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On a development of the comparison principle in the stability theory of motion

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Abstract: In this paper, conditions for different types of stability of the zero solution of a nonautonomous nonlinear comparison equation are established, and on this basis the corresponding stability results for the zero solution of the original system of differential equations are obtained. A development of the comparison principle, related to a generalized estimate of the total derivative of a Lyapunov-type auxiliary function with respect to the system under investigation is considered.

Keywords: Nonautonomous nonlinear comparison equations, stability, scalar and vector Lyapunov functions.

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1 Introduction

The classical application of the direct Lyapunov method (cf. [13]) in the qualitative theory of equations provides for two stages: the first one is the construction of a suitable Lyapunov function (scalar or vector) and the second is the estimation of the total derivative of this function with respect to the equations of perturbed motion. The Lyapunov theorems and/or their generalizations were the base on which the researchers conducted the qualitative analysis of the properties of motion, including results related to nonlinear systems of differential equations of perturbed motion

$$\frac{dx}{dt} = f(t,x), \tag{1.1}$$

where $x \in \mathbb{R}^n$; $f \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}_+ = [0,\infty)$.

In the papers [4, 5, 6, 21] some ideas of the direct method of Lyapunov were developed and important questions related to the existence, uniqueness and stability of solutions of systems of type (1.1) have been considered on the basis of the generalized direct Lyapunov method and differential inequalities. In the paper [8], the ideas of the direct Lyapunov method and differential inequalities have been combined and efficient conditions for the stability of the zero solution of systems of the type (1.1) have been proposed in terms of the stability of the zero solution of a scalar comparison equation. These works contained the main ideas of the scalar comparison principle and had a significant effect on the creation of the comparison principle with vector Lyapunov functions (see [9, 14] and the references therein).

Some difficulties in the analysis of the stability of motion of systems of large dimensions motivated the researchers to use Lyapunov functions with more than one components. In the work [2] the author introduced a certain functional $\Phi(V,t)$ for the system (1.1), where $V = (v_i), i = 1, 2, ..., m$. This can be the norm of V or a Lyapunov function. Let the functional $\Phi(V,t)$ satisfies the operator inequality $P(\Phi(V,t)) \leq 0$ with respect to the solutions of the system (1.1), where $P(\Phi(V,t))$ is a certain operator defined on a partially ordered set. Let the Chaplygin theorem (see [1]) be true for the equation $P(w) = 0$ according to which the inequality $P(z) \leq 0$ implies the estimate $z \leq u$ between a given element z and the solution u of the equation $P(w) = 0$. In this case, we have the estimate $\Phi(y,t) \leq u$. If the functional $\Phi(y,t)$ is chosen so that the "smallness" of u implies "smallness" of y, then the inequality $\Phi(y,t) \leq u$ implies a certain stability type of the zero solution of system (1.1). The comparison principle for Lyapunov scalar functions based on the Gronwall-Bellman inequality as a Chaplygin theorem is an example of the realization of this idea.

In [22] a Lyapunov function is proposed, consisting of more than one component. The proposed two-component Lyapunov function was applied to study the stability of the ship's movement on the course.

The papers [3, 20] laid the foundation of the method of Lyapunov vector functions in the theory stability of systems of large dimensions by means of which important engineering problems of stability and stabilization of the motion of terrestrial and space objects are solved. The comparison principle with Lyapunov vector functions was developed as a result of the work of many authors (cf. [1, 9, 12, 20, 25, 26] and the references therein).

Further development of the direct Lyapunov method is based on the application of matrix-valued auxiliary functions (cf. [10, 17, 18] and the references therein). Such approach made possible the investigation of the stability of multi-dimensional systems in the case, when some independent subsystems can be unstable, which cannot be done by means of Lyapunov vector functions.

The review article [14] presents the main results obtained through the development of the comparison principle on the basis on scalar, vector, and matrix-valued Lyapunov functions and some applications of this approach are also indicated.

The main aim of this article is to investigate conditions for different stability types of the zero solution of a nonautonomous nonlinear comparison equation, and on this basis to obtain the corresponding stability results for the zero solution of the original system of differential equations. At the same time we consider a generalized estimate of the total derivative of the Lyapunov function with respect to the system under investigation.

2 Preliminary Analysis

Consider the system (1.1) with the initial condition

$$x(t_0) = x_0, \tag{2.1}$$

where $f(t, 0) = 0$ for $t \geq t_0$. Assume that for the initial data $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exists a solution $x(t, t_0, x_0)$ of the initial value problem (1.1)–(2.1) for all $t \geq t_0$.

In the next definition we will introduce a class of scalar Lyapunov functions.

Definition 2.1. A function $V \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+), V(t, 0) = 0$ for all $t \geq t_0$ is called *a Lyapunov function*, if V is single valued, positive definite and decreasing for $t \geq t_0$ in some neighborhood of the origin of the phase space, and together with its total derivative $V^{'}$ with respect to (1.1) solves the stability (instability) problem of the state $x = 0$ of system (1.1).

We recall that Lyapunov [13] introduced V-functions and their total derivatives in the following cases:

— in the proof of stability theorems (cf. [13]) V is a sign-definite function, $V^{'}$ with respect to (1.1) is of a constant sign which is opposite to the sign of V, i.e. if $V \geq W$, where W is an independent of t positive definite function, then

$$V^{'} \leq 0, \tag{2.2}$$

where $V^{'} = V_t(t, x) + (f(t, x), \operatorname{grad} V(t, x))$;

— in the proof of instability theorems (cf. [13]) considering the relation

$$V^{'} = \lambda V + W, \tag{2.3}$$

where $\lambda > 0$, and W is either identically zero or a function of constant sign;

— when considering the critical cases, when the form of the Lyapunov functions and their derivatives $V^{'}$ are significantly more complicated.

In [21] for a Lyapunov function V estimates of its total derivative in the form

$$V^{'} \leq f(V), \tag{2.4}$$

where $f(V) > 0$ for $0 < V \leq H, f(0) = 0$ have been considered, and also

$$V^{'} \leq \varphi(t)f(V), \tag{2.5}$$

where $\varphi(t)$ is such that $\int\limits_{t_0}^{t} \varphi(s)ds \le M, M > 0$ is a constant, and

$$V' \le g_1(t)V + g_2(t)V^p,\ p > 1, \tag{2.6}$$

where the functions $g_1(t), g_2(t)$ are positive and integrable for all $t \ge t_0$.

In fact, the estimates (2.2) - (2.6) were a prerequisite for the appearance of comparison equations of the following types

$$\frac{du}{dt} = 0; \quad u(t_0) = u_0 \ge 0; \tag{2.7}$$

$$\frac{du}{dt} = \lambda u + w \quad u(t_0) = u_0 \ge 0, \tag{2.8}$$

where $w \ge 0$;

$$\frac{du}{dt} = g(u), \quad u(t_0) = u_0 \ge 0; \tag{2.9}$$

$$\frac{du}{dt} = \varphi(t)g(u), \quad u(t_0) = u_0 \ge 0; \tag{2.10}$$

$$\frac{du}{dt} = g_1(t)u + g_2(t)u^p, \quad u(t_0) = u_0 \ge 0. \tag{2.11}$$

In his works [5, 6] Conti proposed for the total derivative V' an estimate of the type

$$V' \le W(t, V), \tag{2.12}$$

where W is a real function defined on $\mathbb{R}_+ \times \mathbb{B}_h$, $\mathbb{B}_h = \{x \in \mathbb{R}^n \colon \|x\| < h\}$, $h > 0$. His estimate was the final step towards obtaining the comparison inequality

$$V(t, x(t)) \le r_M(t, t_0, u_0), \tag{2.13}$$

where $r_M(t, \cdot)$ is the maximal solution of the scalar comparison equation

$$\frac{dr}{dt} = W(t, r(t)), \quad r(t_0) = r_0 \ge 0 \tag{2.14}$$

on $t \in [t_0, \beta)$, where $\beta > t_0$ is the right end-point of the common existence interval of the solutions of the given system (1.1) and the comparison equation (2.14).

Definition 2.2. A tuple consisting of a system of differential equations of the type (1.1), a Lyapunov function $V(t, x)$, its total derivative $V'(t, x)$, a majorizing function $W(t, V)$ and a comparison equation of the type (2.14) is the *basis of the comparison principle* in the qualitative theory of differential equations if it enables us to obtain an estimate of the form (2.13) under the condition $V(t_0, x_0) \le r_0$ for all $t \in [t_0, \beta)$.

3 The Comparison Principle by Scalar Lyapunov Functions

We consider a system of differential equations of perturbed motion (1.1) on a domain $D=\{(t,x): t\in\mathbb{R}_+,\ x\in\mathbb{R}^n,\ \|x\|<h\}$, where $n\in N$ and $h>0$.

Next, we need a modification of Theorem 1.6.4 from [11].

Lemma 3.1. *Assume that:*

(1) The functions $g_i:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}_+,\ (t,x)\to g_i(t,x),\ i=1,2$ are nonnegative and integrable on any finite interval and such that for any $p>1$ the initial value problem

$$\frac{du}{dt}=g_1(t,x(t))u(t)+g_2(t,x(t))u^p(t), \tag{3.1}$$

$$u(t_0)=u_0\geq 0, \tag{3.2}$$

has a unique solution $u(t)=u(t,t_0,u_0,x_0)$ depending on the initial data $(t_0,u_0,x_0)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n$, where x_0 is the initial value of the problem (1.1)–(2.1).

(2) The function $v:[t_0,\beta_1)\to\mathbb{R}$ is such that

- *(a) $[t_0,\beta_1),\beta_1>t_0$ is the maximal interval of mutual existence of the solutions $u(t)$ and $x(t)$;*
- *(b) $v(t_0)\leq u_0$;*
- *(c) $\frac{dv}{dt}\leq g_1(t,x(t))v(t)+g_2(t,x(t))v^p(t)$ for all $t\in[t_0,\beta_1)$.*

Then

$$v(t)\leq\frac{u_0\exp\int\limits_{t_0}^{t}g_1(s,x(s))ds}{\left(1-(p-1)u_0^p\int\limits_{t_0}^{t}g_2(s,x(s))\exp\left((p-1)\int\limits_{t_0}^{t}g_1(\tau,x(\tau))d\tau\right)ds\right)^{\frac{1}{p-1}}} \tag{3.3}$$

for all $t\in[t_0,\beta_2)$, $\beta_2\leq\beta_1$, for which

$$(p-1)u_0^p\int\limits_{t_0}^{t}g_2(s,x(s))\exp\left((p-1)\int\limits_{t_0}^{s}g_1(\tau,x(\tau))d\tau\right)ds<1.$$

The proof of Lemma 3.1 is analogous to the proof of Theorem 2 in [16] and is based on the idea of the pseudo-linear representation of the right-hand side of the differential inequality in 2(c).

Corollary 3.2. *If in condition 2(c) of Lemma 3.1 the function $g_2(t, x(t)) = 0$ for all $t \in \mathbb{R}_+$, then the estimate (3.3) is in the form*

$$v(t) \le u_0 \exp\left(\int_{t_0}^{t} g_1(s, x(s))ds\right), \ t \in \mathbb{R}_+. \tag{3.4}$$

Corollary 3.3. *If in condition 2(c) of Lemma 3.1 the function $g_1(t, x(t)) = 0$ for all $t \in \mathbb{R}_+$, then the estimate (3.3) is in the form*

$$v(t) \le \frac{u_0}{\left(1 - (p-1)u_0^{p-1} \int_{t_0}^{t} g_2(s, x(s))ds\right)^{\frac{1}{p-1}}} \tag{3.5}$$

for all $t \in [t_0, \beta_3)$, $\beta_3 \le \beta_1$, for which

$$1 - (p-1)u_0^{p-1} \int_{t_0}^{t} g_2(s, x(s))ds > 0.$$

Remark 3.4. If in Lemma 3.1 the functions $g_i(t, x) = g_i(t)$, $i = 1, 2$, then the assertions of Lemma 3.1 and Corollaries 3.2 and 3.3 remain valid.

Next, suppose that a Lyapunov function $V(t, x) : D \to \mathbb{R}_+$, $V(t, x) \in C^1(D)$, $V(t, x) = 0$ for $x = 0$ is constructed so that

$$V^{'}(t, x) \le g_1(t, x(t))V(t, x(t)) + g_2(t, x(t))V^p(t, x(t)) \tag{3.6}$$

is defined for $(t, x) \in D$. Here, the functions $g_i(t, x) \in C(\mathbb{R}_+ \times \mathbb{R}^n, \ \mathbb{R})$ and have the properties from Lemma 3.1.

Remark 3.5. If in (3.6) we denote $v(t) = V(t, x(t))$, then for the function $V(t, x(t))$ we can obtain an estimate of the type (3.3) for all $t \in [t_0, \beta_2)$, similar to that in Lemma 3.1.

Together with system (1.1) we consider the extended system of $(n+1)$ differential equations

$$\begin{aligned} \frac{dx}{dt} &= f(t, x), \quad x(t_0) = x_0, \\ \frac{du}{dt} &= g_1(s, x(s))u(s) + g_2(s, x(s))u^p(s), \\ &\quad u(t_0) = u_0 \ge 0. \end{aligned} \tag{3.7}$$

Denote $w = (x_2, ..., x_n, u)^T$ and assume that:

(A) The right-hand sides of system (3.7) defined on the domain $\{(t, x, u)\colon\ t \geq 0,$ $u\ \leq H^*,\ \|x\| < \infty\}$, are continuous and satisfy the conditions that guarantee the existence of the solutions $x(t), u(t)$ of system (3.7);

(B) The solution $w(t)$ of system (3.7) is x-continuable, i.e. each solution $w(t)$ is defined for all $t \geq 0$ for which $\mid u(t) \mid \leq H^*$.

Definition 3.6. The zero solution $w(t) = 0$ of the extended system (3.7) is u - stable, if for any $0 < \epsilon < H^*$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that $u(t) < a(\varepsilon)$ for all $t > t_0$ for which $u_0 < \delta$, $\parallel x_0 \parallel < \delta$. Here $a \in K$, where K is a Hahn class of functions.

Lemma 3.7. *Let the conditions of Lemma 3.1 be satisfied, and in addition*

$$\frac{\exp \int\limits_{t_0}^{t} g_1(s, x(s)) ds}{\left(1 - (p-1)\delta^p \int\limits_{t_0}^{t} g_2(s, x(s)) \exp\left((p-1) \int\limits_{t_0}^{t} g_1(\tau, x(\tau)) d\tau\right) ds\right)^{\frac{1}{p-1}}} < \frac{a(\varepsilon)}{\delta(t_0, \epsilon)} \tag{3.8}$$

for all $t \in [t_0, \beta_2)$.

Then the zero solution $w(t) = 0$ of the extended system (3.7) is u - stable.

If the estimate (3.8) is satisfied uniformly with respect to t_0 and $\delta = \delta(\varepsilon)$, then the solution $w(t) = 0$ of the extended system (3.7) is u - stable uniformly in t_0.

Proof. The proof of Lemma 3.7 follows directly from the estimate (3.3), since (3.8) leads to

$$u(t) = u(t, t_0, u_0, x_0) < a(\epsilon) \tag{3.9}$$

for all $t \in [t_0, \beta_2)$.

□

We denote by $g_i^*(t, h) = \sup\{g_i(t, x) : x \in \mathbb{B}_h\}, i = 1, 2$ and formulate the following lemma.

Lemma 3.8. *Let the conditions of Lemma 3.1 be satisfied for the functions $g_i^*(t, h), i = 1, 2$, and in addition, assume that for each $t_0 \geq 0$ there exists $\Delta(t_0) > 0$ such that for $u_0 < \Delta(t_0)$ the inequality*

$$\frac{\Delta(t_0)}{\left(1 - (p-1)\Delta^p(t_0) \int\limits_{t_0}^{t} g_2^*(s, h) \exp\left((p-1) \int\limits_{t_0}^{t} g_1^*(\tau, h) d\tau\right) ds\right)^{\frac{1}{p-1}}} < a(\epsilon) \tag{3.10}$$

is true for all $t \in [t_0, \beta_4)$, $\beta_4 \leq \beta_1$, *and*

$$\int_{t_0}^{t} g_1^*(s,h)\,ds \to -\infty \quad as \quad t \to \infty. \tag{3.11}$$

Then the zero solution $w(t) = 0$ *of the extended system (3.7) is asymptotically* u *- stable.*

If the conditions (3.10) and (3.11) are satisfied uniformly with respect to t_0 *and* $\delta = \delta(\varepsilon) > 0$, *then the solution* $w(t) = 0$ *of the extended system (3.7) is uniformly asymptotically* u *- stable.*

Proof. It follows from (3.10) that

$$u(t) < a(\epsilon)\exp\int_{t_0}^{t} g_1^*(s,h)\,ds$$

for all $t \in [t_0, \beta_4)$ and from (3.11) we have

$$\lim u(t) = 0 \quad \text{as} \quad t \to +\infty.$$

□

The following theorem holds.

Theorem 3.9. *Assume that for system (1.1) there exist functions* $V(t,x)$ *and* $g_i(t,x) \in C(\mathbb{R}_+ \times \mathbb{R}^n,\ \mathbb{R})$, $i = 1, 2$, *that satisfy the conditions of Lemma 3.1 and are such that:*

(1) *The function* $V(t,x)$ *satisfies the inequality* $a(\|x\|) \leq V(t,x)$ *for* $(t,x) \in D$, $a \in K$;
(2) *The inequality (3.6) is satisfied.*

Then the state $x = 0$ *of system (1.1) is stable (uniformly stable) provided that the conditions of Lemma 3.5 are satisfied.*

If, in addition the function $V(t,x)$ *is such that*

$$V(t,x) \leq b(\|x\|) \quad for \quad (t,x) \in D,\ b \in K,$$

then the state $x = 0$ *of system (1.1) is asymptotically stable (uniformly asymptotically stable) provided that the conditions of Lemma 3.6 are satisfied.*

Proof. Since the conditions of Lemma 3.1 are satisfied, then (3.3) holds. The estimate (3.3) and Lemma 3.5 imply the u-stability (uniform u-stability) of the solution $w(t) = 0$ of the extended system (3.7).

From

$$u(t) = u(t, t_0, x_0, u_0) < a(\epsilon)$$

for all $t \in [t_0, \beta_2)$ and Lemma 3.1 (see Remark 3.5), we have

$$V(t, x(t, t_0, x_0)) \le u(t),$$

and hence,

$$a(\|x(t, t_0, x_0)\|) \le V(t, x(t, t_0, x_0)) \le u(t) < a(\epsilon)$$

for all $t \in [t_0, \beta_2)$. Therefore $\|x(t, t_0, x_0)\| < \epsilon$ for all $t \in [t_0, \beta_2)$.

If (3.8) is satisfied uniformly in $t_0 \in \mathbb{R}_+$, then $\delta(\epsilon) > 0$ does not depend on t_0. This proves the stability (uniform stability) of the state $x = 0$ of (1.1). The proof of asymptotic stability (uniform asymptotic stability) is similar on the base of Lemma 3.6. □

4 The Comparison Principle by Vector Lyapunov Functions

The paper [3] investigates the following system of connected differential equations

$$\begin{aligned} \frac{dx}{dt} &= Ax + By + g(x, y), \quad x(0) = x_0, \\ \frac{dy}{dt} &= Cx + Dy + h(x, y), \quad y(0) = y_0, \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, A, B, C, D are constant matrices of appropriate dimensions, $g \in C(\mathbb{R}^n \times \mathbb{R}^m,\ \mathbb{R}^n)$, $h \in C(\mathbb{R}^n \times \mathbb{R}^m,\ \mathbb{R}^m)$, $g(0,0) = h(0,0) = 0$.

For the stability analysis of the state $x = y = 0$ of (4.1) the authors in [3] applied two quadratic forms

$$v_1(x) = x^T R x, \quad v_2(y) = y^T S y, \tag{4.2}$$

which are components of a vector function $V(x, y) = (v_1(x), v_2(y))^T$. The matrices R and S in the quadratic forms (4.2) are positive definite and can be constructed under certain assumptions on the matrices A and C in (4.1).

In this paper, in studying the stability with respect to the first approximation of the system (4.1) the term "Vector Lyapunov Function" has been introduced and an example of a two component vector Lyapunov function has been given.

Next, the following definition will be of an importance. Let D_1 be a subset of $\mathbb{R}^k$.

Definition 4.1. The function $G(t,u,x(t))$ is said to be quasi-monotonically nondecreasing with respect to its second variable, if $u \leq v$, $u, v \in D_1$, $u_i = v_i$, $1 \leq i \leq k$ imply $g_i(t,u,x(t)) \leq g_i(t,v,x(t))$ for all $(t,x(t)) \in D$.

In the paper [20] for a system of the type (1.1), without an explicit allocation of subsystems, it is proposed the use of a finite number of functions

$$V_1(t,x),\ V_2(t,x), \ldots, V_k(t,x) \tag{4.3}$$

for which the total derivatives with respect to system (1.1) satisfy the inequalities

$$V_i'(t,x) \mid_{(1.1)} \leq g_i(t, V_1(t,x), \ldots,\ V_k(t,x), x(t)),\ i = 1, 2, \ldots, k. \tag{4.4}$$

Here $g_i \in W$ (W denotes a class of quasi-monotonically nondecreasing functions defined on a certain domain). In this case an auxiliary system of the type

$$\frac{dy}{dt} = G(t,y), \quad y(t_0) = y_0 \geq 0, \tag{4.5}$$

has been considered, where $y \in \mathbb{R}^k$, $G(t,y) \in W$, which played the role of a comparison system for the original system (1.1).

Next, we will present a generalization of the Comparison Lemma 2.6 from the book [23] for the case of a vector function $V(t,x)$ and a generalized estimate of its total derivative with respect to system (1.1).

Lemma 4.2. *Assume that:*

(1) The function $G \in W$, $G : \mathbb{R}_+ \times \mathbb{R}^k_+ \times \mathbb{R}^n \to \mathbb{R}^k_+$, $(t,u,x) \to G(t,u,x)$ is continuous and bounded on any finite interval of existence of the solutions of (1.1) and $u_M(t) : [t_0, \beta_5) \to \mathbb{R}^k$ is the maximal solution of

$$\frac{du}{dt} = G(t,u,x(t)),$$
$$u(t_0) = u_0 \geq 0,$$

depending on the initial data $(t_0, u_0, x_0) \in \mathbb{R}_+ \times D_1 \times \mathbb{R}^n$, where x_0 is the initial value in the problem (1.1)–(2.1).

(2) The function $w : [t_0, \beta_6) \to \mathbb{R}^k$, $\beta_6 \leq \beta_5$, $(t, w(t)) \in \mathbb{R}_+ \times D_1$, is continuous, and such that

- (*a*) *$[t_0, \beta_6)$ is the maximal interval of mutual existence of the solutions $u_M(t)$ and $x(t)$;*
- (*b*) *$w(t_0) \leq u_0$;*
- (*c*) *$\frac{dw}{dt} \leq G(t, w(t), x(t))$ for all $t \in [t_0, \beta_6)$.*

Then $w(t) \le u_M(t)$ *for all* $t \in [t_0, \beta_6)$.

The proof of Lemma 4.2 is analogous to the proof of Lemma 2.6 in [23].

Lemma 4.3. *Let the vector function* $V \in C^1(\mathbb{R}_+ \times \mathbb{R}^n,\ \mathbb{R}^k)$ *be such that* $V(t, 0) = 0$, $t \in \mathbb{R}_+$. *Assume that:*

(1) *The function* $G \in C(\mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^n,\ \mathbb{R}^k)$ *is quasi-monotonically nondecreasing with respect to* u *for all* $(t, x(t)) \in D$ *with components* $g_s(t, u, x(t))$ *such that* $V_s'(t, x)\mid_{(1.1)} \le g_s(t, V_1(t, x), \dots,\ V_k(t, x),\ x(t)),\ s = 1, 2, \dots, k$ *for* $(t, x) \in D$;

(2) *There exists a maximal solution* $r_M(t) = r_M(t, t_0, x_0, u_0)$ *of the comparison system*

$$\begin{aligned} \frac{du}{dt} &= G(t, u, x(t)) \\ u(t_0) &= u_0 \ge 0 \end{aligned} \tag{4.6}$$

for $t \ge t_0$ *and any solution* $x(t)$ *of (1.1).*

Then $V(t_0, x_0) \le u_0$ *implies*

$$V(t, x(t)) \le r_M(t) \tag{4.7}$$

for all $t \ge t_0$.

The proof of Lemma 4.3 is very similar to the proof of the Comparison Lemma 2.6 in [23], taking into account Lemma 4.2.

Recall that the inequality (4.7) is fulfilled componentwise.

The estimate (4.7) allows us to investigate various dynamic properties of systems of the type (1.1) on the basis of certain dynamic properties of the comparison system (4.6).

The following theorem which is analogous to the statement given in [11] and is based on the use of scalar Lyapunov functions holds.

Theorem 4.4. *For the system (1.1), assume that there exist a vector function* $V(t, x)$ *and a majorizing function* $G(t, V(t, x), x(t))$ *satisfying all conditions of Lemma 4.3 which are such that:*

$$a(\|x\|) \le e^T V(t, x), \quad V(t, 0) = 0, \tag{4.8}$$

for $(t, x) \in \mathbb{R}_+ \times D$, *where* $e^T = (1, 1, \dots, 1) \in \mathbb{R}^k$.

Then

(a) *The stability of the solution* $u = 0$ *of the comparison system (4.6) implies stability of the state* $x = 0$ *of system (1.1);*

(*b*) *The equi-attractivity of the solution* $u = 0$ *of the comparison system (4.6) implies equi-attractivity of the state* $x = 0$ *of system (1.1).*
If, in addition, the function V *is such that*

$$e^T V(t,x) \leq b(\|x\|) \quad \text{for} \quad (t,x) \in \mathbb{R}_+ \times D, \;\; b \in K, \tag{4.9}$$

then

(*c*) *The uniform stability of the solution* $u = 0$ *of the comparison system (4.6) implies uniform stability of the state* $x = 0$ *of system (1.1);*

(*d*) *The uniform attractivity of the solution* $u = 0$ *of the comparison system (4.6) implies uniform attractivity of the state* $x = 0$ *of system (1.1).*

The proof of Theorem 4.4 is very similar to the proof of the Theorem 32.1 in [24], taking into account Lemma 4.3.

Remark 4.5. Theorem 4.4 is well-known in the case when in the inequality (4.4) the functions $g_i(t, V_1(t,x), \ldots, V_k(t,x), x(t)) = g_i(t, V_1(t,x), \ldots, V_k(t,x))$, $i = 1, 2, \ldots, k$ (see [1, 12, 20] and the references therein).

Remark 4.6. If, in the inequality (4.4), the functions $g_i(t, V_1(t,x), \ldots, V_k(t,x), x(t)) = g_i(V_1(x), \ldots, V_k(x))$, $i = 1, 2, \ldots, k$, then the comparison system (4.6) is autonomous, and the conditions for the uniform asymptotic stability of its zero solution are well-known (see [14] and the references therein).

5 Concluding Remarks

Sometimes (see [1, 20]) instead of conditions (4.8) and (4.9), the following conditions are used

$$\max_i V_i(t,x) \geq a(\|x\|)$$

and

$$\max_i V_i(t,x) \leq b(\|x\|).$$

Theorems 3.9 and 4.4 in the present paper combine the results of the works [1, 8, 10, 15] and demonstrate one of the further directions for development of the comparison principle in the theory of stability of motion. Namely, here we give an approach for obtaining conditions for various types of stability of the zero solution of nonlinear nonautonomous comparison equations that are used in the comparison principle. Since a comparison equation (scalar or vector) can be integrated only in some exceptional cases, the problem of a constructive estimation of the Lyapunov functions remains relevant. A new method for solving this problem was proposed in [15] on the basis of a pseudo-linear representation of the nonlinear integral inequality. The new idea was applied in [19] to study the boundedness of solutions of systems with a quadratic nonlinearity. In the present paper, for the first time, we apply this idea in the stability analysis of a scalar equation of a specific type.

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