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The approximation of the square root of the total variation flow

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Abstract: Here, it is discussed the approximation of square root of the total variation flow which is relevant in the image restoring method.

Keywords: bounded variation, maximal monotone operator, square root, total variation flow.

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Homage to professor Constantin Corduneanu to his 90th anniversary

1 Introduction

We consider a degraded (noisy) image u_0 as a real valued function on a given bounded convex domain Ω of $\mathbb{R}^2$ with a piecewise smooth boundary $\partial\Omega$. The restored image is given by $u = u(x)$, $x \in \Omega$. There exists a huge literature on partial differential approach to image restoration and among the most important and efficient methods is that based on total variation flow technique firstly described in the pioneering work [18] (see also [7]–[10], [13]–[15], [17], [19]).

Formally, the total variation flow $u(t) = u(t, u_0)$, $t \geq 0$, is defined by the parabolic boundary value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \operatorname{div}\left(\frac{\nabla_x u}{|\nabla_x u|}\right) = 0 \quad && \text{in } (0,\infty) \times \Omega,\\
&u(0,x) = u_0(x), && x \in \Omega,\\
&u(t,x) = 0, && t \geq 0,\ x \in \partial\Omega.
\end{aligned} \tag{1.1}$$

However, the Cauchy problem (1.1) is not well posed and has meaning if it is replaced by the evolution equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \partial\varphi(u(t)) \ni 0, \quad \text{a.e. } t > 0,\\
&u(0) = u_0,
\end{aligned} \tag{1.2}$$

where $\partial\varphi : L^2(\Omega) \to L^2(\partial\Omega)$ is the subdifferential of the total variation function $\varphi : L^2(\Omega) \to]-\infty, +\infty]$

$$\varphi(u) = \begin{cases} \|Du\| + \displaystyle\int_{\partial\Omega} |\gamma_0(u)| d\mathcal{H}^1 & \text{if } u \in BV(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega), \end{cases} \tag{1.3}$$

where $BV(\Omega)$ is the space of functions $u : \Omega \to \mathbb{R}$ with bounded variation on Ω and $\|Du\|$ is the total variation of u, that is (see, e.g., [2]),

$$\|Du\| = \sup \left\{ \int_\Omega u \text{ div } \varphi \, dx; \ \varphi \in C_0^\infty(\Omega; \mathbb{R}^2), \ \|\varphi\|_\infty \leq 1 \right\},$$

$\gamma_0(u)$ is the trace of u on the boundary $\partial\Omega$ and $d\mathcal{H}^1$ is the Hausdorff measure. (We note that in $2-D$, $BV(\Omega) \subset L^2(\Omega)$.) The subdifferential $\partial\varphi$ is defined by (see, e.g., [3], [11])

$$\partial\varphi(u) = \left\{ v \in L^2(\Omega); \int_\Omega v(x)(u(x) - \bar{u}(x))dx \geq \varphi(u) - \varphi(\bar{u}), \ \forall \bar{u} \in BV(\Omega) \right\}.$$

We set

$$D(\partial\varphi) = \{v \in BV(\Omega); \ \partial\varphi(v) \neq \emptyset\}.$$

Then, for each $u_0 \in L^2(\Omega) = \overline{D(\partial\varphi)}$, the Cauchy problem (1.2) has a unique solution $u = S(t)u_0 \in C([0, \infty); L^2(\Omega))$ such that

$$\frac{du}{dt}(t) \in L^2(\Omega), \ S(t)u_0 \in D(\partial\varphi), \ \forall t > 0. \tag{1.4}$$

(See [6], [11].)

By (1.4) it follows the denoising (restoring) effect of the total variation flow $t \to S(t)u_0$. In fact, for each $t > 0$, the function $x \to u(t, x)$, that is, the restored image, has bounded variation and so with a countable number of discontinuities along rectifiable curves in Ω. This means that the restored image $u(t, \cdot)$ preserves the edges and this is the principal advantage of total variation flow compared with classical nonlinear filters based on Sobolev spaces $W^{1,p}(\Omega)$, $p > 1$. It should be said, however, that (1.4) implies an additional regularity of restored image $u(t, \cdot)$ and not just the bounded variation property, $u(t) \in BV(\Omega)$.

In order to keep the approach within the bounded variation limits, we replace here (1.2) by the equation

$$\begin{aligned} &\frac{du}{dt} + (\partial\varphi)^{\frac{1}{2}}(u) \ni 0, \ t \geq 0, \\ &u(0) = u_0, \end{aligned} \tag{1.5}$$

where $(\partial\varphi)^{\frac{1}{2}}$ is the *square root* of the maximal monotone operator $\partial\varphi$. (See [3]-[5].) Then, the solution $u(t) = S_{\frac{1}{2}}(t)u_0$ is in $D(\partial\varphi)^{\frac{1}{2}} = BV(\Omega)$ for all $t > 0$. Here, we shall compute numerically the solution u to (1.5) via an approximating algorithm inspired by [3] and use it for a denoising procedure.

2 The square root of total variation flow

Here we briefly present, following [3], the definition and the main properties of the square root of a nonlinear maximal monotone operator introduced in [4], [5]. (We also refer to the monograph [16] for a survey on these topics.)

Let H be a real Hilbert space with the scalar product $(\cdot,\cdot)$ and norm $|\cdot|_H$. Let $A : D(A) \subset H \to H$ be a (multivalued) maximal monotone operator in H, that is,

$$\begin{aligned}
&(v_1 - v_2, u_1 - u_2) \geq 0,\ \forall v_i \in Au_i,\ i = 1, 2,\\
&\mathbb{R}(I + A) = H,
\end{aligned}$$

where I is the identity operator and $\mathbb{R}(I+A)$ denotes the range of $I+A$. We assume that $0 \in \mathbb{R}(A)$. Then, by Theorem 2.3 in [3], for each $u_0 \in \overline{D(A)}$ (the closure of the domain $D(A)$ of A), there is a unique solution $u = u(t, u_0) \in C([0,\infty); H)$ to the boundary value problem

$$\begin{aligned}
&\frac{d^2u}{dt^2} \in Au(t),\ \text{a.e. } t > 0,\\
&u(0) = u_0,\ \sup_{t\geq 0} |u(t)|_H < \infty.
\end{aligned} \tag{2.1}$$

which *has a unique solution u such that*

$$\begin{aligned}
&u \in C([0,\infty); H),\ t\,\frac{du}{dt} \in L^\infty(0,T;H),\\
&t^{\frac{1}{2}}\,\frac{du}{dt} \in L^2(0,\infty;H),\ t^{\frac{3}{2}}\,\frac{d^2u}{dt^2} \in L^2(0,\infty;H).
\end{aligned}$$

(We mention also the book [18] for a treatment of more general boundary value problems of this type.)

We set $S_{\frac{1}{2}}(t)u_0 = u(t, u_0)$. Then

$$S_{\frac{1}{2}}(t) : \overline{D(A)} \to \overline{D(A)},\ t \geq 0,$$

is a continuous semigroup of contractions in $\overline{D(A)} \subset H$. By a well known result due to Komura (see [3]), there is a unique maximal monotone operator $A_{\frac{1}{2}} : D(A_{\frac{1}{2}}) \to H$

such that $\overline{D(A_{\frac{1}{2}})} = \overline{D(A)}$ and

$$\lim_{t\to 0} \frac{u_0 - S_{\frac{1}{2}}(t)u_0}{t} = A^0_{\frac{1}{2}}u_0, \ \forall u_0 \in D(A_{\frac{1}{2}}).$$

(Here, $A^0_{\frac{1}{2}}u_0$ is the minimal section of $A_{\frac{1}{2}}u_0$.) This means that $-A_{\frac{1}{2}}$ is the infinitesimal generator of the semigroup —$S_{\frac{1}{2}}(t)$ and the operator $A_{\frac{1}{2}}$ is called the *square root of the operator* A. In the special case $A = \partial\varphi$, where $\varphi : H \to]-\infty, +\infty]$ is a convex and lower semicontinuous function, we have (see Theorem 2.6 in [3])

$$D(A_{\frac{1}{2}}) = D(\varphi) = \{u_0 \in H; \varphi(u_0) < \infty\}, \tag{2.2}$$

$$\frac{1}{2}\,|A^0_{\frac{1}{2}}u_0|^2 = \varphi(u_0) - \varphi_\infty, \ \varphi_\infty = \inf\{\varphi(u); u \in H\}. \tag{2.3}$$

We call $S_{\frac{1}{2}}(t)$ the *square root of the semigroup* $S(t)$ and refer to [3], [12] for its smoothing properties. In fact, it turns out that $t \to S_{\frac{1}{2}}(t)u_0$ is a.e. differentiable on $(0,\infty)$ and maps H into $D(A_{\frac{1}{2}}) = BV(\Omega)$ for all $t > 0$. We note here that $S_{\frac{1}{2}}(t)$ is given by (see [3])

$$S_{\frac{1}{2}}(t)u_0 = \lim_{T\to\infty} u_T(t) \text{ in } H, \ \forall t \geq 0, \tag{2.4}$$

uniformly on compact intervals $[0,T]$, where $u_T \in W^{2,2}(0,T;H)$ is the solution to the two point boundary value problem

$$\begin{aligned} &\frac{d^2u_T}{dt^2} \in Au_T(t), \ \text{a.e. } t \in (0,T), \\ &u_T(0) = u_T(T) = u_0. \end{aligned} \tag{2.5}$$

In the special case where $H = L^2(\Omega)$, $A = \partial\varphi$, φ given by (1.3) and $S(t)$ is the total variation flow process, the corresponding semigroup $S_{\frac{1}{2}}(t) = V_{\frac{1}{2}}(t)$ is called the *square root* (or *fractional*) *of the total variation flow* $S(t)$. For the restoring process, the flow $u_0 \to V_{\frac{1}{2}}(t)u_0$ is more efficient than the total variation flow $S(t)$ because its smoothing effect on the initial data is restricted to $BV(\Omega)$ instead of the more restricted set $D(\partial\varphi)$ and so preserves better the edges and reduces the staircase effects.

3 Approximating the square root of the total variation flow

As seen in (2.4), for each $u_0 \in L^2(\Omega)$, $V_{\frac{1}{2}}(t) = \lim_{T\to\infty} u_T(t)$ in H uniformly in t on compacts, where u_T is the solution to problem (2.5), where $A = \partial\varphi$. Since the nonlinear multivalued mapping $\partial\varphi$ is hard to describe in explicit terms, we approximate equation (2.5) by

$$\begin{aligned}&\frac{\partial^2 u}{\partial t^2}+\varepsilon\Delta_x u(t,x)+\mathrm{div}_x(\mathrm{sign}\nabla_x u(t,x))\ni 0,\ t\in(0,T),\ x\in\Omega,\\ &u(0,x)=u(T,x)=u_0(x),\ \forall x\in\Omega,\\ &u(t,x)=0,\ \forall(t,x)\in(0,T)\times\partial\Omega.\end{aligned}\tag{3.1}$$

Here $\varepsilon>0$ and sign : $\mathbb{R}^2\to 2^{\mathbb{R}^2}$ is defined by

$$\mathrm{sign}\ z=\frac{z}{|z|_2}\ \text{for}\ z\neq 0,\ \mathrm{sign}\ 0=\{v\in\mathbb{R}^2;\ |v|_2\le 1\},\tag{3.2}$$

where $|\cdot|_2$ is the Euclidean norm of $\mathbb{R}^2$.

The boundary value problem (3.1) can be written in the form (2.5)

$$\begin{aligned}&\frac{d^2u}{dt^2}\in A_\varepsilon u,\ \text{a.e.}\ t\in(0,T),\\ &u(0)=u(T)=u_0,\end{aligned}\tag{3.3}$$

where the operator $A_\varepsilon : D(A_\varepsilon)\subset L^2(\Omega)\to L^2(\Omega)$ is defined by

$$A_\varepsilon u=-\varepsilon\Delta u-\mathrm{div}(\mathrm{sign}(\nabla u)),\ \forall u\in D(A_\varepsilon)=H_0^1(\Omega)\cap H^2(\Omega).\tag{3.4}$$

In other words,

$$A_\varepsilon u=\{-\varepsilon\Delta u-\mathrm{div}\,\eta;\ \eta\in L^2(\Omega),\ \eta\in\mathrm{sign}\nabla u\ \text{a.e. in}\ \Omega\}.$$

We have

Lemma 3.1. *For each $\varepsilon>0$, the operator A_ε is maximal monotone in $L^2(\Omega)\times L^2(\Omega)$.*

Proof. Since the monotonicity of A_ε is immediate, we confine to prove that $\mathbb{R}(I+A_\varepsilon)=L^2(\Omega)$, that is, for each $f\in L^2(\Omega)$, the elliptic boundary value problem

$$\begin{aligned}&u-\varepsilon\Delta u-\mathrm{div}(\mathrm{sign}\nabla u)\ni f\quad &&\text{in}\ \Omega,\\ &u=0 &&\text{on}\ \partial\Omega,\end{aligned}\tag{3.5}$$

has a unique solution $u\in H_0^1(\Omega)\cap H^2(\Omega)$. To this end, we approximate the signum mapping (3.2) by $\psi_\lambda(u)=\nabla j_\lambda(u)$, where $j_\lambda:\mathbb{R}^2\to\mathbb{R}$ is defined by

$$j_\lambda(u)=\begin{cases}\dfrac{1}{2\lambda}|u|^2 & \text{if}\ |u|\le\lambda,\\[2mm] |u|-\dfrac{\lambda}{2} & \text{if}\ |u|>\lambda,\end{cases}$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^2$. We recall that, by Corollary 8.2. in [10], we have, for the operator $A_0 = -\Delta$, $D(A_0) = H_0^1(\Omega) \cap H^2(\Omega)$,

$$\int_\Omega j_\lambda((\nabla(1+\varepsilon A_0)^{-1}y)dx \le \int_\Omega j_\lambda(\nabla y)dx, \ \forall y \in H_0^1(\Omega), \ \lambda > 0,$$

or, equivalently,

$$\int_\Omega \Delta u \ \mathrm{div}(\psi_\lambda(\nabla u))dx \ge 0, \ \forall u \in H_0^1(\Omega) \cap H^2(\Omega), \ \lambda > 0. \tag{3.6}$$

(In fact, this implies a remarkable property of the heat flow on convex domains $\Omega \subset \mathbb{R}^d$. Namely, it is a semigroup of contractions in $W_0^{1,1}(\Omega)$ as well as in $BV(\Omega)$.)

Consider now, for each $\lambda > 0$, the equation

$$\begin{aligned} &u_\lambda - \varepsilon\Delta u_\lambda - \mathrm{div}\ \psi_\lambda(\nabla u_\lambda) = f \quad && \text{in } \Omega, \\ &u_\lambda = 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.7}$$

which clearly has a unique solution $u_\lambda \in H_0^1(\Omega) \cap H^2(\Omega)$ given by the convex minimization problem

$$u_\lambda = \arg\min\left\{\int_\Omega \left(\frac{\varepsilon}{2}|\nabla u|^2 + j_\lambda(\nabla u) - fu\right)dx;\ u \in H_0^1(\Omega)\right\}.$$

Taking into account (3.6), it is easily seen that, for $\lambda \to 0$, $u_\lambda \to u$ strongly in $H_0^1(\Omega)$ and weakly in $H^2(\Omega)$ to a solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ to (3.5). This completes the proof of Lemma 3.1. ∎

By Lemma 3.1, it follows that, for each $T > 0$, the boundary value problem (3.1) has a unique solution

$$u_\varepsilon \in W^{2,2}(0,T;H) \cap C([0,T];L^2(\Omega)).$$

Theorem 3.2 below is the main result.

Theorem 3.2. *Let $u_0 \in \overline{D(A)} = L^2(\Omega)$. Then, for $\varepsilon \to 0$, we have*

$$u_\varepsilon \to u_T \ \text{in}\ C([0,T];L^2(\Omega)), \tag{3.8}$$

where u_T is the solution to (2.5) and $A = \partial\varphi$.

Proof. We assume first that $u_0 \in D(\partial\varphi)$. We note that

$$\begin{aligned} u_\varepsilon = \arg\inf\Big\{ \int_0^T \int_\Omega \frac{1}{2}((u_t^2 + \varepsilon|\nabla u|^2) + |\nabla u|)dt\,dx; \\ u \in W^{1,2}(0,T;H);\ u(0) = u(T) = u_0\Big\}, \end{aligned} \tag{3.9}$$

where $H = L^2(\Omega)$. This implies that

$$\frac{1}{2}\int_0^T \int_\Omega ((u_\varepsilon)_t^2 + \varepsilon|\nabla u_\varepsilon|^2)dt\,dx + \int_0^T \int_\Omega |\nabla u_\varepsilon| dt\,dx \le C,\ \forall \varepsilon > 0. \tag{3.10}$$

By (3.1) and (3.10), it also follows that

$$\int_\Omega |u_\varepsilon(t,x)|^2 dx \le C,\ \forall \varepsilon > 0,\ t \in [0,T].$$

(We denote by C several positive constants independent of ε.)

By (3.3) we have

$$\begin{aligned} |A_\varepsilon u_\varepsilon(t)|_H^2 &= \langle A_\varepsilon u_\varepsilon(t), u''_\varepsilon(t)\rangle_2 \\ &= \frac{d}{dt}\left\langle A_\varepsilon u_\varepsilon(t), u'_\varepsilon(t)\right\rangle_2 - \left\langle (A_\varepsilon u_\varepsilon(t))', u'_\varepsilon(t)\right\rangle_2, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_2$ is the scalar product in $L^2(\Omega)$ and $v' = \frac{dv}{dt}$. This yields

$$\int_0^t s|A_\varepsilon u_\varepsilon(s)|_H^2 ds + \langle A_\varepsilon u_\varepsilon(t), u_\varepsilon(t)\rangle_2 \le t\left\langle A_\varepsilon u_\varepsilon(t), u'_\varepsilon(t)\right\rangle,\ \forall t \in [0,T]$$

and, finally,

$$\int_0^T (T-t)t|A_\varepsilon u_\varepsilon(t)|_H^2 dt \le C,\ \forall \varepsilon > 0.$$

Similarly, if multiply (3.4) by $-\Delta u_\varepsilon$, integrate on Ω and take into account that, by (3.6),

$$\int_\Omega \Delta u_\varepsilon \operatorname{div}(\operatorname{sign}\nabla u_\varepsilon)dx \ge 0,$$

we get

$$\varepsilon\int_0^T (T-t)t|\Delta u_\varepsilon(t)|_H^2 dt + \int_0^T (T-t)t|\operatorname{div}(\operatorname{sign}\nabla u_\varepsilon)|_H^2 dt \le C,\ \forall \varepsilon > 0.$$

Hence, on a subsequence, $\varepsilon \to 0$,

$$\begin{aligned} u_\varepsilon &\longrightarrow u_T && \text{weakly in } W^{1,2}_{\mathrm{loc}}((0,T];H) \\ \frac{d^2u_\varepsilon}{dt^2} &\longrightarrow \frac{d^2u_T}{dt} && \text{weakly in } W^{1,2}([0,T];H) \end{aligned} \tag{3.11}$$

and, since $BV(\Omega)$ is compactly embedded in $L^1(\Omega)$, while the total variation φ is lower semicontinuous, we have by (3.10)–(3.11) that

$$u_\varepsilon \longrightarrow u_T \text{ in } C([0,T];L^1(\Omega)) \tag{3.12}$$

$$\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega |\nabla u_\varepsilon(t)|dt\,dx \ge \int_0^T \varphi(u(t))dt. \tag{3.13}$$

Since, by (3.9),

$$\begin{aligned}&\int_0^T\int_\Omega \left(\frac{1}{2}\,(u_\varepsilon)_t^2+\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2+|\nabla u_\varepsilon|\right)dt\,dx\\ &\qquad \le \int_0^T\int_\Omega\left(\frac{1}{2}\,v_t^2+\frac{\varepsilon}{2}\,|\nabla v|^2+|\nabla v|\right)dt\,dx,\end{aligned}$$

for all $v\in W^{1,2}([0,T];L^2(\Omega))\cap L^2(0,T];H_0^1(\Omega))$, by (3.11), (3.13), it follows that

$$u_T=\arg\min\left\{\int_0^T\left(\frac{1}{2}|v_t(t)|_H^2+\varphi(v(t))\right);v(0)=v(T)=u_0\right\}$$

and so u_T is the solution to the equation

$$\begin{aligned}&\frac{d^2u}{dt}\in\partial\varphi(u),\ \text{a.e. in } (0,T),\\ &u(0)=u(T)=u_0,\end{aligned}$$

as claimed.

This convergence result extends by density to all of $u_0\in\overline{D(\partial\varphi)}=H$ and this completes the proof. ∎

By Theorem 3.2, it follows that, *for ε sufficiently small and T large enough, the solution u_ε to the boundary value problem* (3.1) *approximates on* $(0,T)$ *the square root of the total flow* $V_{\frac{1}{2}}(t)$.

We note that the solution u_ε to equation (3.1) can be found by a standard convex minimization procedure by taking into account its equivalent variational formulation (3.9). We expect to give details and applications to image restoring problem in a forthcoming paper.

References

[1] Andreu, F., Caselles, V., Diaz, J.I., Mazon, J.M., Qualitative properties of th total variation flow, *J.Funct. Anal.*, 188 (2002), 516-547.

[2] Attouch, H., Buttazzo, G., Michaille, G., *Variational Analysis in Sobolev and BV Spaces*, MPS-SIAM Series, Philadelphia, 2006.

[3] Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden, 1976.

[4] Barbu, V., Sur un problème aux limites pour une classe des équations différentielles nonlinéaires abstraites du deuxième ordre en t, *C.R. Acad. Sci. Paris*, 274 (1972), 459-462.

[5] Barbu, V., A class of boundary problems for second order abstract differential equations, *J. Faculty Science Univ. Tokyo*, 19 (1972), 295-319.

[6] Barbu, V., *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer, New York, Dordrecht, London, 2009.

[7] Barbu, V., Nonlinear diffusion equations in image processing, *Revue Roumaine Math. Pures Appl.*, (2013).

[8] Barbu, T., Barbu, V., A PDE approach to image restoration problem with observation on a meager domain, *Nonlinear Anal. Real World Appl.*, 13 (2012), 1206-1215.

[9] Barbu, T., Barbu, V., Biga, V., Coca, D., A PDE approach to image denoising and restoration, *Nonlinear Anal. Real World Appl.*, 19 (2009), 1351-1361.

[10] Barbu, V., Röckner, M., Stochastic variational inequalities and applications to the total variation flow perturbed by linear multiplicative noise, (PDF Available) in *Archive Rational Mechanics and Analysis*, 209 (3) (2012), DOI: 10.1007/s00205-013-0632-x.

[11] Brezis, H., *Opérateurs maximaux monotones et semigroups de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[12] Brezis, H., Equations d'évolution du second ordre associées à des opérateurs monotones, *Israel J. Math.,* 12 (1972), 51-60.

[13] Calder, J., Mansouri, A., Yezzi, A., Image diffusion and sharpening via Sobolev gradient flows, *SIAM J. Imaging Sci.*, 3 (2010), 981-1014.

[14] Calder, J., Mansouri, A., Yezzi, A., New possibilities in image diffusion and sharpening via high-order Sobolev gradients, *J. Math. Imaging*, 40 (2011), 248-252.

[15] Chamballe, A., Lions, P.L., Image recovery via total variation minimization and related problems, *Numer. Math.*, 76 (1997), 167-182.

[16] Moroşanu, Gh., *Nonlinear Evolution Equations and Applications*, Springer, Dordrecht, 1984.

[17] Osher, S., Sole, A., Vese, L., Image decomposition and restoration using total variation minimization and the H^{-1}-norm, *Multiscale Model Simulation*, 1 (2003), 349-370.

[18] Rudin, L., Osher, S., Fatemi, E., Nonlinear total variation based noise removal algorithms, *Physica*, D. 60 (1992), 259-268.

[19] Weickert, J., *Anisotropic Diffusion in Image Processing*, B.G. Teubner, Stuttgart, 1998.

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