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Almost periodic solutions in gross-substitute discrete dynamical systems

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Abstract: We consider the existence of almost periodic solutions of a gross-substitute discrete system, which appear as tatonnement processes of mathematical economic models, by using uniformly stable and properties of an almost periodic gross-substitute discrete system.

Keywords: Almost periodic solutions; Almost periodic gross-substitute discrete systems; limiting equations of almost periodic systems.

MSC2010: 39A10, 39A11

Dedicated to Acad. Constantin Corduneanu on the occasion of his 90th birthday

1 Introduction.

In this paper, we consider tatonnement processes with discrete non-autonomounus almost periodic coefficients. The model process is given by a system of difference equations

$$\Delta p(n) = \sigma_i D_i(n, p(n)), \quad i = 1, 2, \cdots, m, \tag{1.1}$$

where Δ is a difference operator, $p(n) = (p_i(n))$ is a price-vector, $D_i(n, p)$ is the excess demand function for the i-th good and σ_i is a positive constant. These equations form a mathematical economic model for the classical law of supply and demand. We assume below conditions that the system (1) is a gross-substitute system that satisfies Walras' law and that $D(n, p)$ is almost periodic in n for uniformly p. As an example of this system, by $\sigma_i = 1$ of (1), we consider the following equations

$$D_i(n, p_i) = (\sum_{\alpha=1}^{M} \sum_{j=1}^{m} a_{ij}^{\alpha}(n) p_j^{\alpha})/p_i, \text{ for all } i,$$

where $a_{ij}^{\alpha} \equiv a_{ij}^{\alpha}(n)$ is almost periodic in n, $a_{ij}^{\alpha} \geq 0$ when $i \neq j$ and $\sum_{j=1}^{m} a_{ij}^{\alpha}(n) \equiv 0$ for all j and α. Autonomous tatonnement processes have been studied extensively in the economic literature (cf. [7, 10]). The stability results and limiting behavior of these systems are well understood (cf. [1, 7, 10, 12]). However, if we wish to build a theory of such economic models which reflects changes due to seasonal adjustments, then it is important to study discrete time-dependent or non-autonomous systems. We describe here the theory, which is adequate to describe the limiting behavior of systems with almost periodic seasonal adjustments. In the example, such above continuous systems would occur if the coefficients $a_{ij}^{\alpha}(n)$ are periodic with incommensurable periods (cf. [6]). Recently, Saito [9] has shown the existence of periodic solutions of ω periodic system (1) by assuming the uniformly stability for the solution of (1). In this paper, we study the discrete almost periodic system for a generalization of the periodic differential system presented in Nakajima [6], and our result is based on Sell and Nakajima [11] for the differential system. In particular, we show that any positively compact solution of system (1) is asymptotically almost periodic solution (cf. [13]). As we see that the positive compactness of solutions guarantee stability. In order to prove that the limiting behavior is almost periodic, we use the inherited property of uniform stability for the difference system. In particular, we use the result which asserts that the positively compact uniformly stable solution of an almost periodic difference equation is a separating proerty for the difference system. As a technical point, we note that the given positively compact solution need not be asymptotically stable. But nevertheless, by the strong structure of gross-substitute system, this solution is asymptotically almost periodic solution (cf. [5, 13]).

2 Preliminaries.

We denote by R^m the real m-dimensional Euclidean space with norm $|x| = \sum_{i=1}^{m} |x_i|$, where $x = (x_1, x_2, \cdots, x_m) \in R^m$. Let $R = (-\infty, \infty)$. $\mathbf{Z}$ is the set of integers, $\mathbf{Z}^+$ is the set of nonnegative integers. Let

$$P_0 = \{x \in R^m |\ x_i > 0 \text{ for } 1 \le i \le m\}.$$

Let $f(n, x) \equiv f = (f_1, f_2, \cdots, f_m) : \mathbf{Z} \times P_0 \to R^m$ be a continuous function for second variable.

We introduce an almost periodic function $f(n, x) : \mathbf{Z} \times U \to R^m$, where U is an open set in R^m.

Definition 1. $f(n, x)$ is said to be almost periodic in n uniformly for $x \in U$, if for any $\epsilon > 0$ and any compact set K in U there exists a positive integer $L^*(\epsilon, K)$ such that any interval of length $L^*(\epsilon, K)$ contains an integer τ for which

$$|f(n + \tau, x) - f(n, x)| \le \epsilon$$

for all $n \in \mathbf{Z}$ and all $x \in K$. Such a number τ in above inequality is called an ϵ-translation number of $f(n,x)$.
In order to formulate a property of almost periodic functions, which is equivalent to the above definition, we discuss the concept of the normality of almost periodic functions. Namely, let $f(n,x)$ be almost periodic in n uniformly for $x \in U$. Then, for any sequence $\{h_k^{'}\} \subset \mathbf{Z}$, there exist a subsequence $\{h_k\}$ of $\{h_k^{'}\}$ and a function $g(n,x)$ such that

$$f(n+h_k,x) \to g(n,x) \tag{2.1}$$

uniformly on $\mathbf{Z} \times K$ as $k \to +\infty$, where K is a compact set in U. There are many properties of the discrete almost periodic functions [2], which are corresponding properties of the continuous almost periodic functions $f(t,x) \in C(R \times U, R^m)$ (cf. [5, 13]).
Moreover, the following concept of asymptotic almost periodicity was introduced by Frechet in the case of continuous function (cf.[13]).
$u(n)$ is said to be asymptotically almost periodic if it is a sum of a almost periodic function $p(n)$ and a function $q(n)$ defined on $I^* = [a,\infty) \subset \mathbf{Z}^+$ which tends to zero as $n \to \infty$, that is,

$$u(n) = p(n) + q(n).$$

Furthermore, $u(n)$ is asymptotically almost periodic if and only if for any sequence $\{n_k\}$ such that $n_k \to \infty$ as $k \to \infty$, there exists a subsequence $\{n_{k_j}\}$ for which $u(n+n_{k_j})$ converges uniformly on I^* as $j \to \infty$.
We denote by $T(f)$ the function space consisting of all translates of f, that is, $f_\tau \in T(f)$, where

$$f_\tau(n,x) = f(n+\tau,x), \qquad \tau \in \mathbf{Z}. \tag{2.2}$$

Let $H(f)$ denote the uniform closure of $T(f)$ in the sense of (2). $H(f) = Cl\{f_\tau | \tau \in \mathbf{Z}\}$ is called the hull of f. In particular, we denote by $\Omega(f)$ the set of all limit functions $g \in H(f)$ such that for some sequence $\{n_k\}$, $n_k \to +\infty$ as $k \to \infty$ and $f(n+n_k,x) \to g(n,x)$ uniformly on $\mathbf{Z} \times S$ for any compact subset S in R^m. Clearly, $f \in H(f)$. By (3), if $f : \mathbf{Z} \times U \to R^m$ is almost periodic in n uniformly for $x \in U$, so is a function in $\Omega(f)$.

We consider the system of difference equation

$$\Delta x(n) = f(n,x(n)) \tag{2.3}$$

on $\mathbf{Z} \times P \subset P_0$, $P = \{x| \ |x| < B^* \text{ for some } B^* > 0\}$ is called a (bounded) gross-substitute system if the following four assumptions are satisfied:

$(H1)$ For any compact set $K \subseteq P$ there is a constant $1 > L = L(K) > 0$ such that $|f(n,x) - f(n,y)| \leq L|x-y|$ for $n \in \mathbf{Z}$ and all $x, y \in K$,
$(H2)$ For any $i = 1, 2, \cdots, m$, we have $f_i(n,x) \leq f_i(n,y)$ for any $x, y \in P$ with $x_i = y_i$ and $x_j \leq y_j \ (1 \leq j \leq m)$,
$(H3)$ $\sum_{i=1}^m f_i(n,x) = 0$ for all $n \in \mathbf{Z}$ and $x \in P$
and
$(H4)$ $f(n,x)$ is almost periodic in n uniformly for $x \in P$.

Remark 1. The hypothesis $(H2)$ is the standard definition for a gross-substitute system [7], and also condition $(H2)$ is usually called Kamke type condition (cf. [9]). The hypothesis $(H3)$ is basically Walras' law. In term of equation (1), Walras' law is sometimes stated as $\sum_{i=1}^m p_i D_i(n,p) = 0$ for $n \in \mathbf{Z}$ and $p \in P$. However, the change of variables $p \to x$ defined by

$$x_i = p_i^2/\sigma_i \quad \text{for } 1 \leq i \leq m$$

shows that the latter is equivalent to $(H3)$, since system (1) is transformed into the system (4), where $f(n,x) = (f_i(n,x))$ for $1 \leq i \leq m$ has the form

$$f_i(n,x) = \sqrt{\sigma_i x_i} D_i(n, \sqrt{\sigma_1 x_1}, \sqrt{\sigma_2 x_2}, \cdots, \sqrt{\sigma_m x_m})$$

and satisfies system (1) and $(H3)$. In economic theory, Walras' law is an assertion of the equality of supply and demand. Since we assume only all four of the above conditions for system (4), we have lumped these sins together under the single title of an almost periodic gross-substitute system.

Let $p(n) = (p_i(n))$ be a solution of system (4) defined on $[n_0, \infty)$ for some $n_0 \in \mathbf{Z}^+$. Then $p(n)$ is said to be positively compact (short compact) if $p(n)$ remains in a compact subset of P for all $n \geq n_0$, that is, there are positive constants α and β, $0 < \alpha \leq \beta$, such that

$$\alpha \leq p_i(n) \leq \beta \quad \text{for } n \geq n_0 \text{ and } 1 \leq i \leq m.$$

Then, we remark that the boundedness of solutions does not necessarily imply the compactness.

3 Theorem of uniformly stable.

We first define the stability of an almost periodic gross-substitute system (4).

Definition 2. The bounded solution $\phi(n)$ of system (4) is said to be uniformly stable with respect to compact set K (in short, US w.r.t.K) if for any $\epsilon > 0$ and any $n_0 \geq 0$, there exists a $\delta(\epsilon) > 0$, such that $|\phi(n_0) - x(n_0)| \leq \delta(\epsilon)$ implies $|\phi(n) - x(n)| \leq \epsilon$ for all $n \geq n_0$, where $x(n)$ is a solution of (4) such that $x(n_0) \in K$.

We next have the following theorem, preliminaries lemma and proposition which are essentially the same as [8, 13] for the differential equations. To make this paper concrete for difference systems, we give the proofs, except for lemma. We omit the proof of lemma, as we can see the proof of Lemma 3 in [9].

Lemma 1. Let $\phi(n)$ through x_0 at $n = 0$ be a positively compact solutions of (4). Assume that for all $x_1, x_2 \in P$, we have $\Delta|\phi_1(n) - \phi_2(n)| \leq 0$, where $\phi_i(n)$ are distinct solutions of (4) through x_1 and x_2 at $n = 0$, respectively. Then, $\phi(n)$ is US w.r.t.K of (4).

For every $g \in \Omega(f)$, we consider the solution $\psi(n)$ of limiting equation

$$\Delta x(n) = g(n, x(n)), \tag{3.1}$$

and we first show that an inherited property of US w.r.t.K in almost periodic system (4).

Proposition 1. Suppose that for every $g \in \Omega(f)$, the solution of (5) is unique for the initial value problem. If the compact solution $\phi(n)$ of system (4) is US w.r.t.K, then $\psi(n)$ of (5) through x_0 at $n = 0$ is US w.r.t.K in $\Omega(f)$.

Proof. Let $\{s_k\}$ be a sequence such that $s_k > 0, f(n+s_k, x) \to g(n, x)$ uniformly on $\mathbf{Z} \times K$ and $\phi(s_k) \to x_0$ as $k \to \infty$. We now set $\phi^k(n) = \phi(n + s_k)$, which is a US solution with respect to K of

$$\Delta x(n) = f(n + s_k, x(n)) \tag{3.2}$$

through $(0, \phi(s_k))$ with the same pair $(\epsilon, \delta(\epsilon))$ as the one for uniformly stability of $\phi(n)$, and $|\phi^k(n)| \leq B$ for $n \geq 0$, where B is a positive constant of compactness. Therefore, $\{\phi^k(n)\}$ is uniformly bounded on $\mathbf{Z}^+$, and hence there exists a subsequence, which we shall denote by $\{\phi^k(n)\}$ again, such that $\phi^k(n)$ converges to $\psi(n)$ uniformly on any compact interval on $\mathbf{Z}^+$. For a fixed $n_0 \in \mathbf{Z}^+$, if k is sufficiently large, we have

$$|\phi^k(n_0) - \psi(n_0)| < \frac{1}{2}\delta(\frac{\epsilon}{2}), \tag{3.3}$$

where we can assume that $\epsilon < B^* - B$, where B^* is a boundary for the boundedness. Let y_0 be such that

$$|y_0 - \psi(n_0)| < \frac{1}{2}\delta(\frac{\epsilon}{2}). \tag{3.4}$$

and let $x(n)$ be the solution of (4) such that $x(n_0 + s_k) = y_0$. Then $x^k(n) = x(n+s_k)$ is a solution of (6) and $x^k(n_0) = y_0$. Since $\phi^k(n)$ is US w.r.t.K and $|\phi^k(n_0) - x^k(n_0)| < \delta(\frac{\epsilon}{2})$, we have

$$|\phi^k(n) - x^k(n)| < \frac{\epsilon}{2} \quad \text{for all } \ n \le n_0. \tag{3.5}$$

Since $|x^k(n)| \le B + \frac{\epsilon}{2}$ for all $n \ge n_0$ and large k, a subsequence of sequence $\{x^k(n)\}$ converges to the solution $y(n)$ of (5) through y_0 at $n = n_0$, which is uniquely determined, uniformly on any compact interval $[n_0, n_0 + N^*]$, where $N^* \subset \mathbf{Z}^+$. We denote by $\{x^k(n)\}$ this subsequence again. Thus, if k is sufficiently large,

$$|x^k(n) - y(n)| < \frac{\epsilon}{4} \quad \text{and } \ |\phi^k(n) - \psi(n)| < \frac{\epsilon}{4} \quad \text{on } \ [n_0, n_0 + N^*]. \tag{3.6}$$

It follows from (9) and (10) that $|\psi(n) - y(n)| < \epsilon$ on $[n_0, n_0 + N^*]$. Since N^* is arbitrary, $|\psi(n) - y(n, n_0, y_0)| < \epsilon$ for all $n \ge n_0$ if $|\psi(n_0) - y_0| < \frac{1}{2}\delta(\frac{\epsilon}{2})$, where $y(n, n_0, y_0)$ is the solution of (5) through y_0 at $n = n_0$. This proves Proposition 1, that is $\phi(n)$ has the inherited property of US w.r.t.K.

Theorem 1 (cf. [Theorem A in 11], [13]). Let system (4) be the almost periodic gross-substitute system. Let $\phi(n)$ through x_0 at $n = 0$ be a positively compact uniformly stable solution of (4). Then $\Omega(\phi, f)$ is a nonempty compact semi-separating property set. Moreover, if for some $g \in H(f)$, the section $\Omega(\phi)$

$$\Omega(\phi) = \{\psi \in P| \ (\psi, g) \in \Omega(\phi, f)\}$$

has only finitely many points, then for each $(\psi, g) \in \Omega(\phi, f)$, the solution $\psi(n)$ of (5) through y_0 at $n = 0$ is almost periodic in n.

Proof. From the definition of Ω and assumption $(H4)$, it is clear that $\Omega(\phi, f)$ is a nonempty compact set. By assumption $(H1)$, we can easily show the uniqueness of solution for system (4), and note, in this special case, that the uniqueness of solution is inherited property since the assumption $(H1)$ is more stronger than the uniqueness of solution. Then, it follows from Lemma 18.2 in [13] that US w.r.t K is a semi-separating property for system (4), and moreover, by using Proposition 1 and assumptions of this theorem, solution $\psi(n)$ of system (5) is US w.r.t K in

$\Omega(f)$. Thus, by the same reason of above lemma, US w.r.t K is a semi-separating property for system (5) in $\Omega(f)$. Therefore, it follows immediately from Lemma 18.3 in [13] that for every $g \in \Omega(f)$, system (5) has only a finite number of solutions in compact set K which are US w.r.t.K and separation constant $\lambda(\phi_1, \phi_2)$ can be chosen independently of solutions and system (4). Now, we can show that the solution $\psi(n)$ of (5) through y_0 at $n = 0$ is asymptotically almost periodic on $(-\infty, 0]$. Let ϕ be a solution of (4) in compact set K with the inherited property of US w.r.t. K and let $\lambda_0 > 0$ be the separation constant. From above sentence, note that we can assume this λ_0 is independent of solutions and system (4). We shall show that ϕ satisfies the condition in Lemma 18.1 in [13] with interval $I \subset \mathbf{Z}$ replaced by $(-\infty, 0]$ and $\lambda = \frac{\lambda_0}{2}$. Let $\gamma_k^{'}$ be a sequence such that $\gamma_k^{'} \to -\infty$ as $k \to \infty$. Then there exists a sequence $\gamma \subset \gamma^{'}$ such that

$$\begin{aligned}
&T_\gamma f = g \quad \text{uniformly on } \mathbf{Z} \times K \\
&\textit{and} \\
&T_\gamma \phi = \psi \quad \text{exists uniformly on compact subset on } \mathbf{Z}.
\end{aligned}$$

Let $\alpha = \{\alpha_k\}$, $\alpha_k < 0$, $\beta^{'} \subset \gamma$ and $\beta^{''} \subset \gamma$ such that $T_{\alpha+\beta'}\phi = \xi$ and $T_{\alpha+\beta''}\phi = \eta$ exists. We can assume that ξ and η are solutions of US w.r.t K of $\Delta x(n) = h(n, x(n))$, where

$$h = T_\alpha g = T_\alpha T_\gamma f.$$

Thus, we have $\xi \equiv \eta$ or $|\xi(n) - \eta(n)| \geq \lambda_0 = 2\lambda$ on $(-\infty, 0]$. Therefore, the solution ψ is asymptotically almost periodic on $(-\infty, 0]$, and there is an almost periodic solution of (5) in K by the same argument for a differential equation of Theorem 16.1 in [13]. This completes the proof of theorem.

4 Lemmas and theorem of almost periodic solutions.

Let $x(n) = \phi_1(n)$ through x_0 at $n = 0$ and $y(n) = \phi_2(n)$ through y_0 at $n = 0$ be two solutions of a gross-substitute system (4). Assume that both these solutions are defined on a common interval $I \subset \mathbf{Z}$. At this point, we do not require that $f(n, x)$ be almost periodic in n uniformly for x to prove these lemmas. For $n \in I$, we define the following five subsets of $\{i|1 \le i \le m\}$;

$$
\begin{aligned}
&P_n = \{i|\ x_i(n) \ge y_i(n)\},\\
&Q_n = \{i|\ x_i(n) \le y_i(n)\},\\
&A_n = \{i|\ \exists h_i \in \mathbf{Z}^+ \text{ with } x_i(s) > y_i(s) \text{ for } n < s < n + h_i\},\\
&B_n = \{i|\ \exists h_i \in \mathbf{Z}^+ \text{ with } x_i(s) < y_i(s) \text{ for } n < s < n + h_i\}\\
&\textit{and}\\
&C_n = \{1, 2, \cdots, m\} - (A_n \cup B_n).
\end{aligned}
$$

We next define the $m \times m$ matrix $A(n) = (a_{ik}(n)),\ \ 1 \le i, k \le m$, by

$$
\begin{aligned}
a_{ik}(n) \ &= \ f_i(n, x_1(n), x_2(n), \cdots, x_{k-1}(n), x_k(n), y_{k+1}(n), \cdots, y_m(n))\\
&- \ f_i(n, x_1(n), x_2(n), \cdots, x_{k-1}(n), y_k(n), y_{k+1}(n), \cdots, y_m(n)).
\end{aligned}
$$

Notice that these five sets and the terms $a_{ik}(n)$ depend on n and the ordered pair $(x(\cdot), y(\cdot))$.
We show the following five lemmas to prove main theorem:

Lemma 2. The following statements are valid:

(A) $\quad k \in C_n \Rightarrow x_k(n) = y_k(n),\ \ \Delta x_k(n) = \Delta y_k(n)$ and $a_{ik}(n) = 0$ for all i,

(B) $\quad k \in A_n \Rightarrow a_{ik}(n) \ge 0$ for all $i \ne k$,

(C) $\quad k \in B_n \Rightarrow a_{ik}(n) \le 0$ for all $i \ne k$,

(D) $\quad \sum_{i=1}^{m} a_{ik}(n) = 0$ for all k,

(E) $\quad k \in A_n \Rightarrow \sum_{i \in A_n} a_{ik}(n) \le 0,$

(F) $\quad k \in B_n \Rightarrow \sum_{i \in B_n} a_{ik}(n) \ge 0,$

(G) $\quad \Delta[x_i(n) - y_i(n)] = \sum_{i \in A_n} a_{ik}(n) + \sum_{i \in B_n} a_{ik}(n)$ for all i,

(H) $\quad \sum_{i,k \in A_n} a_{ik}(n) \le 0$ and $\sum_{i,k \in B_n} a_{ik}(n) \ge 0$

and

(I) $\quad \sum_{i \in A_n, k \in B_n} a_{ik}(n) \le 0$ and $\sum_{i \in B_n, k \in A_n} a_{ik}(n) \ge 0.$

Proof. (A) follows immediately from the definition of C_n. (B) and (C) are direct consequences of $(H2)$. (D) follows from $(H3)$. If $k \in A_n$ then (B) implies that

$\sum_{i\in B_n} a_{ik}(n)+\sum_{i\in C_n} a_{ik}(n) \geq 0$. Then, statement (E) follows from (D). Statement (F) is proved similarly. It is easily seen that $\Delta[x_i(n)-y_i(n)] = \sum_{k=1}^m a_{ik}(n)$ for all i. Then, statement (G) follows from (A). Statement (H) follows immediately from (E) and (F). Finally since A_n and B_n are disjoint, statement (I) follows from (B) and (C).

Lemma 3. We have $\Delta|x(n)-y(n)| \leq 0$ on I.

Proof. From Lemma 2 $((A),(G),(H),(I))$, we have

$$\begin{aligned}
&\Delta|x(n)-y(n)| = \sum_{i=1}^m \Delta|x_i(n)-y_i(n)| \\
&= \sum_{i\in A_n} \Delta[x_i(n)-y_i(n)] - \sum_{i\in B_n} \Delta[x_i(n)-y_i(n)] \\
&= \sum_{i\in A_n} [\sum_{k\in A_n} a_{ik}(n) + \sum_{k\in B_n} a_{ik}(n)] - \sum_{i\in B_n} [\sum_{k\in A_n} a_{ik}(n) + \sum_{k\in B_n} a_{ik}(n)] \leq 0.
\end{aligned}$$

Lemma 4. Assume that we have $\Delta|x(n)-y(n)| = 0$ on I. Then the following statements are valid:

$$(A^*) \qquad \sum_{i,k\in A_n} a_{ik}(n) = 0 \text{ and } \sum_{i,k\in B_n} a_{ik}(n) = 0,$$

$$(B^*) \qquad \sum_{i\in A_n, k\in B_n} a_{ik}(n) = 0 \text{ and } \sum_{i\in B_n, k\in A_n} a_{ik}(n) = 0,$$

$$(C^*) \qquad i \in A_n, \text{ and } k \in B_n \Rightarrow a_{ik}(n) = a_{ki}(n) = 0,$$

$$(D^*) \qquad i \in A_n \Rightarrow \Delta[x_i(n)-y_i(n)] = \sum_{k\in A_n} a_{ik}(n) \geq a_{ii},$$

$$(E^*) \qquad i \in B_n \Rightarrow \Delta[x_i(n)-y_i(n)] = \sum_{k\in B_n} a_{ik}(n) \leq a_{ii}$$

and

$$(F^*) \qquad \sum_{i\in A_n} \Delta[x_i(n)-y_i(n)] = 0 \text{ and } \sum_{i\in B_n} \Delta[x_i(n)-y_i(n)] = 0.$$

Proof. We can use to refer to the corresponding statements of Lemma 2 from $(A),(B)$ and etc. In the proof of Lemma 3, it was show that $\Delta|x(n)-y(n)|$ can be written as the sum of four non-positive terms, namely
$\sum_{i,k\in A_n} a_{ik}(n)$, $-\sum_{i,k\in B_n} a_{ik}(n), \sum_{i\in A_n,k\in B_n} a_{ik}(n)$ and $-\sum_{i\in A_n,k\in B_n} a_{ik}(n)$. Since

$\Delta|x(n) - y(n)| = 0$ each of these term must be zero, which prove (A^*) and (B^*). Statement (C^*) follows from $(B^*), (B)$ and (C). Statement (D^*) follows from $(G), (C^*)$ and (B). Likewise statement (E^*) follows from $(G), (C^*)$ and (C). Finally statement (F^*) follows from $(A^*), (D^*)$ and (E^*).

Lemma 5. Assume that $\Delta|x(n)-y(n)| = 0$ on I. If $i \in A_{n_0}$ then $x_i(n)-y_i(n) > 0$ and $i \in A_n$ for all $n > n_0$. Likewise if $i \in B_{n_0}$, then $x_i(n) - y_i(n) < 0$ and $i \in B_n$ for all $n > n_0$.

Proof. We prove the statement concerning A_n. The argument for B_n is similar. if $i \in A_{n_0}$, then there is an $h_i > 0$ such that $x_i(n) > y_i(n)$ for $n_0 < n < n_0 + h_i$. Now define

$$n_1 = \sup\{n \in I \mid x_i(s) > y_i(s) \text{ for all } s, n_0 < s < n\}.$$

It will suffice to show that $n_1 \notin I$, then one has $x_i(n_1) = y_i(n_1)$ and $x_i(s)-y_i(s) > 0$ for $n_0 < s < n_1$. However from (D^*) in Lemma 4 and the hypothesis $(H1)$ one has $\Delta[x_i(n) - y_i(n)] \geq a_{ii}(n) \geq -L[x_i(n) - y_i(n)]$ for $n_0 < n < n_1$. The Gronwall inequality then implies that $[x_i(n) - y_i(n)] \geq (1 - L)^{n-s}[x_i(s) - y_i(s)]$ for all $n_0 \leq s < n$. if s is chosen so that $n_0 < s < n_0 + h_i$, then $[x_i(s) - y_i(s)] > 0$. Hence $[x_i(n) - y_i(n)] > 0$ for all $n > n_0$, which contradicts the fact that $x_i(n_1) = y_i(n_1)$.

Lemma 6. Assume that $\Delta|x(n) - y(n)| = 0$ on I. Pick $s, n \in I$ with $s \leq n$. Then, we have

$$A_s \subseteq A_n, \;\; B_s \subseteq B_n, \;\; A_n \subseteq P_s \text{ and } B_n \subseteq Q_s.$$

Proof. The inequalities $A_s \subseteq A_n$ and $B_s \subseteq B_n$ follow from Lemma 5. If $i \notin P_s$, then $x_i(s) < y_i(s)$ and $i \in B_s$. Consequently, we have $i \in B_n$ by Lemma 5. Hence $i \notin A_n$ since A_n and B_n are disjoint. In order words, we have $A_n \subseteq P_s$. The proof that $B_n \subseteq Q_s$ is similar.

Remark 2. We can prove some other relationship under the assumption that $\Delta|x(n) - y(n)| = 0$ on I. Specifically the following statements are valid:

$$(A_*) \qquad k \in A_n \Rightarrow \sum_{i \in A_n} a_{ik}(n) = 0,$$

$$(B_*) \qquad k \in B_n \Rightarrow \sum_{i \in B_n} a_{ik}(n) = 0$$

and

$$(C_*) \qquad i \in C_n \text{ and } k \in A_n \cup B_n \Rightarrow a_{ik}(n) = 0.$$

It is also possible to show that $\Delta|x(n)-y(n)|=0$ on I if and only if statement (C^*) of Lemma 4 is valid on I.

The main object of this paper is to prove the following result:

Theorem 2. Let equation (4) be an almost periodic gross-substitute system. If there exists a positively compact solution $x(n)$ of (4), then there exists an almost periodic solution $\phi(n)$ of (4) that satisfies

$$|x(n)-\phi(n)| \to 0, \text{ as } n \to +\infty,$$

that is, $x(n)$ is positively almost periodic solution of (4).

Proof. Let $\phi(n)$ be a positively compact solution of (4). If follows from Lemma 1 and 3 that $\phi(n)$ is US w.r.t.K. Then by Theorem 1, $\Omega(\phi, f)$ is a nonempty compact set. Therefore every section $\Omega(\phi)=\{\psi \in P|(\psi, g) \in \Omega(\phi, f)\}$ is a nonempty compact set in P. We can now show that the section $\Omega(\phi)$ contains a single point, that is x. It will then follows from Theorem 1 that the solution $\phi(n)$ is almost periodic in n. Choose $x \in \Omega(\phi)$. Define $U: \Omega(\phi) \to R$ and $V: \Omega(\phi) \to R$ by

$$U(y)=\sum_{i=1}^{m} \max\left(x_i-y_i, 0\right)$$
and
$$V(y)=\sum_{i=1}^{m} \min\left(x_i-y_i, 0\right).$$

U and V are continuous functions defined on $\Omega(\phi)$. Furthermore, we have $V(y) \leq 0 \leq U(y)$ for all $y \in \Omega(\phi)$. We can use the following fact:

Lemma 7. The set $\Omega(\phi)$ contains the single point x if and only if we have $U(y)=V(y)=0$ for all $y \in \Omega(\phi)$.

Since U and V are continuous functions on a compact set, they assume their maximum and minimum values on $\Omega(\phi)$. Thus there are values $y, z \in \Omega(\phi)$ such that

$$(i)\ \ 0 \leq U(r) \leq U(y)$$
and
$$(ii)\ \ V(z) \leq V(r) \leq 0,$$

for all $r \in \Omega(\phi)$. Let $U_0 = U(y)$. We can now show that $U_0 = 0$, by contradiction. A similar argument shows that $V(z) = 0$. Then by Lemma 7, we have $\Omega(\phi) = \{x\}$. Let $x(n) = \phi(n)$ through x at $n = 0$ and $y(n) = \phi(n)$ through y at $n = 0$ be the corresponding solutions of (4). Since both $x(n)$ and $y(n)$ remain in a compact set K in P for all n, they are defined for all $n \in \mathbf{Z}$. Now define the corresponding five sets P_n, Q_n, A_n, B_n and C_n as well as the terms $a_{ik}(n)$, $1 \le i, k \le m$. we can now assume the validity of the following;

Lemma 8. $\Delta|x(n) - y(n)| = 0$ on Z.

Proof. To do this, perhaps, the simplest argument is based on the fact that system (4) has a separation condition. For general theory, it then follows from Ellis' Theorem [4], and this implies that the solutions $x(n_k) \to x$ and $y(n_k) \to y$. Now if $\Delta|x(n) - y(n)| \not\equiv 0$ it follows from Lemma 1 that there is a $v \in Z, v > 0$ such that $|x(n) - y(n)| \le |x(v) - y(v)| < |x(0) - y(0)|, n \ge v$. Now choose $n_k \to +\infty$ so that $x(n_k) \to x(0)$ and $y(n_k) \to y(0)$. Then we have the contradiction $|x(0) - y(0)| = \lim_{k\to\infty} |x(n_k) - y(n_k)| < |x(0) - y(0)|$. If $v \le 0$, we simply repeats the above argument with a suitable translate of $x(n)$ and $y(n)$.

Since A_n is monotone in n by Lemma 6, it follows that there is a set $A \subseteq \{i|1 \le i \le m\}$ and an $n_0 \ge 0$ such that $A_n = A$ for all $n \ge n_0$. It also follows from Lemma 6 that $A \subseteq P_n$ for all $n \in \mathbf{Z}$.
We set $w(n) = \sum_{i\in A_n}[x_i(n) - y_i(n)]$. Then $\Delta w(n) = 0$ by (F^*) in Lemma 4. Hence $w(n) = w(0)$ for all $n \ge 0$. Since $\{i|x_i > y_i\} \subseteq A_0$ it follows from our choice of y that $w(0) = U_0$. Now choose a sequence $n_k \to +\infty$ such that $x(n_k) \to y$ and $y(n_k) \to r$, where $r \in \Omega(\phi)$. Since $x_i(n_k) - y_i(n_k) > 0$ for $i \in A$ by Lemma 5, it follows that $y_i - r_i \ge 0$ for all $i \in A$. Since $A \subseteq P_0$ at $s = 0$ by Lemma 6, it follows that $x_i - y_i \ge 0$ for all $i \in A$. Since $w(n_k) = w(0) = U_0$, it follows that

$$\sum_{i\in A}(y_i - r_i) = U_0. \tag{4.1}$$

Next since $A_0 \subseteq A \subseteq P_0$ we have

$$\sum_{i\in A}(x_i - y_i) = U_0. \tag{4.2}$$

By adding (11) and (12) together we have $\sum_{i\in A}(x_i - r_i) = 2U_0$. However $x_i - r_i \ge 0$ for $i \in A$. Therefore, we have $2U_0 = \sum_{i\in A}(x_i - r_i) \le U(r) \le U(y) = U_0$, which is impossible if $U_0 > 0$. Hence $U_0 = 0$.
It then follows from Theorem 1 that $\phi(n)$ is almost periodic solution of (4). In order

to show that $|\hat{\phi}(n) - \phi(n)| \to 0$ as $n \to +\infty$, we can use Lemma 3. Since we have $\Delta|\hat{\phi}(n) - \phi(n)| \leq 0$ for all $n \geq 0$, define β by

$$\beta = \lim_{n \to +\infty} |\hat{\phi}(n) - \phi(n)|.$$

Now choose a sequence $n_k \to +\infty$ so that $\phi(n_k) \to r$ as $k \to +\infty$. Since $f_{n_k} \to f$ it follows that $r \in \Omega(\phi)$ and consequently $r = x$. Consequently, we have $\beta = 0$. Then, this completes the proof of theorem.

Remark 3. Since $\Omega(\phi)$ is a 1-fold covering of the base space $H(\phi)$, it can easily be shown that the frequency module of the almost periodic solution $\phi(n)$ of (4) is contained in the frequency module of f (cf. [5, 13]). We will omit these details since they are similar results in the continuous case using the standard arguments of differential equations.

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