Libertas Mathematica (new series)
Volume 38(2018), No. 2, 157-174

Topics in Functional Differential Equations

Mehran Mahdavi

Abstract: In this paper we provide a concise presentation of the book "Functional Differential Equations: Advances and Applications" by Constantin Corduneanu, Yizeng Li, and Mehran Mahdavi, a research monograph, published by Wiley [7]. The book contains five chapters, an Appendix, and a bibliography which includes more than five hundred fifty references. In each chapter, except the first one, there is a section, bibliographical notes, in which numerous references and their relationships to our work are provided. The book presents only part of the results available in the literature, mainly mathematical ones, without any claim related to the coverage of the whole field of functional differential equations or functional equations. The book also includes many applications of the results.

Keywords: Functional differential equations, solutions, existence, uniqueness, stability, oscillatory motion, almost periodicity, neutral equations, Fourier Analysis

MSC2010: 34K05, 34K13, 34K20, 34K25, 34K40, 45B05

1 Introduction and Brief History

A functional differential equation or a functional equation is a relationship which contains an unknown element, usually a function, which has to be determined, or at least partially identifiable by some of its properties. Solving a functional differential equation means finding a *solution*, that is, the unknown element in the relationship. Sometimes one finds several solutions (solution set), while in other cases the equation may not have a solution, particularly when one provides the class/space to which it should belong.

Functional differential equations are classified as classical and non-classical types. The classical types include the ordinary differential equations, the integral equations of Volterra or Fredholm and the integro-differential equations. The classical types have been thoroughly investigated since Newton's times. The non-classical types include equations involving operators and delay type (finite or infinite) differential equations.

Three pioneers of the non-classical theory of functional differential equations are: A. D. Myshkis, N. N. Krasovskii, and J. K. Hale. Myshkis is the author of the first book, entirely dedicated to functional differential equations, in the category of delay type (finite or infinite), see Myshkis [15]. Krasovskii introduced the method of *Liapunov functionals* (not just functions!), which permitted a true advancement in the theory of functional differential equations, especially in the nonlinear case and stability problems. See Krasovskii [11]; this book is an English translation of the 1959 Russian edition. Hale created a vast theory in the study of functional differential equations, by constantly using the modern tools of Functional Analysis, both linear and nonlinear. See Hale [9], in this book Hale used, for the first time, the theory of semigroups of linear operators on a Banach function space. This approach allowed Hale to develop a theory of linear systems with finite delay, in the time-invariant framework, dealing with adequate concepts that naturally generalize those of ordinary differential equations with constant coefficients (*e.g.*, characteristic values of the system/equation).

In the field of applications of functional differential equations, the book by Kolmanovskii and Myshkis [12] illustrates a great number of applications to science (including biology), engineering, business/economics, environmental sciences, and medicine, including the stochastic factors.

The classical categories are related to the use of the so-called Niemytskii operator, defined by the formula $(Fu)(t) = f(t, u(t))$, with $t \in R$ or in an interval of R, while in the case of functional differential equation, the right-hand side of the equation

$$\dot{x}(t) = (Fx)(t)$$

implies a more general type of operator F. For instance, using Hale's notation, one can take $(Fx)(t) = f(t, x(t), x_t)$, where $x_t(s) = x(t+s)$, $-h \leq s \leq 0$ represents a restriction of the function $x(t)$, to the interval $[t-h, t]$. This is the finite delay case. Another choice is

$$(Fx)(t) = (Vx)(t),\ t \in [t_0, T],$$

where V represents an *abstract Volterra operator*, also known as *causal operator*, which means that $(Vx)(t)$, $t \in (t_0, t_0 + a]$, $t_0 \in R$, $a > 0$, is determined by the values of x on the interval $[t_0, t]$.

The formal definition of the causality property of an operator is the following: $V : C([t_0, t_0 + a], R^n) \to C([t_0, t_0 + a], R^n)$ will be called *causal* if for each $s \in (t_0, t_0 + a]$, from $x(t) = y(t)$ on $[t_0, s]$, one obtains $(Vx)(t) = (Vy)(t)$ on the same interval.

2 Overview of the First Chapter

The first chapter is introductory and includes the historical background of the subject, classification of various types of functional differential equations, and the description of mathematical tools we have used throughout the book. For instance, we have described various *function spaces* which were used in subsequent chapters. We mention two examples. The space $C([a,b],R^n)$ which denotes the Banach space of continuous maps from $[a,b]$ into R^n, with the norm

$$|x|_C = \sup |x(t)|;\ t \in [a,b],$$

where $|\cdot|$ is the Euclidean norm in R^n. This space is frequently encountered in problems related to functional differential equations, especially when we look for continuous solutions. The space of measurable functions/elements are of great importance in the development of modern analysis and have appeared at the beginning of the past century, primarily due to Lebesgue's discovery of measure theory. Actually, the first function spaces amply investigated in the literature are known as Lebesgue's spaces or L^p-spaces.

The space $L^p(R,R^n)$, $p \geq 1$, is the linear space of all measurable maps from R into R^n, such that $\int_R |x(s)|^p\, ds < \infty$, the ds representing the Lebesgue measure on R. The norm of this space is

$$|x|_{L^p} = \left\{ \int_R |x(s)|^p\, ds \right\}^{\frac{1}{p}},\ 1 \leq p < \infty.$$

The case $p = \infty$ is characterized by

$$|x|_{L^\infty} = ess - \sup_{t\in R} |x(t)| < \infty.$$

The theory of these Banach spaces, whose elements are, in fact, equivalence classes of functions (*i.e.*, two functions are equivalent, if and only if they coincide, except on set of points of Lebesgue measure zero) is largely diffused in many books/textbooks available.

3 Overview of the Second Chapter

In this chapter we provided various results on the local existence of solutions for functional differential equations within the classes of continuous or measurable functions (real-valued or with values in R^n). The problem of global existence (*i.e.*, on a preassigned interval) was also discussed.

The comparison method was used to obtain global existence results. The comparison method, in the study of ordinary differential equations, has been used extensively in the theory of existence, stability, and almost all chapters of this field. Later on, it also served in the investigation of various classes of functional differential equations.

The comparison method, in the framework adopted by Conti [1], (see also Sansone and Conti [18]), was built up by the systematic use, under *the most general conditions* imposed on the equation and auxiliary items, of two concepts: the Liapunov's function/functional and the differential inequalities.

We also dealt with the existence of solutions in spaces of measurable functions like Lebesgue spaces L^p, $1 \leq p \leq \infty$.

Our approach consisted of applying general existence results, in most cases obtained by fixed-point methods, to various classes of functional equations (ordinary differential equations, integral equations in a single variable, integro-differential equations, finite or infinite delay equations, and equations involving operators acting on function spaces, such as causal operators).

Some results were concerned with the existence of solutions with special properties such as positiveness, or constant sign, boundedness on unbounded intervals, asymptotic behavior, belonging to various function spaces, uniqueness, and dependence on data.

In this chapter we also provided some applications. For instance, there is an application to optimal control theory based on the existence of a unique element with minimal norm in each closed convex set in a Banach space.

We now state a global existence result which was obtained by means of the Tychonoff fixed point theorem and was taken from Corduneanu [2]. The statement and proof of this theorem are given in the mentioned reference.

Theorem 3.1. *Consider the functional differential equation*

$$\dot{x}(t) = (Vx)(t),\ t \in [0,T), \tag{3.1}$$

with the initial condition $x(0) = x^0 \in R^n$, under the following assumptions:

(i) *V is a causal, continuous operator from $C([0,T), R^n)$ into $L^1_{loc}([0,T), R^n)$.*

(ii) *There exist real-valued functions $A(t)$ and $B(t)$, defined on $[0,T)$, with $A(t)$ continuous and positive and $B(t)$ locally integrable, such that*

$$\int_0^t B(s)\,ds \leq A(t) - A(0),\ t \in [0,T), \tag{3.2}$$

while $x(t) \in C([0,T), R^n)$ and $|x(t)| \leq A(t)$ imply

$$|(Vx)(t)| \leq B(t),\ a.e.\ on\ [0,T). \tag{3.3}$$

Then, there exists a solution $x(t) \in C([0,T), R^n)$ of our problem, provided $|x^0| \le A(0)$. This solution will satisfy the estimate $|x(t)| \le A(t)$, $t \in [0,T)$.

We now consider an illustration of theorem 3.1, to the case of linear functional differential equations of the form

$$\dot{x}(t) = (Lx)(t) + f(t),\ t \in [0,T), \tag{3.4}$$

under the following assumptions:

(1) $L : C([0,T), R^n) \to C([0,T), R^n)$ is a linear, causal and continuous operator.

(2) $f(t) \in L^1_{loc}([0,T), R^n)$.

We will now prove the existence, on the whole interval $[0,T)$, of a solution to (3.4) by checking the validity of the assumptions of theorem 3.1.

Since assumption (1) in the theorem is obviously satisfied, there remains to construct the functions $A(t)$ and $B(t)$.

Since L is a continuous operator, we can write this condition in the form $|Lx|_{L^1} \le \alpha(t)\,|x|_C$, $t \in [0,T)$, with $\alpha(t) > 0$ a nondecreasing function on $[0,T)$. Hence, with $B(t) = \alpha(t)\,A(t) + |f(t)|$, one should have

$$\int_0^t [\alpha(s)\,A(s) + |f(s)|]\,ds \le A(t) - A(0),\ t \in [0,T],$$

choosing $x^0 = \theta \in R^n$, the null element of R^n. The general case, with $|x^0| < A(0)$, does not present any difficulty. This inequality can be rewritten as follows:

$$A(t) \ge A(0) + \int_0^t |f(s)|\,ds + \int_0^t \alpha(s)\,A(s)\,ds. \tag{3.5}$$

Inequality (3.5) will be the consequence of the similar inequality of the form

$$A(t) \ge c(t) + \int_0^t \alpha(s)\,A(s)\,ds,\ t \in [0,T), \tag{3.6}$$

with $c(t) = A(0) + \int_0^t |f(s)|\,ds > 0$. Inequality (3.6) can be reduced to an equivalent form by the substitution

$$y(t) = \int_0^t \alpha(s)\,A(s)\,ds,\ t \in [0,T). \tag{3.7}$$

One obtains $\dot{y}(t) = \alpha(t)\,A(t)$, $t \in [0,T)$, which leads, together with (3.6) to the inequality

$$\dot{y}(t) \ge \alpha(t)\,y(t) + \alpha(t)\,c(t),\ t \in [0,T). \tag{3.8}$$

Since $y(0) = 0$, according to (3.7), we observe that the solution of the equation $\dot{y}(t) = \alpha(t)\, y(t) + \alpha(t)\, c(t)$, with $y(0) = 0$, given by

$$y(t) = e^{\int_0^t \alpha(s)\, ds} \int_0^t e^{-\int_0^t \alpha(u)\, du} \alpha(s)\, c(s)\, ds,$$

is positive on $[0, T)$. Thus, according to (3.8), $\dot{y}(t)$ is positive on $[0, T)$ because both terms on the right-hand side are positive. Since (3.7) implies $A(t) = \frac{\dot{y}(t)}{\alpha(t)} > 0$ on $[0, T)$, the constructed functions $A(t)$ and $B(t)$, $t \in [0, T)$, satisfy all requirements of theorem 3.1.

This conclusion ends the discussion of equation (3.4), having proven the global existence of solutions.

Mahdavi [13], [14] discussed the basics of linear and neutral functional differential equations of causal type. Corduneanu and Mahdavi [4], [5], [6] proved existence results for neutral functional equations with causal operators.

4 Overview of the Third Chapter

Third chapter, the most extended, deals with problems of stability, particularly for ordinary differential equations (in which case the theory is the most advanced and can provide models for other classes of functional differential equations). In this chapter, the interest is of concern not only to mathematician, but also to other scientists, engineers, economists, and others deeply engaged in related applied fields. The concept of *partial stability* is also covered in this chapter.

The concept of partial stability, also called by many authors stability with respect to part of the variables, emerged in the work of Liapunov in the nineteenth century, but it received special attention only after the 1950s. See Rumiantsev [17], for instance, which was mostly motivated by the mechanical interpretations of stability. There are many engineering-oriented publications from which one can easily realize that, in investigating stability of such systems, there are only part of the parameters involved, presenting significance for the whole systems.

Intuitively, the stability property of a solution to a functional differential equation means the continuous dependence of the solutions, in the neighborhood of the given one, with respect to the perturbations (initial or permanent). Unlike the case when the dependence is considered on a finite interval (which is a simpler problem), the *stability* involves the dependence on a whole semi-axis, which is a feature that is conducing to more complex problems.

For general functional differential equations, unlike ordinary differential equations, we do not possess stability results with the same degree of generality.

Consider the functional differential equation

$$\dot{x}(t) = (Vx)(t),\ t \in R_+ \tag{4.1}$$

for which we provided global existence in section 3. Now we make a rather general assumption, namely,

(a) $V : L_{loc}(R_+, R^n) \to L_{loc}(R_+, R^n)$

satisfies the generalized Lipschitz condition

$$\int_0^t |(Vx)(s) - (Vy)(s)|\, ds \leq A(t) \int_0^t |x(s) - y(s)|\, ds,\ t \in R_+. \tag{4.2}$$

It is obvious that (4.2) implies the continuity of V on $L_{loc}(R_+, R^n)$, as well as its *causality.*

The initial condition could be the Cauchy condition

$$x(0) = x^0 \in R^n, \tag{4.3}$$

or the functional type initial condition

$$x(t) = \begin{cases} \phi(t), & t \in [0, t_0),\ \ t_0 > 0, \\ x^0, & t = t_0. \end{cases} \tag{4.4}$$

The necessity of assigning the value of $x(t)$ at $t = t_0$ stems from the fact that, in case $\phi(t)$ is only measurable, a situation might arise in which $\phi(t)$ is known only almost everywhere on $[0, t_0]$. Hence, by assigning $x(t_0) = x^0$, we deal with precise data. The solution of (4.1), verifying (4.4), will be denoted by $x(t; t_0, \phi, x^0)$, and its existence and uniqueness is assured by the condition (a) formulated above for the operator V in (4.1), under initial condition (4.4), with

$$\phi \in L^1([0, t_0], R^n). \tag{4.5}$$

For stability of the solution $x = \theta$ of the system (4.1), we assume $(V\theta)(t) = \theta$, $t \in R_+$. The following definitions will be used:

Stability: For each $\epsilon > 0$ and $t_0 > 0$, there exists $\delta = \delta(\epsilon, t_0) > 0$, such that

$$|x^0| < \delta \quad and \quad \int_0^b |\phi(s)|\, ds < \delta, \tag{4.6}$$

imply

$$|x(t; t_0, \phi, x^0)| < \epsilon \ \ for \ \ t \geq t_0.$$

Asymptotic Stability: This means stability as defined before and, for each $t_0 > 0$, there exists $\eta(t_0) > 0$, such that

$$|x(t; t_0, \phi, x^0)| \to 0 \ \ as \ \ t \to \infty, \tag{4.7}$$

provided

$$|x^0| < \eta \ , \quad \int_0^{t_0} |\phi(s)| \, ds < \eta. \tag{4.8}$$

Uniform Stability: Stability as before and the independence of $\delta(\epsilon, t_0)$ in $t_0 > 0$; that is, $\delta(\epsilon, t_0) \equiv \delta(\epsilon)$.

Uniform Asymptotic Stability: Uniform stability, as defined earlier, and existence of $\delta_0 > 0$ and $T(\epsilon) > 0$ for $\epsilon > 0$, such that

$$|x^0| < \delta_0 \ \ and \ \ \int_0^{t_0} |\phi(s)| \, ds < \delta_0, \tag{4.9}$$

imply

$$|x(t; t_0, \phi, x^0)| < \epsilon \ \ for \ \ t \geq t_0 + T(\epsilon). \tag{4.10}$$

Let us mention that similar definitions can be formulated for other stability concepts, such as exponential asymptotic stability or stability under permanent perturbations (in such cases, of the operator V).

Before we formulate a result on stability for the solution $x(t) = \theta$, $t \geq 0$, of system (4.1), we need to define a property for functionals. Namely, the property of positive definiteness could be obtained if the Liapunov's functional $W : L_{loc}(R_+, R^n) \to R_+$ satisfies the following property:

(A) Let us consider the functional $W : L_{loc}(R_+, R^n) \to R_+$, with $W(t, \theta) = 0$, $t \in R$. We say that W has property A, if for each $\epsilon > 0$ and $t_0 > 0$, there exists $\delta = \delta(\epsilon, t_0) > 0$ such that

$$(Wx)(t) < \delta \ \ for \ \ t \geq t_0, \tag{4.11}$$

implies

$$|x(t)| < \epsilon \ \ for \ \ t \geq t_0. \tag{4.12}$$

We now state a stability result for the system (4.1), using the comparison method. The proof of this theorem is in our book, Corduneanu, Li, Mahdavi [7]. We mention the fact that the comparison method primarily relies on the theory of differential inequalities, a branch of investigation that appeared later in the theory of differential equations.

Theorem 4.1. *Consider system (4.1), with initial data (4.4) and V satisfying* (a). *Assume the following conditions are satisfied:*

(i)) There exists a Fréchet differentiable functional $W : L_{loc}(R_+, R^n) \to R_+$, *such that* $(W)(\theta) = 0$ *on* R_+.

(ii) There exists a function $\alpha : R^2 \to R_+$, *continuous and such that* $\alpha(0,0) = 0$, *and*

$$(Wx)(t_0) \leq k(t_0)\,\alpha(|x^0|, |\phi|_L), \tag{4.13}$$

with $k(t_0) > 0$, *along the solution* $x(t; t_0, \phi, x^0)$.

(iii) The comparison inequality

$$(W'x)(t)[(Vx)(t)] \leq \omega(t, (Wx)(t)),\ t \in R_+ \tag{4.14}$$

or on the largest interval of existence of $x(t)$.

(iv) The comparison equation

$$\dot{y}(t) = \omega(t, y(t)),\ t \geq 0, \tag{4.15}$$

enjoys the properties of existence and uniqueness through each point (t, y_0), $t_0 \geq 0$, $0 \leq y_0 < k_0$, $k_0 > 0$ *being a constant (sufficiently large).*

Then, the solution $x = \theta$ *of (4.1) is stable (asymptotically stable) if the solution* $y = 0$ *of (4.15) is stable (asymptotically stable).*

5 Overview of the Fourth Chapter

An important property of motion, encountered in nature and man-made systems, is known as oscillation or oscillatory motion. Historically, the periodic oscillations (of a pendulum, for instance) have been investigated by mathematicians and physicists.

Gradually, more complicated oscillatory motions have been observed, leading to the apparition of *almost periodic* oscillations/vibrations. In the third decade of the twentieth century, Harald Bohr (1887-1951) constructed a wider class than the periodic one, called *almost periodic*.

In the last decade of the twentieth century, motivated by the needs of researchers in applied fields, even more complex *oscillatory motions* have emerged. In the books by Osipov [16] and Zhang [21], [22], new spaces of oscillatory functions/motions have been constructed and their applications illustrated.

Chapter four deals with oscillatory properties, especially of almost periodic type (which includes periodic). The choice of spaces of almost periodic functions are not, as usual, the classical Bohr type, but a class of spaces forming a scale, starting with the simplest space (Poincaré) of those almost periodic functions whose Fourier series are absolutely convergent and finishes with the space of Besicovitch, the richest one known, for which we have enough meaningful tools.

The oscillatory solutions (*e.g.*, periodic and almost periodic) constitute a wide preoccupation of researchers, and several monographs have beeb dedicated to the subject. Recently, relatively new classes of almost periodic solutions have made their way into the literature (see Shubin [19], [20], Corduneanu [3], and others). For most of these classes, the series approach can be applied, even in nonlinear cases.

In this chapter, we define the AP_r-almost periodic functions and establish basic properties (the case of the function defined on R). Of course, we have in mind applications to various classes of functional equations, namely ordinary differential equations, integral equations, and convolution equations. The convolution extends from the classical cases, to functions in AP_r-almost periodic spaces. This chapter also provides several examples of functional differential or integro-differential equations, with regard to the existence of AP_r-almost periodic solutions, solutions in Besicovitch spaces of almost periodic functions, and also in the classical case (Bohr).

The oscillatory motion is encountered often in various applied fields. One of the most generally known models is that of the motion of a simple/mathematical pendulum. This motion is fully described by the equation

$$x(t) = A\,\sin(\omega\,t + \delta), \tag{5.1}$$

where $A > 0$ is known as the amplitude of oscillation/motion, $\frac{\omega}{2\pi}$ represents the frequency, and δ describes the phase displacement.

Taking into account Euler's formula $e^{i\,\lambda\,t} = \cos\lambda\,t + i\,\sin\lambda\,t$, $\lambda, t \in R$, the complex form of the equation (5.1) will be

$$x(t) = A_1\,e^{i\,\lambda\,t} + A_2\,e^{-i\,\lambda\,t}, \tag{5.2}$$

where A_1 and A_2 are complex numbers. It is obvious that the right-hand side of (5.2) represents a special case of a complex-valued function of the form

$$T(t) = A_1\,e^{i\,\lambda_1\,t} + \cdots + A_k\,e^{i\,\lambda_k\,t}, \tag{5.3}$$

with the $A_j \in \mathcal{C}$ and $\lambda_j \in R$, $j = 1, 2, \ldots, k$, with $\lambda_j \neq \lambda_k$ for $j \neq k$.

A function of the form (5.3), for any $k \in N$, is called a *trigonometric polynomial.* Now we describe how we construct the spaces of *almost periodic functions.* The set of all trigonometric polynomials will be denoted by $\mathcal{T}$. It is easy to check that $\mathcal{T}$ is an algebra over the complex field $\mathcal{C}$.

Starting from $\mathcal{T}$ we will construct a family of Banach function spaces by using different norms on this linear space. We shall consider the following type of norms on $\mathcal{T}$:

$$\|T\|_r = \left(\sum_{j=1}^{k} |A_j|^r \right)^{\frac{1}{r}}, \ 1 \leq r \leq 2. \tag{5.4}$$

It can be easily shown that $\|T\|_r$ is a norm on the linear space $\mathcal{T}$. The normed space obtained by endowing $\mathcal{T}$ with the norm given by (5.4) will be denoted by $\mathcal{T}_r$. One can check that $\mathcal{T}_r$ is not a Banach space (normed and complete), which raises difficulties when we want to operate in such a space. By completing these normed spaces we obtain Banach function spaces. This will be our approach in constructing spaces of almost periodic functions.

The Banach function space obtained by the completion of $\mathcal{T}_r$ will be denoted by $AP_r(R,\mathcal{C})$, $1 \leq r \leq 2$. The norm on $AP_r(R,\mathcal{C})$, generated by (5.4), is defined by the following formula:

$$|f|_r = \left(\sum_{k=1}^{\infty} |A_k|^r \right)^{\frac{1}{r}}, \ 1 \leq r \leq 2. \tag{5.5}$$

The connection between f and its norm must be understood in the following sense: the formal series

$$A_1 \, e^{i\,\lambda_1\,t} + A_2 \, e^{i\,\lambda_2\,t} + \cdots + A_n \, e^{i\,\lambda_n\,t} + \cdots , \tag{5.6}$$

is such that its partial sums

$$T_n(t) = \sum_{k=1}^{n} A_k \, e^{i\,\lambda_k\,t}, \tag{5.7}$$

provides a sequence which converges to f in the norm $|\cdot|_r$:

$$|f(t) - T_n(t)|_r \to 0 \ \ as \ \ n \to \infty. \tag{5.8}$$

Since (5.5) makes sense only in case of convergence, we have

$$\sum_{k=n+1}^{\infty} |A_k|^r \to 0 \ \ as \ \ n \to \infty. \tag{5.9}$$

We see that the formal series (5.6), in general non-convergent in the usual sense (like pointwise), enjoys the property

$$\sum_{k=1}^{\infty} |A_k|^r \ \ converges. \tag{5.10}$$

It is useful to examine in some detail the case $r = 1$, which implies the fact that the numerical series

$$\sum_{k=1}^{\infty} |A_k| \ \textit{converges}. \tag{5.11}$$

The condition (5.11) implies the absolute and uniform convergence, on R, of the complex trigonometric series (5.6). Indeed, from the equalities

$$|A_k\, e^{i\,\lambda_k\,t}| = |A_k|,\ k \in N,\ t \in R, \tag{5.12}$$

we obtain the equality

$$f(t) = \sum_{k=1}^{\infty} A_k\, e^{i\,\lambda_k\,t},\ t \in R, \tag{5.13}$$

which allows us to get the coefficients A_k in terms of the function $f(t)$. Indeed, if we multiply both sides of (5.13) by $e^{-i\,\lambda_j\,t}$ and take into account the formula

$$\lim_{\ell\to\infty} (2\,\ell)^{-1} \int_{-\ell}^{\ell} e^{i\,\lambda\,t}\, dt = \begin{cases} 0 & \lambda \neq 0, \\ 1 & \lambda = 0, \end{cases} \tag{5.14}$$

then integrating both sides of the equality, we obtain the following:

$$\lim_{\ell\to\infty} (2\,\ell)^{-1} \int_{-\ell}^{\ell} f(t)\, e^{-i\,\lambda_j\,t}\, dt = A_j,\ j \in N. \tag{5.15}$$

Hence, the series in (5.13) is the Fourier (generalized !) series of $f(t) \in AP_r(R, \mathcal{C})$, which is a space of *almost periodic functions*, apparently used for the first time by H. Poincaré in his treatise *Nouvelles Methodes de la Mécanique Celeste*.

Poincaré's space, $AP_1(R, \mathcal{C})$, has been used by authors in applications to the oscillatory behavior of solutions to some classes of functional differential equations. It is a convenient feature to obtain the solution as an absolutely and uniformly (on R) convergent Fourier series.

6 Overview of the Fifth Chapter

The fifth chapter contains results of any nature, available for the so-called *neutral equations*. There are several types of functional differential equations belonging to this class, which can be roughly defined as the class of those equations that are not solved with respect to the highest-order derivative involved.

For those types of equations, this book proves existence results, some kinds of behavior (*e.g.*, boundedness), and stability of the solutions (especially asymptotic stability).

The investigation of neutral functional differential equations, both with continuous and discrete argument, has been developed rapidly throughout the past 50-60 years, and is sometimes comparable with the case of ordinary functional equations.

Neutral functional equations display a large variety of types and have been encountered in mathematical research long time ago. For instance, the very simple system $\dot{x} = y - z$, $\dot{y} = z - x$, $\dot{z} = x - y$, which is in normal form (solved with respect to the derivatives), leads to the integral combination $x\,\dot{x} + y\,\dot{y} + z\,\dot{z} = 0$, or $\frac{d}{dt}(x^2 + y^2 + z^2) = 0$, from which we get the first integral $x^2 + y^2 + z^2 = c$, $c \geq 0$; but the system that resulted from above, namely $\dot{x} = y - z$, $\dot{y} = z - x$, $(x^2 + y^2 + z^2)^{\cdot} = 0$ will be termed, necessarily, as a *neutral* system of ordinary differential equations.

When investigating problems by means of functional differential equations in applied areas, it is possible that a combination of unknown functions could be easier observed and its properties investigated. This would certainly lead to a neutral equation/system. A common example is from mechanics, when the energy of a system is investigated.

The material in chapter five covers only a *limited* number of neutral systems of functional equations.

Neutral functional differential equations are of the form

$$\frac{d}{dt}(Vx)(t) = (Wx)(t) \tag{6.1}$$

where V and W are causal operators acting on various function spaces whose elements are continuous maps from R (or an interval $I \in R$), or belong to some measurable function spaces. When we deal with spaces of measurable functions, it is understood that the solutions are meant in the Carathéodory sense.

By choosing the opeartor V in different manners, we obtain many types of neutral equations, including the "normal" ones

$$\frac{dx}{dt} = (Wx)(t), \tag{6.2}$$

which corresponds to $V = I$. Other choices encountered in the literature are

$$(Vx)(t) = x(t) + c\,x(t-h),$$

$$(Vx)(t) = x(t) + \sum_{k=1}^{m} c_k\,x(t-h_k),$$

$$(Vx)(t) = x(t) + (\nu_0\,x)(t),$$

$$(Vx)(t) = x(t) + g(t, x_t), \tag{6.3}$$

where x_t stands for the function $x_t(u) = x(t+u)$, $u \in [-h, 0)$. The last case was introduced by Hale [10], and has since diffused into the mathematical literature.

In this chapter, we have obtained existence results for certain first-order and second-order neutral equations. To obtain existence results for second-order equations, we reduced them to first-order equations. We now state a result (see our book [7] for its proof).

Theorem 6.1. *Consider the neutral functional differential equation*

$$\frac{d}{dt}[x(t) + (Vx)(t)] = (Wx)(t), \ t \in [0, T], \tag{6.4}$$

with the initial condition

$$x(0) = x_0, \tag{6.5}$$

under the following hypotheses:

(H_1) *The operators V and W are continuous causal operators on the space $C([0,T], R^n)$.*

(H_2) *V is compact and has the fixed initial value property*

$$(Vx)(0) = \theta \in R^n, \ x \in C([0,T], R^n). \tag{6.6}$$

(H_3) *The operator W takes bounded sets into bounded sets of $C([0,T], R^n)$.*

Then, there exists a solution of equation (6.4), satisfying the initial condition (6.5), defined on an interval $[0, a]$, with $a \le T$.

We state another result for a second-order causal neutral functional differential equation (see our book [7] for its proof).

Theorem 6.2. *Consider the second-order causal neutral functional differential equation*

$$\frac{d}{dt}[\dot{x}(t) + (L\dot{x})(t)] = (Vx)(t), \ t \in [0, T), \tag{6.7}$$

under the following assumptions:

(1) *L is a linear, causal, continuous and compact operator on the space $L^p([0,T], R^n)$, $1 < p < \infty$.*

(2) *V is acting on $L^p([0,T], R^n)$ and is causal, continuous, taking bounded sets into bounded sets.*

Then, there exists $\delta > 0$, $\delta \le T$, such that the equation (6.7), with initial conditions $x(0) = x_0 \in R^n$, $\dot{x}(0) = \nu_0 \in R^n$, has a solution $x(t) \in AC([0,\delta], R^n)$, satisfying a.e. on $[0,\delta]$ the equation, such that $\dot{x}(t) \in L^p([0,T], R^n)$, while $\dot{x}(t) + (L\dot{x})(t) \in AC([0,\delta], R^n)$.

7 Overview of the Appendix

The appendix introduces the readers to what is known about generalized Fourier series of the form

$$\sum_{k=1}^{\infty} a_k \, e^{i\,\lambda_k(t)}, \tag{7.1}$$

with $a_k \in \mathcal{C}$ and $\lambda_k(t)$ some real-valued function on R, these functions being at least locally integrable. Such series intervene in studying various applied problems and appears naturally to classify them as belonging to the third stage of development of the Fourier analysis (after periodicity and almost periodicity). The presentation is descriptive, less formal, and somewhat a survey of problems occurring in the construction of new spaces of oscillatory functions.

The Fourier Analysis is a vast field of knowledge with many connections in recent development of the mathematical theory of vibrations/oscillations and waves.

The *first stage*, amply illustrated by the work of Euler, Fourier, Riemann, Dini, Drichlet, Fejér and many other distinguished mathematicians of the past, is still in development and is related, mainly, to the investigation of cases involving *periodicity* of phenomena.

The *second stage* in this development of vibratory/oscillating processes is related to cases of *almost periodicity*, a concept due to H. Bohr and developed by many followers. The almost periodicity concept, obviously more comprehensive than that of periodicity, does not suffice in describing the vibratory or wave phenomena, which are neither periodic nor almost periodic.

The *third stage* of Fourier analysis has in view primarily, those phenomena of oscillation/vibration whose description involves series of the form (7.1) and the functions (possibly generalized) which are characterized by those type of series. The case in (7.1) when $\lambda_k(t)$ are linear functions in t leads to the series related to the almost periodic functions. The periodic case occurs only when $\lambda_k(t) = \lambda_k\, t$, $k \geq 1$, $\lambda_k \in R$, $\lambda_k = k\,\omega$, $k \in Z$, $\omega \neq 0$.

In section 5 when we presented the oscillatory properties, we made reference to the works of Osipov and Zhang who are the pioneers of the spaces of oscillatory functions. Also, we mentioned the works of Shubin and Corduneanu, the last contribution proposing a method of constructing new spaces of oscillatory functions more general than the almost periodic ones.

8 Conclusion

Since the topics discussed in this book are rather specialized with respect to the general theory of functional differential equations, the book can serve as source material for graduate students in mathematics, science, and engineering. In many applica-

tions one encounters functional differential equations that are not of a classical type and, therefore, are only rarely taken into consideration for teaching. The list of references in this book contains many examples of this situation. For instance, the case of equations in population dynamics is a good illustration (see Gopalsamy [8]). Also the book by Kolmanovskii and Myshkis [12] provides a large number of applications for functional differential equations, which may interest many categories of readers.
Acknowledgment. The author wishes to express his gratitude and thank the referee for thoroughly reading the paper and making appropriate corrections.

References

[1] R. Conti, *Sulla Prolungabilità delle soluzioni di un sistema di equazioni differenziali ordinarie*, Bollettino Dell'Unione Mathematica Italiana, 11 (**3**), 510–514, 1956.

[2] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.

[3] C. Corduneanu, *A Scale of Almost Periodic Function Spaces*, Differential and Integral Equations, 24 (**1-2**), 1–27, 2011.

[4] C. Corduneanu and M. Mahdavi, *On Neutral Functional Differential Equations with Causal Operators*, In Proceedings of the Third Workshop of the Inter. Inst. General Systems Science: Systems Science and Its Applications, 43–48, Tianjin People's Publishing House, Tianjin, 1998.

[5] C. Corduneanu and M. Mahdavi, *On Neutral Functional Differential Equations with Causal Operators, II*, Chapman & Hall/CRC Research Notes in Mathematics: Integral Methods in Science and Engineering, 102–106, Chapman & Hall/CRC, London, 2000.

[6] C. Corduneanu and M. Mahdavi, *Neutral Functional Equations in Discrete Time*, Proceedings of the International Conference on Nonlinear Operations, Differential Equations and Applications, Babes-Bolyai University of Cluj-Napoca, Romania, Volume III, 33–40, Cluj-Napoca, Romania, 2002.

[7] C. Corduneanu, Y. Li, and M. Mahdavi, *Functional Differential Equations: Advances and Applications*, Wiley, New Jersey, 2016.

[8] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic, Dordrecht, 1992.

[9] J. K. Hale, *Linear Functional Differential Equations with Constant Coefficients*, Contributions to Differential Equations, (**2**), 291–319, 1963.

[10] J. K. Hale, *Theory of Functional Differential Equations*, volume 3, Springer-Verlag, New York, 1977.

[11] N. N. Krasovskii, *Stability of Motion*, Stanford University Press, Stanford, 1963. in Russian, 1959.

[12] V. B. Kolmanovskii and A. D. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer, Dordrecht, 1999.

[13] M. Mahdavi, *Linear Functional Differential Equations with Abstract Volterra Operators*, Differential and Integral Equations, 8 (**6**), 1517–1523, 1995.

[14] M. Mahdavi, *Neutral Equations with Causal Operators*, Integral Methods in Science and Engineering, 161–166, Birkhäuser, Boston, 2002.

[15] A. D. Myshkis, *Linear Differential Equations with Retarded Argument*, Nauka, Moscow, 1972. Russian, German Edition: Deutscher Verlag Der Wissenschaften, Berlin, 1955.

[16] V. F. Osipov, *Bohr-Fresnel Almost Periodic Functions*, Peterburg Gos. University, St. Petersburg, 1992.

[17] V. V. Rumiantsev, *On the Stability of Motion with Respect to Part of the Variables*, Vestnik Moscow University, Mathematics-Mechanics series, Number 4, 9–16, Russian, 1957.

[18] G. Sansone and R. Conti, *Nonlinear Differential Equations*, Macmillan, New York, 1964.

[19] M. A. Shubin, *Differential and Pseudodifferential Operators in Spaces of Almost Periodic Functions*, Matematicheskii Sbornik, 137 (**4**), 560–587, 1974.

[20] M. A. Shubin, *Almost Periodic Functions and Partial Differential Operators*, Uspekhi Matematicheskikh Nauk, 33 (**2**), 3–47, 1978

[21] Chuanyi Zhang, *Almost Periodic Type Functions and Ergodicity*, Kluwer Academic, Dordrecht, 2003.

[22] Chuanyi Zhang, *Strong Limit Power Functions*, Journal of Fourier Analysis and Applications, (**12**), 291–307, 2006.

Mehran Mahdavi
Department of Mathematics, Bowie State University, 14000 Jericho Park Road, Bowie, Maryland 20715, U.S.A.
E-mail: mmahdavi@bowiestate.edu