Optimization problems in classes of rearrangements for (p,q)-Laplace equations

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Abstract: This paper is concerned with maximization and minimization of a functional associated with solutions of (p,q)-Laplace equations depending on functions which belong to a class of rearrangements. We prove existence and uniqueness results, and present some features of optimal solutions.

Keywords: Rearrangements, (p,q)-Laplacian, Energy Integral, Optimization.


1 Introduction

Let Ω be a bounded smooth domain in \( \mathbb{R}^N \), and let \( 1 < q < p \). We consider the boundary value problem

\[-\Delta_p u - \Delta_q u = f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.\]

The non-homogeneous differential operator \( \Delta_p + \Delta_q \) is called (p,q)-Laplacian. As observed in [17], it stems from a wide range of important applications including biophysics [10], plasma physics [19], reaction-diffusion equations [7], as well as models of elementary particles [2]. In the last decades there has been a great interest in the investigation of these problems mainly concerning existence and multiplicity of solutions, eigenvalues, ground-state solutions [1, 6, 11].

In the present paper we consider the case \( f(x,u) = g(x)|u|^{\alpha-1} \), where \( 1 \leq \alpha < q \) and \( g(x) \) is a measurable bounded non-negative function which is positive in a subset with a positive measure. For \( v \in H_0^{1,p}(\Omega) \) we define

\[ I(v) = \int_{\Omega} \left( \frac{1}{p} |\nabla v|^p + \frac{1}{q} |\nabla v|^q - \frac{1}{\alpha} g|v|^\alpha \right) dx, \]
and consider the classical minimization problem

$$\inf_{v \in H^{1,p}_0(\Omega)} I(v). \quad (1.1)$$

The proof of the existence of a solution to this problem is standard. Let \( v_i \in H^{1,p}_0(\Omega) \) be a sequence such that

$$\tilde{I} := \lim_{i \to \infty} I(v_i),$$

where \( \tilde{I} \) is the value of the inferior of \( I(v) \). By using Poincaré and Hölder inequalities we find

$$\int_\Omega g|v_i|^\alpha dx \leq C \int_\Omega |\nabla v_i|^\alpha dx \leq C \left( \int_\Omega |\nabla v_i|^p dx \right)^{\frac{\alpha}{p}}.$$

Here and in what follows we denote by \( C \) constants (possibly different) independent of \( i \). It follows that

$$I(v_i) \geq \frac{1}{p} \int_\Omega |\nabla v_i|^p \, dx - C \left( \int_\Omega |\nabla v_i|^p \, dx \right)^{\frac{\alpha}{p}}.$$

By using the well known inequality

$$(\epsilon B)^{\frac{1}{\epsilon}} \leq \frac{\alpha}{p} (\epsilon B)^{\frac{\alpha}{p}} + \frac{p - \alpha}{p} \left( \frac{1}{\epsilon} \right)^{\frac{p}{p-\alpha}}$$

for \( B = C \left( \int_\Omega |\nabla v_i|^p dx \right)^{\frac{\alpha}{p}} \) and for a suitable value of \( \epsilon \) we find

$$C \left( \int_\Omega |\nabla v_i|^p dx \right)^{\frac{\alpha}{p}} \leq \frac{1}{2p} \int_\Omega |\nabla v_i|^p dx + C.$$

(Clearly, the two constants \( C \) in above are different.) Hence, we have

$$I(v_i) \geq \frac{1}{2p} \int_\Omega |\nabla v_i|^p \, dx - C.$$

It follows that the value of \( \tilde{I} \) is finite. We also find that

$$\int_\Omega |\nabla v_i|^p dx \leq C.$$

Hence, a subsequence of \( \{v_i\} \) (denoted again \( \{v_i\} \)) converges weakly in \( H^{1,p}(\Omega) \) and in \( H^{1,q}(\Omega) \) to some function \( u \in H^{1,p}_0(\Omega) \). By Rellich’s Theorem, \( \{v_i\} \) converges to
u strongly in $L^p(\Omega)$ as well as in $L^q(\Omega)$ and in $L^\alpha(\Omega)$. Therefore, we have

$$\tilde{I} \leq I(u) = \int_\Omega \left( \frac{1}{p} |\nabla u|^p + \frac{1}{q} |\nabla u|^q - \frac{1}{\alpha} g |u|^\alpha \right) dx$$

$$\leq \liminf_{i \to \infty} \int_\Omega \frac{1}{p} |\nabla v_i|^p dx + \liminf_{i \to \infty} \int_\Omega \frac{1}{q} |\nabla v_i|^q dx - \lim_{i \to \infty} \frac{1}{\alpha} g |v_i|^\alpha dx$$

$$\leq \lim_{i \to \infty} \int_\Omega \left( \frac{1}{p} |\nabla v_i|^p + \frac{1}{q} |\nabla v_i|^q - \frac{1}{\alpha} g |v_i|^\alpha \right) dx = \tilde{I}.$$

Hence, $u$ is a minimizer in (1.1). Since $I(u) = I(|u|)$, we may assume that a non-negative minimizer exists.

We note that a minimizer $u$ of the functional $I(v)$ is a maximizer of the functional

$$-I(v) = \int_\Omega \left( \frac{1}{\alpha} g |v|^\alpha - \frac{1}{p} |\nabla v|^p - \frac{1}{q} |\nabla v|^q \right) dx.$$

It is easy to show that a non-negative minimizer $u$ of problem (1.1) is a solution to the following boundary value problem:

$$\int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \cdot \nabla \phi - g u^{\alpha-1} \phi) dx = 0 \quad \forall \phi \in H^{1,p}_0(\Omega). \quad (1.2)$$

Problem (1.2) has been discussed in [15, 16]. In particular, since $g$ is non-negative and bounded, any solution $u$ belongs to $C^{1,\sigma}(\Omega)$ for some $0 < \sigma < 1$. Another important fact is that either $u(x) \equiv 0$ or $u(x) > 0$ in $\Omega$ (see [18], Theorem 2.5.1 page 74 and Corollary 7.1.3 page 163).

Under our assumptions (in particular $\alpha < q < p$), a minimizer of (1.1) is non trivial. Indeed, if $v \in H^{1,p}(\Omega)$, $v > 0$, and $\epsilon > 0$ small enough we have

$$I(\epsilon v) = \epsilon^\alpha \int_\Omega \left( \frac{\epsilon^{p-\alpha}}{p} |\nabla v|^p + \frac{\epsilon^{q-\alpha}}{q} |\nabla v|^q - \frac{1}{\alpha} g(x)v^\alpha \right) dx < 0.$$

The following uniqueness result is crucial for our purposes.

**Theorem 1.1.** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$, let $g \in L^\infty(\Omega)$ and let $1 \leq \alpha < q < p$. If $u, v \in H^{1,p}_0(\Omega) \cap C^0(\overline{\Omega})$ are positive solutions to problem (1.2) then $u(x) = v(x)$ in $\Omega$.

**Proof.** This proof is inspired by the proof of Theorem 3.2 in [9]. See also [13], page 160. Define $A = \{x \in \Omega : u(x) > v(x)\}$. If we prove that $A$ is empty, the assertion of the theorem follows. We argue by contradiction, assuming $A$ is not
empty. For $\epsilon > 0$, define $u_\epsilon = u + \epsilon$ and $v_\epsilon = v + \epsilon$. Note that in $A$ we have $u_\epsilon(x) > v_\epsilon(x)$. Using

$$\phi_1(x) = \max\left[\frac{u_\epsilon^\alpha(x) - v_\epsilon^\alpha(x)}{u_\epsilon^{\alpha-1}(x)}, 0\right]$$

as test function in the equation for $u$ we obtain

$$\int_A |\nabla u|^{p-2}\nabla u \cdot \nabla \left(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\right) dx + \int_A |\nabla u|^{q-2}\nabla u \cdot \nabla \left(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\right) dx = \int_A g(x)(u_\epsilon^\alpha - v_\epsilon^\alpha) \left(\frac{u}{u_\epsilon}\right)^{\alpha-1} dx.$$ 

Using

$$\phi_2(x) = \max\left[\frac{u_\epsilon^\alpha(x) - v_\epsilon^\alpha(x)}{v_\epsilon^{\alpha-1}(x)}, 0\right]$$

as test function in the equation for $v$ we obtain

$$\int_A |\nabla v|^{p-2}\nabla v \cdot \nabla \left(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\right) dx + \int_A |\nabla v|^{q-2}\nabla v \cdot \nabla \left(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\right) dx = \int_A g(x)(v_\epsilon^\alpha - u_\epsilon^\alpha) \left(\frac{v}{v_\epsilon}\right)^{\alpha-1} dx.$$ 

Subtracting the latter equality from the first one we get

$$\int_A |\nabla u|^{p-2}\nabla u \cdot \nabla \left(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\right) dx + \int_A |\nabla u|^{q-2}\nabla u \cdot \nabla \left(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\right) dx + \int_A |\nabla v|^{p-2}\nabla v \cdot \nabla \left(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\right) dx + \int_A |\nabla v|^{q-2}\nabla v \cdot \nabla \left(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\right) dx = L_\epsilon,$$

where

$$L_\epsilon = \int_A g(x)(u_\epsilon^\alpha - v_\epsilon^\alpha) \left(\frac{u}{u_\epsilon}\right)^{\alpha-1} dx + \int_A g(x)(v_\epsilon^\alpha - u_\epsilon^\alpha) \left(\frac{v}{v_\epsilon}\right)^{\alpha-1} dx.$$

We claim that the sum of the second and the fourth integrals in the left hand side of (1.3) in non-negative. Indeed, since

$$\nabla \left(\frac{u_\epsilon^\alpha - v_\epsilon^\alpha}{u_\epsilon^{\alpha-1}}\right) = \nabla u + (\alpha - 1)\left(\frac{v_\epsilon}{u_\epsilon}\right)^\alpha \nabla u - \alpha \left(\frac{v_\epsilon}{u_\epsilon}\right)^{\alpha-1} \nabla v$$

and

$$\nabla \left(\frac{v_\epsilon^\alpha - u_\epsilon^\alpha}{v_\epsilon^{\alpha-1}}\right) = \nabla v + (\alpha - 1)\left(\frac{u_\epsilon}{v_\epsilon}\right)^\alpha \nabla v - \alpha \left(\frac{u_\epsilon}{v_\epsilon}\right)^{\alpha-1} \nabla u$$
we find
\[ \int_A |\nabla u|^{q-2} \nabla u \cdot \nabla \left( \frac{v_\varepsilon^\alpha - v_\varepsilon^\alpha}{u_\varepsilon^{\alpha-1}} \right) \, dx + \int_A |\nabla v|^{q-2} \nabla v \cdot \nabla \left( \frac{v_\varepsilon^\alpha - u_\varepsilon^\alpha}{v_\varepsilon^{\alpha-1}} \right) \, dx \]
\[ = \int_A \left\{ |\nabla u|^q \left( 1 + (\alpha - 1) \left( \frac{v_\varepsilon}{u_\varepsilon} \right)^\alpha \right) - \alpha |\nabla u|^{q-2} \left( \frac{v_\varepsilon}{u_\varepsilon} \right)^{\alpha-1} \nabla u \cdot \nabla v \right\} \, dx \tag{1.5} \]
\[ + \int_A \left\{ |\nabla v|^q \left( 1 + (\alpha - 1) \left( \frac{v_\varepsilon}{u_\varepsilon} \right)^\alpha \right) - \alpha |\nabla v|^{q-2} \left( \frac{u_\varepsilon}{v_\varepsilon} \right)^{\alpha-1} \nabla v \cdot \nabla u \right\} \, dx. \]
Now, for \( X, Y \in \mathbb{R}^N \), we recall the inequality
\[ |X|^{q-2} X \cdot Y \leq \frac{1}{s} |X|^q + \frac{1}{q} |Y|^q \] with \( \frac{1}{s} + \frac{1}{q} = 1 \),
and replace the vector \( X \) by \( \lambda X \), where \( \lambda \) is a positive real number. We get
\[ |X|^{q-2} \lambda^{q-1} X \cdot Y \leq \frac{1}{s} \lambda^q |X|^q + \frac{1}{q} |Y|^q. \]
With \( \lambda = \left( \frac{u_\varepsilon}{v_\varepsilon} \right)^{\frac{\alpha-1}{q-1}} \), \( X = \nabla u \), \( Y = \nabla v \) we find
\[ |\nabla u|^{q-2} \left( \frac{v_\varepsilon}{u_\varepsilon} \right)^{\alpha-1} \nabla u \cdot \nabla v \leq \frac{1}{s} \left( \frac{v_\varepsilon}{u_\varepsilon} \right)^{s(\alpha-1)} |\nabla u|^q + \frac{1}{q} |\nabla v|^q. \]
Similarly, with \( \lambda = \left( \frac{u_\varepsilon}{v_\varepsilon} \right)^{\frac{\alpha-1}{q-1}} \), \( X = \nabla v \), \( Y = \nabla u \) we find
\[ |\nabla v|^{q-2} \left( \frac{u_\varepsilon}{v_\varepsilon} \right)^{\alpha-1} \nabla v \cdot \nabla u \leq \frac{1}{s} \left( \frac{u_\varepsilon}{v_\varepsilon} \right)^{s(\alpha-1)} |\nabla v|^q + \frac{1}{q} |\nabla u|^q. \]
In view of these inequalities, by (1.5) we get
\[ \int_A |\nabla u|^{q-2} \nabla u \cdot \nabla \left( \frac{u_\varepsilon^\alpha - v_\varepsilon^\alpha}{u_\varepsilon^{\alpha-1}} \right) \, dx + \int_A |\nabla v|^{q-2} \nabla v \cdot \nabla \left( \frac{v_\varepsilon^\alpha - u_\varepsilon^\alpha}{v_\varepsilon^{\alpha-1}} \right) \, dx \]
\[ \geq \int_A \left\{ |\nabla u|^q \left[ 1 + (\alpha - 1) \left( \frac{v_\varepsilon}{u_\varepsilon} \right)^\alpha - \frac{\alpha}{s} \left( \frac{v_\varepsilon}{u_\varepsilon} \right)^{s(\alpha-1)} \right] - \alpha \right\} \, dx \]
\[ + |\nabla v|^q \left[ 1 + (\alpha - 1) \left( \frac{u_\varepsilon}{v_\varepsilon} \right)^\alpha - \frac{\alpha}{s} \left( \frac{u_\varepsilon}{v_\varepsilon} \right)^{s(\alpha-1)} \right] \, dx. \]
Since the function
\[ \psi(t) = 1 + (\alpha - 1)t^\alpha - \frac{\alpha}{s} t^{s(\alpha-1)} - \frac{\alpha}{q} \]
is non-negative for \( t > 0 \) we find
\[ \int_A |\nabla u|^{q-2} \nabla u \cdot \nabla \left( \frac{u_\varepsilon^\alpha - v_\varepsilon^\alpha}{u_\varepsilon^{\alpha-1}} \right) \, dx + \int_A |\nabla v|^{q-2} \nabla v \cdot \nabla \left( \frac{v_\varepsilon^\alpha - u_\varepsilon^\alpha}{v_\varepsilon^{\alpha-1}} \right) \, dx \geq 0 \]
as claimed. Therefore, from (1.3) we find
\[ \int_A |\nabla u|^{p-2}\nabla u \cdot \nabla \left( \frac{u_{\epsilon}^{\alpha} - v_{\epsilon}^{\alpha}}{u_{\epsilon}^{\alpha-1}} \right) dx + \int_A |\nabla v|^{p-2}\nabla v \cdot \nabla \left( \frac{v_{\epsilon}^{\alpha} - u_{\epsilon}^{\alpha}}{v_{\epsilon}^{\alpha-1}} \right) dx \leq L_{\epsilon}, \quad (1.6) \]
where \( L_{\epsilon} \) is defined as in (1.4). The left hand side of (1.6) can be rewritten as in (1.5) with \( p \) in place of \( q \), that is
\[ \int_A |\nabla u|^{p-2}\nabla u \cdot \nabla \left( \frac{u_{\epsilon}^{\alpha} - v_{\epsilon}^{\alpha}}{u_{\epsilon}^{\alpha-1}} \right) dx + \int_A |\nabla v|^{p-2}\nabla v \cdot \nabla \left( \frac{v_{\epsilon}^{\alpha} - u_{\epsilon}^{\alpha}}{v_{\epsilon}^{\alpha-1}} \right) dx = \int_A \left\{ \left| \nabla u \right|^{p} \left[ 1 + (\alpha - 1) \left( \frac{u_{\epsilon}}{u_{\epsilon}} \right)^{\alpha} \right] - \alpha \left| \nabla u \right|^{p-2} \left( \frac{v_{\epsilon}}{u_{\epsilon}} \right)^{\alpha-1} \nabla u \cdot \nabla v \right\} dx + \int_A \left\{ \left| \nabla v \right|^{p} \left[ 1 + (\alpha - 1) \left( \frac{u_{\epsilon}}{v_{\epsilon}} \right)^{\alpha} \right] - \alpha \left| \nabla v \right|^{p-2} \left( \frac{u_{\epsilon}}{v_{\epsilon}} \right)^{\alpha-1} \nabla v \cdot \nabla u \right\} dx. \quad (1.7) \]
Using the inequality
\[ \left| X \right|^{p-2}X \cdot Y \leq \frac{\left| X \right|^{p}}{r} \frac{1}{p} \left| Y \right|^{p}, \quad \frac{1}{r} + \frac{1}{p} = 1 \]
we find
\[ \left| \nabla u \right|^{p-2} \left( \frac{u_{\epsilon}}{u_{\epsilon}} \right)^{\alpha-1} \nabla u \cdot \nabla v \leq \frac{1}{r} \left( \frac{u_{\epsilon}}{u_{\epsilon}} \right)^{s(\alpha-1)} \left| \nabla u \right|^{p} + \frac{1}{p} \left| \nabla v \right|^{p} \]
and
\[ \left| \nabla v \right|^{p-2} \left( \frac{u_{\epsilon}}{v_{\epsilon}} \right)^{\alpha-1} \nabla v \cdot \nabla u \leq \frac{1}{r} \left( \frac{u_{\epsilon}}{v_{\epsilon}} \right)^{s(\alpha-1)} \left| \nabla v \right|^{p} + \frac{1}{p} \left| \nabla u \right|^{p}. \]
Using these inequalities, from (1.6) and (1.7) we find
\[ \int_A \left\{ \left| \nabla u \right|^{p} \left[ 1 + (\alpha - 1) \left( \frac{u_{\epsilon}}{u_{\epsilon}} \right)^{\alpha} \right] - \alpha \left( \frac{u_{\epsilon}}{u_{\epsilon}} \right)^{q(\alpha-1)} - \frac{\alpha}{p} \right\} + \int_A \left\{ \left| \nabla v \right|^{p} \left[ 1 + (\alpha - 1) \left( \frac{u_{\epsilon}}{v_{\epsilon}} \right)^{\alpha} \right] - \alpha \left( \frac{u_{\epsilon}}{v_{\epsilon}} \right)^{r(\alpha-1)} - \frac{\alpha}{p} \right\} dx \leq L_{\epsilon}. \]
Putting
\[ \varphi(t) = 1 + (\alpha - 1) t^{\alpha} - \frac{\alpha}{r} t^{r(\alpha-1)} - \frac{\alpha}{p}, \]
we have
\[ \int_A \left\{ \left| \nabla u \right|^{p} \varphi \left( \frac{u_{\epsilon}}{u_{\epsilon}} \right) + \left| \nabla v \right|^{p} \varphi \left( \frac{u_{\epsilon}}{v_{\epsilon}} \right) \right\} dx \leq L_{\epsilon}. \quad (1.8) \]
Since
\[ \lim_{\epsilon \to 0} u_{\epsilon} = u, \quad \lim_{\epsilon \to 0} v_{\epsilon} = v, \quad \lim_{\epsilon \to 0} L_{\epsilon} = 0, \]
as $\epsilon \to 0$, (1.8) yields

$$\int_A \left\{|\nabla u|^p \varphi\left(\frac{u}{v}\right) + |\nabla v|^p \varphi\left(\frac{u}{v}\right)\right\} dx \leq 0. \quad (1.9)$$

We have $\varphi(1) = 0$ and $\varphi'(t) = \alpha(\alpha - 1)t^{\alpha p - 2p + 1}(t^{p - \alpha} - 1)$, hence $\varphi(t) > 0$ for $t \neq 1$. Since $\frac{u}{v} > 1$ in $A$, by (1.9) we must have $|\nabla u| = |\nabla v| = 0$ in $A$. Hence, $\nabla (u - v) = 0$ in $A$ and $u - v = 0$ on $\partial A$. Then, $u(x) = v(x)$, contradicting the definition of $A$. The theorem follows.

If $F \subset \mathbb{R}^N$ is a measurable set we denote with $|F|$ its Lebesgue measure. We say that two measurable functions $g_1(x)$ and $g_2(x)$ defined in $\Omega$ have the same rearrangement if (see [3, 14])

$$|\{x \in \Omega : g_1(x) \geq t\}| = |\{x \in \Omega : g_2(x) \geq t\}| \forall t \in \mathbb{R}.$$

Let $g_0(x)$ be a non-negative bounded function defined in $\Omega$. We assume that $g_0(x) > 0$ in a subset of positive measure. We denote by $\mathcal{G} = \mathcal{G}(g_0)$ the class of functions $g$ which have the same rearrangement as $g_0$. Moreover, we denote by $\overline{\mathcal{G}}$ the closure of $\mathcal{G}$ in the weak* topology of $L^\infty(\Omega)$.

With $g \in \overline{\mathcal{G}}$, we consider the functional

$$J(g) = \int_\Omega \left(\frac{1}{\alpha} g u^\alpha - \frac{1}{p} |\nabla u|^p - \frac{1}{q} |\nabla u|^q\right) dx, \quad (1.10)$$

where $u$ is the variational positive solution to problem (1.2). Observe that this function $u$ satisfies

$$\int_\Omega \left(\frac{1}{\alpha} g u^\alpha - \frac{1}{p} |\nabla u|^p - \frac{1}{q} |\nabla u|^q\right) dx = \sup_{v \in H^1_0(\Omega), v \geq 0} \int_\Omega \left(\frac{1}{\alpha} g v^\alpha - \frac{1}{p} |\nabla v|^p - \frac{1}{q} |\nabla v|^q\right) dx,$$

and that, by Theorem 1.1, the superior is unique.

We are interested in the maximization and the minimization of the functional $J(g)$ for $g \in \overline{\mathcal{G}}$.

## 2 Optimization

We make use of the following results.

**Lemma 2.1.** Let $g : \Omega \to \mathbb{R}$ and $w : \Omega \to \mathbb{R}$ be measurable functions, and suppose that every level set of $w$ has zero measure. Then there exists a non-decreasing function $\varphi$ such that $\varphi(w)$ is a rearrangement of $g$. Furthermore, there exists a non-increasing function $\psi$ such that $\psi(w)$ is a rearrangement of $g$. 

Proof. The first assertion follows from Lemma 2.9 of [4]. The second assertion follows applying the first one to $-w$. \hfill \square

Recall that $\overline{\mathcal{G}}$ denotes the closure of $\mathcal{G}$ in the weak* topology of $L^\infty(\Omega)$. It is well known that $\overline{\mathcal{G}}$ is convex and weakly sequentially compact (see for example [4], Lemma 2.2).

**Lemma 2.2.** Let $\mathcal{G}$ be the set of rearrangements of a fixed function $g_0 \in L^\infty(\Omega)$, and let $w \in L^1(\Omega)$. If there is a non-decreasing function $\varphi$ such that $\varphi(w) \in \mathcal{G}$ then

$$\int_\Omega gw\,dx \leq \int_\Omega \varphi(w)w\,dx \quad \forall g \in \overline{\mathcal{G}},$$

and the function $\varphi(w)$ is the unique maximizer relative to $\overline{\mathcal{G}}$. Furthermore, if there is a non-increasing function $\psi$ such that $\psi(w) \in \mathcal{G}$ then

$$\int_\Omega gw\,dx \geq \int_\Omega \psi(w)w\,dx \quad \forall g \in \overline{\mathcal{G}},$$

and the function $\psi(w)$ is the unique minimizer relative to $\overline{\mathcal{F}}$.

Proof. The first assertion follows from Lemma 2.4 of [4]. The second assertion follows from the first one putting $\psi(t) = \varphi(-t)$.

\hfill \square

**Theorem 2.3.** Let $\Omega$, $g_0$, $\mathcal{G}$ and $\overline{\mathcal{G}}$ be as in above. If $J(g)$ is defined as in (1.10) for $g \in \overline{\mathcal{G}}$ then:

i) $J(g)$ is weakly continuous with respect to the weak* topology of $L^\infty(\Omega)$;

ii) $J(g)$ is Gâteaux differentiable with derivative $\frac{1}{\alpha}u^\alpha$ (here $u$ is the variational positive solution to problem (1.2) corresponding to $g$);

iii) $J(g)$ is strictly convex on $\overline{\mathcal{G}}$.

Proof. Let $g_i \in \overline{\mathcal{G}}$, $g_i \rightharpoonup g$. We shall prove that

$$\lim_{i \to \infty} J(g_i) = J(g). \quad (2.1)$$

Indeed, if $u_i$ is the solution to (1.2) corresponding to $g_i$, putting $\phi = u_i$ in that equation we find

$$\int_\Omega (|\nabla u_i|^p + |\nabla u_i|^q)\,dx = \int_\Omega g_i u_i^\alpha\,dx.$$  

Since $g_i$ is uniformly bounded, using Poincaré and Hölder inequalities we find

$$\int_\Omega |\nabla u_i|^p\,dx \leq C \int_\Omega |\nabla u_i|^\alpha \leq C \left( \int_\Omega |\nabla u_i|^p\,dx \right)^{\frac{\alpha}{p}}$$

\hfill \square
and
\[ \int_{\Omega} |\nabla u_i|^p \, dx \leq C. \] (2.2)

Recall that we denote by \( C \) constants independent of \( i \). By (2.2), it follows that a subsequence (denoted again \( \{ u_i \} \)) converges weakly in \( H^{1,p}(\Omega) \) (as well as in \( H^{1,q}(\Omega) \)) and strongly in \( L^\alpha(\Omega) \) to some function \( z \in H^{1,p}_0(\Omega) \).

We claim that
\[ \lim_{i \to \infty} \int_{\Omega} (g_i - g)(u_i^\alpha - z^\alpha) \, dx = 0. \]

Indeed, since
\[ |(g_i - g)(u_i^\alpha - z^\alpha)| \leq C\alpha |u_i - z| \cdot |u_i + z|^{\alpha - 1} \]
we have
\[ \left| \int_{\Omega} (g_i - g)(u_i^\alpha - z^\alpha) \, dx \right| \leq C\alpha \|u_i - z\|_{L^\alpha(\Omega)} \|u_i + z\|^{\alpha - 1}_{L^\alpha(\Omega)}. \]

Since \( u_i \to z \) strongly in \( L^\alpha(\Omega) \), the claim follows.

Since
\[ \lim_{i \to \infty} \int_{\Omega} (g_i - g)z^\alpha \, dx = 0, \]
and since
\[ 0 = \lim_{i \to \infty} \int_{\Omega} (g_i - g)(u_i^\alpha - z^\alpha) \, dx = \lim_{i \to \infty} \int_{\Omega} (g_i - g)u_i^\alpha \, dx - \lim_{i \to \infty} \int_{\Omega} (g_i - g)z^\alpha \, dx \]
we find
\[ \lim_{i \to \infty} \int_{\Omega} (g_i - g)u_i^\alpha \, dx = 0. \] (2.3)

Obviously, if \( u \) is the positive solution to problem (1.2) corresponding to \( g \) we also have
\[ \lim_{i \to \infty} \int_{\Omega} (g_i - g)u^\alpha \, dx = 0. \] (2.4)

To conclude the proof of assertion i), we write
\[
J(g) + \int_{\Omega} \frac{1}{\alpha} (g_i - g)u^\alpha \, dx \\
= \int_{\Omega} \left( \frac{1}{\alpha} g_i u^\alpha - \frac{1}{p} |\nabla u|^p - \frac{1}{q} |\nabla u|^q \right) \, dx \\
\leq \int_{\Omega} \left( \frac{1}{\alpha} g_i u_i^\alpha - \frac{1}{p} |\nabla u_i|^p - \frac{1}{q} |\nabla u_i|^q \right) \, dx = J(g_i) \\
= \int_{\Omega} \frac{1}{\alpha} (g_i - g)u_i^\alpha \, dx + \int_{\Omega} \left( \frac{1}{\alpha} g u_i^\alpha - \frac{1}{p} |\nabla u_i|^p - \frac{1}{q} |\nabla u_i|^q \right) \, dx \\
\leq \int_{\Omega} \frac{1}{\alpha} (g_i - g)u_i^\alpha \, dx + J(g).
\] (2.5)
By (2.5), (2.3) and (2.4) we find (2.1), and assertion i) of the theorem is proved.

To prove assertion ii), we claim that the function \( z \) mentioned in above is equal to \( u \), the variational positive solution to problem (1.2) corresponding to \( g \). Indeed from

\[
\int_{\Omega} |\nabla z|^p \, dx \leq \liminf_{i \to \infty} \int_{\Omega} |\nabla u_i|^p \, dx,
\]

\[
\int_{\Omega} |\nabla z|^q \, dx \leq \liminf_{i \to \infty} \int_{\Omega} |\nabla u_i|^q \, dx,
\]

and (2.1) we find

\[
J(g) = \int_{\Omega} \left( \frac{1}{\alpha} g u^\alpha - \frac{1}{p} |\nabla u|^p - \frac{1}{q} |\nabla u|^q \right) \, dx
\]

\[
= \lim_{i \to \infty} J(g_i) = \lim_{i \to \infty} \int_{\Omega} \left( \frac{1}{\alpha} g_i u_i^\alpha - \frac{1}{p} |\nabla u_i|^p - \frac{1}{q} |\nabla u_i|^q \right) \, dx
\]

\[
\leq \lim_{i \to \infty} \int_{\Omega} \frac{1}{\alpha} g_i u_i^\alpha \, dx - \liminf_{i \to \infty} \int_{\Omega} \frac{1}{p} |\nabla u_i|^p \, dx - \liminf_{i \to \infty} \int_{\Omega} \frac{1}{q} |\nabla u_i|^q \, dx
\]

\[
\leq \int_{\Omega} \left( \frac{1}{\alpha} g z^\alpha - \frac{1}{p} |\nabla z|^p - \frac{1}{q} |\nabla z|^q \right) \, dx \leq J(g).
\]

It follows that

\[
\int_{\Omega} \left( \frac{1}{\alpha} g u^\alpha - \frac{1}{p} |\nabla u|^p - \frac{1}{q} |\nabla u|^q \right) \, dx = \int_{\Omega} \left( \frac{1}{\alpha} g z^\alpha - \frac{1}{p} |\nabla z|^p - \frac{1}{q} |\nabla z|^q \right) \, dx.
\]

By the uniqueness of the maximizer of

\[
-I(v) = \int_{\Omega} \left( \frac{1}{\alpha} g v^\alpha - \frac{1}{p} |\nabla v|^p - \frac{1}{q} |\nabla v|^q \right) \, dx
\]

for \( v \geq 0, v \in H_0^{1,p}(\Omega) \), we must have \( u = z \), as claimed.

In view of the latter result, (2.3) implies

\[
\lim_{i \to \infty} \int_{\Omega} g_i u_i^\alpha \, dx = \lim_{i \to \infty} \int_{\Omega} g u_i^\alpha \, dx = \int_{\Omega} g u^\alpha \, dx.
\]

Let \( t_i > 0 \) be a sequence of real numbers such that \( t_i \to 0 \) as \( i \to \infty \). Let \( h \in \mathcal{G} \) and let \( g_i = g + t_i (h - g) \). Then, by (2.5) we find

\[
J(g) + t_i \int_{\Omega} \frac{1}{\alpha} (h - g) u^\alpha \, dx \leq J(g + t_i (h - g)) \leq J(g) + t_i \int_{\Omega} \frac{1}{\alpha} (h - g) u_i^\alpha \, dx.
\]
It follows that
\[
\int_{\Omega} \frac{1}{\alpha} (h - g) u^\alpha \, dx \leq \frac{J(g + t_i(h - g)) - J(g)}{t_i} \leq \int_{\Omega} \frac{1}{\alpha} (h - g) u_i^\alpha \, dx. \tag{2.6}
\]

Since \(g_i \to g\) as \(i \to \infty\), \(u_i \to u\) in \(L^\alpha(\Omega)\). Therefore,
\[
\lim_{i \to \infty} \int_{\Omega} \frac{1}{\alpha} (h - g) u_i^\alpha \, dx = \int_{\Omega} \frac{1}{\alpha} (h - g) u^\alpha \, dx.
\]

Since the sequence \(t_i\) is arbitrary, from the latter equation and (2.6) we find
\[
\lim_{t \to 0^+} \frac{J(g + t(h - f)) - J(g)}{t} = \int_{\Omega} \frac{1}{\alpha} (h - g) u^\alpha \, dx.
\]

This proves assertion ii).

Let \(0 < t < 1\). If \(g_1, g_2 \in \overline{G}\) and \(g_t = tg_1 + (1 - t)g_2\), we have
\[
\int_{\Omega} \left( \frac{1}{\alpha} g_t v^\alpha - \frac{1}{p} |\nabla v|^p - \frac{1}{q} |\nabla v|^q \right) \, dx \\
= t \int_{\Omega} \left( \frac{1}{\alpha} g_1 v^\alpha - \frac{1}{p} |\nabla v|^p - \frac{1}{q} |\nabla v|^q \right) \, dx \\
+ (1 - t) \int_{\Omega} \left( \frac{1}{\alpha} g_2 v^\alpha - \frac{1}{p} |\nabla v|^p - \frac{1}{q} |\nabla v|^q \right) \, dx.
\]

If we take the superior for \(v \in H_0^{1,p}(\Omega)\) in both sides of this equation we find
\[
J(g_t) \leq tJ(g_1) + (1 - t)J(g_2).
\]

To prove strict convexity, assume equality holds in the latter inequality for some \(0 < t < 1\). If \(u_t, u_1\) and \(u_2\) are the solutions corresponding to \(g_t, g_1\) and \(g_2\) respectively, then we have
\[
t \int_{\Omega} \left( \frac{1}{\alpha} g_1 u_t^\alpha - \frac{1}{p} |\nabla u_t|^p - \frac{1}{q} |\nabla u_t|^q \right) \, dx \\
+ (1 - t) \int_{\Omega} \left( \frac{1}{\alpha} g_2 u_t^\alpha - \frac{1}{p} |\nabla u_t|^p - \frac{1}{q} |\nabla u_t|^q \right) \, dx \\
= t \int_{\Omega} \left( \frac{1}{\alpha} g_1 u_1^\alpha - \frac{1}{p} |\nabla u_1|^p - \frac{1}{q} |\nabla u_1|^q \right) \, dx \\
+ (1 - t) \int_{\Omega} \left( \frac{1}{\alpha} g_2 u_2^\alpha - \frac{1}{p} |\nabla u_2|^p - \frac{1}{q} |\nabla u_2|^q \right) \, dx.
\]
It follows that
\[
\int_{\Omega} \left( \frac{1}{\alpha} g_1 u_t^\alpha - \frac{1}{p} |\nabla u_t|^p - \frac{1}{q} |\nabla u_t|^q \right) dx
\]
\[
= \int_{\Omega} \left( \frac{1}{\alpha} g_1 u_1^\alpha - \frac{1}{p} |\nabla u_1|^p - \frac{1}{q} |\nabla u_1|^q \right) dx
\]
and
\[
\int_{\Omega} \left( \frac{1}{\alpha} g_2 u_t^\alpha - \frac{1}{p} |\nabla u_t|^p - \frac{1}{q} |\nabla u_t|^q \right) dx
\]
\[
= \int_{\Omega} \left( \frac{1}{\alpha} g_2 u_2^\alpha - \frac{1}{p} |\nabla u_2|^p - \frac{1}{q} |\nabla u_2|^q \right) dx.
\]
By uniqueness we must have \( u_t = u_1 = u_2 \), and
\[-\Delta_p u_t - \Delta_q u_t = g_1 u_t^{\alpha - 1} \text{ a.e. in } \Omega \]
and
\[-\Delta_p u_t - \Delta_q u_t = g_2 u_t^{\alpha - 1} \text{ a.e. in } \Omega. \]
Since \( g_1(x) \geq 0 \) and \( g_1(x) \not\equiv 0 \), by the strong maximum principle \( u_t(x) > 0 \). It follows that \( g_1(x) = g_2(x) \) a.e. in \( \Omega \), which yields strict convexity of \( J(g) \).

The theorem is proved. \( \square \)

**Theorem 2.4.** Let \( \Omega, g_0, \mathcal{G} \) and \( \mathcal{G} \) be as in above. Let \( J(g) \) be defined as in (1.10) for \( g \in \mathcal{G} \).

i) The problem
\[
\max_{g \in \mathcal{G}} J(g)
\]
has a solution \( \hat{g} \). Moreover, if \( \hat{g} \) is any solution and if \( \hat{u} = u_{\hat{g}} \) then we have \( \hat{g} = \varphi(\hat{u}) \)
where \( \varphi \) is some non-decreasing function.

ii) The problem
\[
\min_{g \in \mathcal{G}} J(g)
\]
has a unique solution \( \tilde{g} \). Moreover, if \( \tilde{u} = u_{\tilde{g}} \) we have \( \tilde{g} = \psi(\tilde{u}) \), where \( \psi \) is some non-increasing function.

**Proof.** By Theorem 2.1, \( J(g) \) is weakly continuous and strictly convex. Therefore, assertion i) follows by Theorem 7 of [3].

Let us prove assertion ii). The uniqueness follows by the strict convexity of \( J(g) \). To prove existence, let
\[
\tilde{J} = \inf_{g \in \mathcal{G}} J(g),
\]
and let \( \{g_i\} \) be a sequence such that

\[
\bar{J} = \lim_{i \to \infty} J(g_i).
\]

Since \( \mathcal{G} \) is weakly compact we can assume that for some subsequence of \( \{g_i\} \) (again denoted \( \{g_i\} \)) there is \( \hat{g} \in \mathcal{G} \) with \( g_i \rightharpoonup \hat{g} \) in the weak* topology of \( L^\infty(\Omega) \). By Theorem 2.3, we have \( \bar{J} = J(\hat{g}) \). Let us show that \( \hat{g} \in \mathcal{G} \). If \( g \in \mathcal{G} \), if \( 0 < t < 1 \) and if \( g_t = tg + (1 - t)\hat{g} \), since \( J(g) \) is Gâteaux differentiable (by Theorem 2.3) we have,

\[
J(\hat{g}) \leq J(g_t) = J(\hat{g}) + t \int_{\Omega} (g - \hat{g}) \frac{1}{\alpha} u^\alpha \, dx + o(t) \quad \text{as} \quad t \to 0.
\]

Hence,

\[
\int_{\Omega} (g - \hat{g}) \frac{1}{\alpha} u^\alpha \, dx \geq 0.
\]

Equivalently,

\[
\int_{\Omega} g u^\alpha \, dx \geq \int_{\Omega} \hat{g} u^\alpha \, dx \quad \forall g \in \mathcal{G}.
\]  (2.7)

On the other hand, by equation (1.2) it follows that the function \( u \) (and the function \( u^\alpha \)) cannot have flat zones in the set

\[
E = \{ x \in \Omega : \hat{g}(x) > 0 \}.
\]

Consider first the case \( |E| = |\Omega| \). Then, by Lemma 2.1 there is a non-increasing function \( \psi \) such that \( \psi(u^\alpha) \) is a rearrangement of \( \hat{g} \). By (2.7) and Lemma 2.2 we must have \( \hat{g} = \psi(u^\alpha) \in \mathcal{G} \).

If \( |E| < |\Omega| \), since \( \hat{g} \in \mathcal{G} \), by Lemma 2.14 of [4] we have

\[
|E| \geq |\{ x \in \Omega : g_0(x) > 0 \}|.
\]

Then there is \( g_1 \in \mathcal{G} \) such that its support is contained in \( E \). By Lemma 2.1 there is a non-increasing function \( \psi_1(t) \) such that \( \psi_1(u^\alpha) \) is a rearrangement of \( g_1 \) on \( E \).

Define

\[
m = \inf_{x \in \Omega \setminus E} u^\alpha(x).
\]

By using (2.7) one proves that \( u^\alpha(x) < m \) in \( E \) (see [5, 8, 9] for details). Now define

\[
\tilde{\psi}(t) = \begin{cases} 
\psi_1(t) & \text{if } 0 \leq t < m \\
0 & \text{if } t \geq m.
\end{cases}
\]

The function \( \psi(t) \) is non-increasing and \( \psi(u^\alpha) \) is a rearrangement of \( g_1(x) \) in \( \Omega \). Indeed, the functions \( g_1 \) and \( \psi(u^\alpha) \) have the same rearrangement on \( E \), and both vanish on \( \Omega \setminus E \). By Lemma 2.2 we must have \( \hat{g} = \psi(u^\alpha) \in \mathcal{G} \). The theorem is proved. \( \square \)
References


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