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A note on admissible maps satisfying compactness conditions on countable sets

Donal O'Regan

Abstract: We present a general Mönch type fixed point result for the multivalued admissible maps in the sense of Gorniewicz.

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Dedicated to Professor Constantin Corduneanu on the occasion of his 90th birthday

1 Introduction

Fixed point theory for admissible maps in the sense of Gorniewicz have been discussed extensively in the literature; we refer the reader to [2] and the references therein. In 1980, Mönch [3] presented a fixed point result which extends Schauder and Sadovskii's fixed point results and Mönch's result was particularly useful in establishing existence results in differential equations. Mönch theorem was extended by many authors [4, 5, 6] and in particular O'Regan and Precup [5] presented a Mönch fixed point theorem for Kakutani maps. In this paper we present a Mönch type result for admissible maps in the sense of Gorniewicz.

Now we introduce the class of maps considered in Section 2. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p,q). Given two diagrams (p,q) and (p',q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are continuous maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. Note if (p,q), $(p_1,q_1) \in D(X,Y)$ (where $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ and $X \overset{p_1}{\Leftarrow} \Gamma' \overset{q_1}{\to} Y$) and $(p,q) \sim (p_1,q_1)$ then it is easy to see (use $q \circ g = q_1$ and $p \circ g = p_1$ where $g : \Gamma' \to \Gamma$) that for $x \in X$ we have $q_1\,(p_1^{-1}(x)) = q\,(p^{-1}(x))$. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\,p^{-1}\,(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ. Let $\phi \in M(X,Y)$ and (p,q) a representative of ϕ. We define $\phi(X) \subseteq Y$ by $\phi(X) = q\,(p^{-1}(X))$. Note $\phi(X)$ does not depend on the representative of ϕ.

A map $\phi : X \to 2^Y$ is said to be admissible [2] and we write $\phi \in Ad(X,Y)$ provided there exists a Hausdorff topological space Γ and a selected pair (p,q) (i.e. $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X$) of ϕ (i.e. $(p,q) \subset \phi$ i.e. $q\,(p^{-1}(x)) \subset \phi(x)$ for every $x \in X$).

2 Fixed Point Theory

We present our main result.

Theorem 2.1. *Let X be a metrizable topological vector space and $\phi \in Ad(X,X)$. Suppose there exists a Hausdorff topological space Γ and a selected pair (p,q) (i.e. $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X$) of ϕ with*

$$\begin{cases} A \subseteq \Gamma, \ \ A = p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(A)\right)\right) \ \ \text{with} \ \ C \subseteq A \\ \text{countable and} \ \ \overline{p(C)} = \overline{co}\left(\{x_0\} \cup q(C)\right), \\ \text{implies} \ \ \overline{co}\left(q(C)\right) \ \ \text{is compact} \end{cases} \tag{2.1}$$

where $x_0 \in p(\Gamma)$. Finally assume

$$\begin{cases} \text{for any nonempty convex compact subset } K \text{ of } X \text{ and} \\ \text{any } \psi \in Ad(K,K) \text{ we have that } \psi \text{ has a fixed point.} \end{cases} \tag{2.2}$$

Then ϕ has a fixed point.

Remark 2.2. In the proof below we see that X metrizable can be replaced by any space with the following properties: (i). X is such that the closure of a subset Ω of X is compact if and only if Ω is sequentially compact, and (ii). for any convex set $D \subseteq X$ if $x \in \overline{D}$ then there exists a sequence x_1, $x_2, \ldots.$ in D with x_n converging to x.

Remark 2.3. In (2.1) in fact $\overline{co}\,(q(C))$ is compact implies $\overline{co}\,(q(A))$ is compact (see the proof below).

Remark 2.4. Conditions to guarantee (2.2) can be found in [2].

Proof. Let p, q be as described in the statement of Theorem 2.1 and let $\mathcal{F}$ be the family of all subsets D of Γ with $p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D)\right)\right) \subseteq D$. Note $\mathcal{F} \neq \emptyset$ since $\Gamma \in \mathcal{F}$ (recall p is surjective). Let

$$D_0 = \cap_{D \in \mathcal{F}} D \quad \text{and} \quad D_1 = p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_0)\right)\right).$$

We now show $D_1 = D_0$. Now for <u>any</u> $D \in \mathcal{F}$ we have since $D_0 \subseteq D$ that

$$D_1 = p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_0)\right)\right) \subseteq p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D)\right)\right) \subseteq D,$$

so as a result $D_1 \subseteq D_0$. Also since $D_1 \subseteq D_0$ we have $q(D_1) \subseteq q(D_0)$ so

$$p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_1)\right)\right) \subseteq p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_0)\right)\right) = D_1,$$

and as a result $D_1 \in \mathcal{F}$, so $D_0 \subseteq D_1$. Consequently

$$D_0 = p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_0)\right)\right). \tag{2.3}$$

We now claim

$$\overline{co}\,(q(D_0)) \ \text{ is compact.} \tag{2.4}$$

Suppose the claim is false. Then there exists a sequence y_1, y_2, ... in $\overline{co}\left(\{x_0\} \cup q(D_0)\right)$ without a convergent subsequence. Let $A_0 = \{y_1, y_2, \ldots.\}$. Each y_n is the limit of a sequence of finite convex combination of points from $\{x_0\} \cup q(D_0)$ so there exists a countable set $Q_0 \subseteq \{x_0\} \cup q(D_0)$ with $y_n \in \overline{co}\,(Q_0)$ for each n. In particular there exists a countable set $C_1 \subseteq D_0$ with $y_n \in \overline{co}\left(\{x_0\} \cup q(C_1)\right)$ for each n; note $A_0 \subseteq \overline{co}\left(\{x_0\} \cup q(C_1)\right)$.

Next we construct a countable set $C_2 \subseteq D_0$ with $C_1 \subseteq C_2$, $p(C_1) \subseteq \overline{co}\left(\{x_0\} \cup q(C_2)\right)$ and $\overline{co}\left(\{x_0\} \cup q(C_1)\right) \subseteq \overline{p(C_2)}$. To see this first note

$$C_1 \subseteq D_0 = p^{-1}\left(\overline{co}\left(\{x_0\} \cup q(D_0)\right)\right)$$

so $p(C_1) \subseteq \overline{co}\left(\{x_0\} \cup q(D_0)\right)$. Now $p(C_1)$ is countable (since p is single valued and C_1 is countable). Note each $x \in p(C_1)$ is the limit of a sequence of finite convex combination of points from $\{x_0\} \cup q(D_0)$. Then there exists a countable set $Q_1 \subseteq \{x_0\} \cup q(D_0)$ with $p(C_1) \subseteq \overline{co}\,(Q_1)$. In particular there exists a countable set $A_2 \subseteq D_0$ with

$$p(C_1) \subseteq \overline{co}\,(\{x_0\} \cup q(A_2)). \tag{2.5}$$

Next note since $C_1 \subseteq D_0$ that $p^{-1}\left(\overline{co}\,(\{x_0\} \cup q(C_1))\right) \subseteq p^{-1}\left(\overline{co}\,(\{x_0\} \cup q(D_0))\right) = D_0$ (see (2.3)) so

$$\{w \in \Gamma :\, p(w) \in \overline{co}\,(\{x_0\} \cup q(C_1))\} \subseteq D_0.$$

Now since p is surjective then

$$\overline{co}\,(\{x_0\} \cup q(C_1)) = \overline{co}\,(\{x_0\} \cup q(C_1)) \cap p(\Gamma) \subseteq p(D_0);$$

to see this note if $x \in \overline{co}\,(\{x_0\} \cup q(C_1)) \cap p(\Gamma)$ then there exists $y \in \Gamma$ with $x \in \overline{co}\,(\{x_0\} \cup q(C_1))$ and $x = p(y)$, and note $p(y)\,(= x) \in \overline{co}\,(\{x_0\} \cup q(C_1))$ so from the above $y \in D_0$ i.e. $x = p(y)$, $y \in D_0$ i.e. $x \in p(D_0)$. Thus $\overline{co}\,(\{x_0\} \cup q(C_1)) \subseteq p(D_0)$. Next note $co\,(\{x_0\} \cup q(C_1))$ is separable (recall the convex hull of a countable set is separable) so there exists a countable set $Q_0 \subseteq X$ with $Q_0 \subseteq co\,(\{x_0\} \cup q(C_1)) \subseteq \overline{Q_0}$ and since $\overline{co}\,(\{x_0\} \cup q(C_1)) \subseteq p(D_0)$ we have $Q_0 \subseteq p(D_0)$. Thus there exists a countable set $B_2 \subseteq D_0$ with $Q_0 \subseteq p(B_2)$ and as a result

$$\overline{co}\,(\{x_0\} \cup q(C_1)) = \overline{Q_0} \subseteq \overline{p(B_2)}. \tag{2.6}$$

Let $C_2 = C_1 \cup A_2 \cup B_2$. Note $C_1 \subseteq C_2$, $C_2 \subseteq D_0$ (since $A_2 \subseteq D_0$, $C_1 \subseteq D_0$ and $B_2 \subseteq D_0$) and since $A_2 \subseteq C_2$ and $B_2 \subseteq C_2$ we have from (2.5) and (2.6) that

$$p(C_1) \subseteq \overline{co}\,(\{x_0\} \cup q(C_1)) \;\text{ and }\; \overline{co}\,(\{x_0\} \cup q(C_1)) \subseteq \overline{p(C_2)}.$$

Proceed (as above) and we obtain countable sets $C_3, C_4, ...$ with $C_n \subseteq D_0$ for $n \in \{1, 2, ...\}$, $C_n \subseteq C_{n+1}$ for $n \in \{1, 2, ...\}$,

$$p(C_n) \subseteq \overline{co}\,(\{x_0\} \cup q(C_{n+1})) \;\text{ for }\; n \in \{1, 2...\}$$

and

$$\overline{co}\,(\{x_0\} \cup q(C_n)) \subseteq \overline{p(C_{n+1})} \;\text{ for }\; n \in \{1, 2...\}.$$

Let $C = \cup_{n=1}^{\infty} C_n$. For each $x \in p(C) = p(\cup_{n=1}^{\infty} C_n)$ we have $x \in p(C_n)$ for some $n \in \{1, 2, ...\}$ so

$$x \in \overline{co}\,(\{x_0\} \cup q(C_{n+1})) \subseteq \overline{co}\,(\{x_0\} \cup q(C)).$$

Thus

$$p(C) \subseteq \overline{co}\,(\{x_0\} \cup q(C)). \tag{2.7}$$

Also since $C_1 \subseteq C_2 \subseteq .....$ (so $q(C_1) \subseteq q(C_2) \subseteq ...$) we have

$$\begin{aligned} co\,(\{x_0\} \cup q(C)) &= co\,(\{x_0\} \cup q\,(\cup_{n=1}^{\infty} C_n)) = co\,(\{x_0\} \cup [\cup_{n=1}^{\infty} q(C_n)]) \\ &\subseteq \cup_{n=1}^{\infty} co\,(\{x_0\} \cup q(C_n)) \subseteq \cup_{n=1}^{\infty} \overline{p(C_{n+1})} \subseteq \overline{p(C)} \end{aligned}$$

since $p(C_n) \subseteq p(C)$ for $n \in \{1, 2, ...\}$. Thus

$$\overline{co}\,(\{x_0\} \cup q(C)) \subseteq \overline{p(C)}$$

and this together with (2.7) yields

$$\overline{p(C)} = \overline{co}\,(\{x_0\} \cup q(C)).$$

Now (2.1) guarantees that $\overline{co}\,(\{x_0\} \cup q(C))$ is compact. This is a contradiction since $\overline{co}\,(\{x_0\} \cup q(C))$ contains the sequence $\{y_n\}$ (note $A_0 \subseteq \overline{co}\,(\{x_0\} \cup q(C_1)) \subseteq \overline{co}\,(\{x_0\} \cup q(C)))$ which has no convergent subsequence.

Thus (2.4) holds i.e $\overline{co}(q(D_0))$ is compact. For convenience let $K = \overline{co}\,(q(D_0))$. Note from (2.3) that $p^{-1}(K) \subseteq D_0$ so $q\,(p^{-1}(K)) \subseteq q(D_0) \subseteq K$. Also note $K \stackrel{p_0}{\Leftarrow} p^{-1}(K) \stackrel{q_0}{\to} K$ where p_0 and q_0 denote contractions of the appropriate maps p and q (see [1 pp 214]). Thus $\phi \in Ad(K, K)$. Now (2.2) guarantees that ϕ has a fixed point. □

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Donal O'Regan
School of Mathematics, Statistics and Applied Mathematics
National University of Ireland, Galway
Ireland
E-mail: donal.oregan@nuigalway.ie