C}(A)$ when A is a finitely generated module.

Proposition 3.2. (cf. [41, Proposition 3.2]) *Let* A *be a finitely generated module over a ring* R *and let* $G = \mathbb{C}(A)$. *Suppose if* $(A) = \{M_1, M_2, \ldots, M_n\}$ *(i.e.* $|(A)|$ *is finite), then* $S_{M_i} \nsubseteq S_{M_j}$ *for* $i \neq j$. *Then the following hold:*

(a) *If* $|(A)| = n$, *where* $1 < n < \infty$, *then* G *is an* n*-partite graph.*

(b) *If* G *is an* n*-partite graph, then* $|(A)| \leq n$. *In this case, if* G *is not an* $(n-1)$*-partite graph, then* $|(A)| = n$.

Proof. The proof is mainly similar to the proof of Proposition 2.3 in [26]. Part (a) follows directly by constructing $V_1, V_2, \ldots, V_n$ as defined in the proof of Theorem 3.1.

For Part (b), let $V_1, V_2, \ldots, V_n$ be The n parts of vertices of $\mathbb{C}(A)$. Assume to the contrary that $|(A)| > n$ and let $M_1, \ldots, M_{n+1} \in (A)$. For any i, choose $B_i \in M_i \setminus \bigcup_{j \neq i} M_j$. Then it is easy To see That $\{B_1, B_2, \ldots, B_{n+1}\}$ is a clique in $\mathbb{C}(A)$. By The Pigeon Hole Principle, Two of B_i's should belong To one of V_i's, That is a contradiction. Therefore $|(A)| \leq n$. Now suppose That $\mathbb{C}(A)$ is not $(n-1)$-partite and $|(A)| = m < n$. By (a), the graph will be m-partite and This is a contradiction. Note that when an n-partite graph can not be reduced to an $(n-1)$-partite graph by joining two of its parts (i.e. union of two parts), then it is impossible to reduce it to an m-partite graph for any $m \leq n-1$ by joining more than two parts of it. □

The following example is a module version of [41, Example 3.3] which provides an important class of graphs, namely the n-Boolean graphs. The definition of a Boolean graph was proposed by Wu and Lu in [44] and we mention it in the following example.

Example 3.3. (cf. [41, Example 3.3]) Let R be a ring and S_i nonisomorphic simple R-modules, where $i = 1, \ldots, n$, with $n \geq 2$. Let $A = \bigoplus_{i=1}^{n} S_i$ be the direct sum of S_i's. Then $\mathbb{C}(A)$, which is isomorphic to the zero-divisor graph of the ring $(\mathbb{Z}_2)^n$, is called the n-Boolean graph. It is easy to see that both the clique number and the chromatic number of the n-Boolean graph is n, and the n-Boolean graph has only one maximal clique. Obviously, the 2-Boolean graph is the complete graph K_2 and for the 3-Boolean graph, see Fig. 1 in [41, Example 2.5].

The following theorem shows that if $\mathbb{C}(R)$ is a finite Boolean graph, then, similar to [41, Theorem 3.4], $\mathbb{C}(A)$ is a finite Boolean graph and A is a finite direct sum of simple modules provided that A is a content [in particular, free, or more generally, projective] R-module.

Theorem 3.4. (cf. [41, Theorem 3.4]) *Let R be a commutative ring and $\mathbb{C}(R)$ an n-Boolean graph, where $2 \leq n < \infty$. Let A be a content [in particular, free, or more generally, projective] R-module over the ring R with $|(A)| = m$, where $2 \leq m < \infty$ and for any two maximal submodules M and N of A, $S_M \nsubseteq S_N$. Then $\mathbb{C}(A)$ is isomorphic to an m-Boolean graph, $\omega(\mathbb{C}(A)) = m$, and A is isomorphic to a direct sum of m summands of simple modules. Moreover, if $m = n$, then $\mathbb{C}(A)$ is isomorphic to $\mathbb{C}(R)$.*

Proof. By Theorem 3.4(2) in [41], $R \cong \Pi_{i=1}^{n} F_i$, where each F_i is a field, for $i = 1, \ldots, n$. Thus $J(R) = 0$ and hence $J(A) = 0$ since A is a content module and by Proposition 1.17, $J(R)A = J(A)$. Now by Chinese Remainder Theorem (Proposition 1.12), $A = A/J(A) \cong A/M_1 \oplus A/M_2 \oplus \cdots \oplus A/M_m$. In this case, $\mathbb{C}(A)$ is isomorphic to an m-Boolean graph by Lemma 3.7 and Example 3.3. Thus $\omega(\mathbb{C}(A)) = m$ and there is only one maximal clique in $\mathbb{C}(A)$. The "in particular part", follows from the fact that every free, or more generally, every projective module is content by [33]. □

The following theorem, similar to [41, Theorem 3.4], shows that $\mathbb{C}(A)$ is a finite Boolean graph and A is a finite direct sum of simple modules provided that the Jacobson radical of A is zero.

Theorem 3.5. (cf. [41, Theorem 3.4]) *Let R be a commutative ring and let A be an R-module over the ring R with $|(A)| = m$, where $2 \leq m < \infty$ and for any two maximal submodules M and N of A, $S_M \nsubseteq S_N$. If $J(A) = 0$, then $\mathbb{C}(A)$ is isomorphic to an m-Boolean graph, $\omega(\mathbb{C}(A)) = m$, and A is isomorphic to a direct sum of m summands of simple modules.*

Proof. The proof is similar to the proof of the above theorem by using the Chinese Remainder Theorem (Proposition 1.12) since $J(A) = 0$ by hypothesis. □

• For simple graphs G and H, recall that a graph homomorphism from G to H is a map $\varphi : G \to H$ such that for distinct $u, v \in V(G)$, $\{u, v\} \in E(G)$ implies $\varphi(u) \neq \varphi(v)$ and $\varphi(u) — \varphi(v) \in E(H)$ (i.e., edge goes to edge). Moreover, if further H is a subgraph of G and the restriction of φ on H is the identity map, then H is called a *retract* of G.

Before closing this section by two results related to a retract of $\mathbb{C}(A)$, respectively, (proposition 3.11 and Corollary 3.12), we need the following (five) lemmas.

Lemma 3.6. (cf. Theorem 1.10 of Chapter 4 in [22]) *If R is a ring and B is a submodule of an R-module A, then there is a one-to-one correspondence between the set of all submodules of A containing B and the set of all submodules of A/B, given by $C \mapsto C/B$. Hence every submodule of A/B is of the form C/B, where C is a submodule of A which contains B.*

By using the above result we can prove the following lemma. Similarly, Corollary 2.10 of [4] states that A factor module M/K is simple if and only if K is a maximal submodule of M.

Lemma 3.7. *A/M is a simple module if and only if M is a maximal submodule of A.*

Lemma 3.8. (cf. Theorem 1.9 of Chapter 4 in [22]) *Let B and C be submodules of a module A over a ring R.*

(i) *There is an R-module isomorphism $B/(B \cap C) \cong (B + C)/C$;*

(ii) *if $C \subseteq B$, then B/C is a submodule of A/C, and there is an R-module isomorphism $\frac{(A/C)}{(B/C)} \cong A/B$.*

By the above result and Lemma 3.7, we have the following.

Lemma 3.9. *There is a one-to-one correspondence between the maximal submodules of A and $A/J(A)$. That is, M is a maximal submodule of A if and only if $M/J(A)$ is a maximal submodule of $A/J(A)$.*

Lemma 3.10. *Let $f : A \to A/J(A)$ be the canonical epimorphism of modules. Then $f^{-1}(J(A/J(A))) = J(A)$.*

Proof. $f^{-1}(J(A/J(A))) = f^{-1}(\bigcap M_i/J(A)) = \bigcap f^{-1}(M_i/J(A)) = \bigcap M_i = J(A)$. Note that the inverse function of any map between two sets preserves the intersection operation and inclusion relation. □

Finally, by using the preceding (five) lemmas (implicitly), we show that $\mathbb{C}(A/J(A))$ is a retract of $\mathbb{C}(A)$ and close this section by showing that for an R-module A with only finitely many maximal submodules, the graph $\mathbb{C}(A)$ can be retracted to an n-Boolean graph (Corollary 3.12).

Proposition 3.11. (cf. [41, Proposition 3.6]) *Let A be a module over a ring R. Then $\mathbb{C}(A/J(A))$ is a retract of (A).*

Proof. For the proof, we use the preceding (five) lemmas implicitly. By verifying the definition of graph retract directly, note that $B + C = A$ if and only if $[(B + J(A))/J(A)] + [(C + J(A))/J(A)] = A/J(A)$ (i.e. edge preserving), and this shows that $\mathbb{C}(A)$ contains a subgraph which is isomorphic to $\mathbb{C}(A/J(A))$. The result then follows. □

By this proposition and the Chinese Remainder Theorem (Proposition 1.12), the following corollary is easily obtained.

Corollary 3.12. (cf. [41, Corollary 3.7]) *For a module A over a ring R, let $G = \mathbb{C}(A)$. If $|(A)| = n < \infty$, then the n-Boolean graph is a retract of G.*

4 $\mathbb{C}(A)$ of Modules with Two Maximal Submodules

In this section, we study the graph $\mathbb{C}(A)$ of modules A with exactly two maximal submodules. We will show that in such a situation, $\mathbb{C}(A)$ is a complete bipartite graph (Lemma 4.1) and consequently, $\mathrm{gr}(\mathbb{C}(A))$ is 4 or ∞ (Corollary 4.3) provided that A is finitely generated. Moreover, For a finitely generated module A, $\mathbb{C}(A)$ is a (complete) bipartite graph if and only if A has exactly two maximal submodules (Theorem 4.5). Theorem 4.7 gives a necessary and sufficient condition for $\mathrm{diam}(\mathbb{C}(A)) = 1$ when A is a finitely generated module. Finally, in Theorems 4.9 and 4.10, we study the conditions under which $\mathbb{C}(A)$ is a complete graph if and only if A is a direct sum of two simple R-modules.

The following lemma shows that $\mathbb{C}(A)$ is a complete bipartite graph when A is a finitely generated module with exactly two maximal submodules.

Lemma 4.1. (cf. [41, Lemma 4.1]) *Let A be a finitely generated module over a ring R and assume $|(A)| = 2$. Then $\mathbb{C}(A)$ is one of the following:*

(a) *The complete graph $K_2 = K_{1,1}$.*

(b) *A star graph $K_{1,n}$, where $2 \leq n < \infty$ is a positive integer.*

(c) *A complete bipartite graph* $K_{m,n}$, *where* m *and* n *are positive integers with* $2 \leq m < \infty$ *and* $2 \leq n < \infty$.

Proof. We can assume that $(A) = \{M_1, M_2\}$. In this situation, it is enough to prove that for any two distinct submodules B_1 and B_2, neither of which is contained in $J(A)$, if they are contained in M_1 and M_2, respectively, then we must have $B_1 + B_2 = A$. In fact, if $B_1 + B_2 \neq A$, then $B_1 + B_2 \subseteq M_1$ or $B_1 + B_2 \subseteq M_2$ (Proposition 1.5). In either case, there is a contradiction. □

By Lemma 4.1 and properties of complete bipartite graphs, we can easily get the following two corollaries.

Corollary 4.2. (cf. [41, Corollary 4.2]) *Let* A *be a finitely generated module over a ring* R *with exactly two maximal submodules and let* $G = \mathbb{C}(A)$. *Then* $diam(G) = 1$ *or 2.*

Corollary 4.3. (cf. [41, Corollary 4.3]) *Let* A *be a finitely generated module over a ring* R *with exactly two maximal submodules and let* $G = \mathbb{C}(A)$. *Then* $gr(G) = 4$ *or* ∞.

By Lemma 2.5 and Corollary 4.3, we have the following theorem.

Theorem 4.4. (cf. [41, Theorem 4.4]) *Let* A *be a finitely generated module over a ring* R *and let* $G = \mathbb{C}(A)$. *Then* $gr(G) = 3$, *4 or* ∞.

For a finitely generated module A over a ring R, the following theorem shows that $\mathbb{C}(A)$ is a (complete) bipartite graph if and only if A has exactly two maximal submodules.

Theorem 4.5. (cf. [41, Theorem 4.5]) *For a finitely generated module* A *over a ring* R, *the following statements are equivalent:*

(a) $\mathbb{C}(A)$ *is a complete bipartite graph;*

(b) $\mathbb{C}(A)$ *is a bipartite graph;*

(c) A *has only two maximal submodules, i.e.* $|(A)| = 2$.

Proof. (a) ⇒ (b) is obvious and (c) ⇒ (a) is easily obtained by Lemma 4.1. (b) ⇒ (c) will be proved below. If $|(A)| = 1$, then the graph $\mathbb{C}(A)$ is the empty graph. If $|(A)| > 2$, by Lemma 2.5, we have that $\mathbb{C}(A)$ has a subgraph K_3. The chromatic number of K_3 is 3; so the chromatic number of $\mathbb{C}(A)$ is greater than or equal to 3. This contradicts the fact that the chromatic number of a bipartite graph is 2. □

We will use the following lemma for the proof of the next two results.

Lemma 4.6. *Let A be a module over a ring R with B a submodule of A and I an ideal of R. Suppose P is a prime submodule of A and $IB \subseteq P$. Then either $I \subseteq S_P$ or $B \subseteq P$.*

Proof. Suppose $IB \subseteq P$ and $B \nsubseteq P$. Thus, there exists $b \in B \setminus P$. Now for each $r \in I$, $rb \in P$ implies $r \in S_P$ (by definition of a prime submodule) and hence $I \subseteq S_P$. □

The following theorem gives a necessary and sufficient condition for $\mathrm{diam}(\mathbb{C}(A)) = 1$ when A is a finitely generated module.

Theorem 4.7. (cf. [41, Theorem 4.6]) *Let A be a finitely generated module over a ring R and let $G = \mathbb{C}(A)$. Suppose for any maximal submodules M and N of A, $S_M \nsubseteq S_N$. Then G is complete if and only if diam$(G) = 1$ if and only if $G = K_2$.*

Proof. We just argue for the case when $\mathrm{diam}(G) = 1$ implies $G = K_2$. If $|(A)| = 1$, then the graph $\mathbb{C}(A)$ is the empty graph; so its diameter cannot be 1. If $|(A)| > 2$, then there are at least three different maximal submodules in A, say M_1, M_2, and M_3. It is easy to verify that $S_{M_1}M_2 \nsubseteq J(A)$ since by primeness of M_3, either $S_{M_1} \subseteq S_{M_3}$ or $M_2 \subseteq M_3$ (Lemma 4.6). Consider $\mathrm{d}(S_{M_1}M_2, M_1)$. As $S_{M_1}M_2 \subseteq M_1$ ($M_2 \subseteq A$ implies $S_{M_1}M_2 \subseteq S_{M_1}A \subseteq M_1$ by definition of S_{M_1}), $S_{M_1}M_2 + M_1 = M_1 \neq A$, $\mathrm{d}(S_{M_1}M_2, M_1) \neq 1$. Also, $M_3 + M_1 = A$ and $M_3 + S_{M_1}M_2 = A$ (Proposition 1.4) shows that $\mathrm{d}(S_{M_1}M_2, M_1) = 2$. By the definition of the diameter of a graph, we get $\mathrm{diam}(G) \geq 2$. This contradicts the condition $\mathrm{diam}(G) = 1$. So $|(A)| = 2$. At last, by Lemma 4.1, the only possibility for G with its diameter 1 is the complete graph K_2. □

The following proposition considers the conditions for $\mathrm{diam}(\mathbb{C}(A)) = 2$ when $J(A)$ is a prime submodule of A.

Proposition 4.8. (cf. [42, Proposition 3.3]) *Let A be a module over a ring R and assume for each submodule $B < A$, $S_B \nsubseteq S_{J(A)}$ whenever $B \nsubseteq J(A)$. Then diam$(\mathbb{C}(A)) = 2$ provided that $\mathbb{C}(A)$ is not a complete graph and $J(A)$ is a prime submodule of A.*

Proof. Clearly, $2 \leq \mathrm{diam}(\mathbb{C}(A)) \leq 3$ by Theorem 2.3 and hypothesis. Let $J(A)$ be a prime submodule of A. As the diameter of a graph is the supreme distance between two vertices, so if we want to prove $\mathrm{diam}(\mathbb{C}(A)) = 2$, it suffices to prove the distance of any two vertices of $\mathbb{C}(A)$ is 1 or 2. For any two proper submodules $B, C \nsubseteq J(A)$, by the prime property of $J(A)$ and Lemma 4.6 and hypothesis, we get $S_B C \nsubseteq J(A)$. If $B + C = A$, then $\mathrm{d}(B, C) = 1$. Now if $B + C \neq A$, thus $\mathrm{d}(B, C) \neq 1$. Since $J(A)$ is the intersection of all maximal submodules of A, there is at least one maximal submodule M of A, such that $S_B C \nsubseteq M$. Thus $S_B C + M = A$ (Proposition 1.4). Hence we get $B+M = A$ (since $S_B C \subseteq S_B A \subseteq B$) and $C+M = A$ (since $S_B C \subseteq C$), which shows that $\mathrm{d}(B, C) = 2$. □

Finally, we close this paper with the following two theorems in which $\mathbb{C}(A)$ is a complete graph if and only if A is the direct sum of two simple modules.

Theorem 4.9. (cf. [41, Theorem 4.6]) *Let R be a commutative ring and $\mathbb{C}(R)$ an n-Boolean graph with $2 \leq n < \infty$. Let A be a finitely generated free R-module over the ring R and let $G = \mathbb{C}(A)$. Suppose for any maximal submodules M and N of A, $S_M \nsubseteq S_N$. Then the following statements are equivalent:*

(a) *G is a complete graph.*

(b) *$diam(G) = 1$.*

(c) *$G = K_2$.*

(d) *$A = S_1 \oplus S_2$, where S_1 and S_2 are simple R-modules.*

Proof. Clearly, (a), (b), and (c) are equivalent from Theorem 4.7 and (d) implies (a) is obvious. Now (c) implies (d) since by Proposition 3.2, A has only two maximal submodules and Theorem 3.4 completes the proof. □

Theorem 4.10. (cf. [41, Theorem 4.6]) *Let R be a commutative ring and let A be a finitely generated R-module over the ring R and let $G = \mathbb{C}(A)$. Suppose $J(A) = 0$ and for any maximal submodules M and N of A, $S_M \nsubseteq S_N$. Then the following statements are equivalent:*

(a) *G is a complete graph.*

(b) *$diam(G) = 1$.*

(c) *$G = K_2$.*

(d) *$A = S_1 \oplus S_2$, where S_1 and S_2 are simple R-modules.*

Proof. By using Theorem 3.5, the proof is similar to the proof of the above theorem. □

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Elham Mehdi-Nezhad
Department of Mathematical Sciences,
Department of Mathematical Sciences,
Stellenbosch University,
South Africa.
E-mail: mhdelh001@myuct.ac.za

A. M. Rahimi
Department of Mathematical Sciences,
University of South Africa,
P.O. Box 392,
0003 Pretoria, South Africa,
and
School of Mathematics,
Institute for Research in Fundamental Sciences (IPM),
P.O. Box 19395-5746,
Tehran, Iran
E-mail: amrahimi@ipm.ir