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Threshold Maximal Principles and Error Bound Properties

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Abstract: Some threshold versions of the Brezis-Browder ordering principle [Adv. Math., 21 (1976), 355-364] are proposed. Further applications of these to error bound properties are then given.

Keywords: Quasi-order, threshold Brezis-Browder principle, maximal point, pseudometric, complete pair, error bound, nonconvex minimization theorem.

MSC2010: 49J53, 47J30

1 Introduction

Let (X, d) be a *metric* space; and $\varphi : X \to R \cup \{\infty\}$ be some function. Denote, for each α, β in $R \cup \{-\infty, \infty\} = [-\infty, \infty]$ with $\alpha \leq \beta$:

$$[\alpha \perp \varphi \lhd \beta] = \{x \in X; \alpha \perp \varphi(x) \lhd \beta\};$$

here, $(\perp, \lhd)$ is any of the relations $\{<, \leq\}$. In particular, when $\alpha = -\infty$, $(\perp) = (\leq)$, the resulting set will be denoted as $[\varphi \lhd \beta]$ or $[\beta \rhd \varphi]$, where $(\rhd)$ is the *dual* of $(\lhd)$; and, if $\beta = \infty$, $(\lhd) = (\leq)$, the set in question is written as $[\alpha \perp \varphi]$ or $[\varphi \top \alpha]$, where $(\top)$ is the dual of $(\perp)$. Further, for each nonempty part Y of X, let $\text{dist}(., Y)$ stand for the *distance to* Y function:

$$\text{dist}(x, Y) = \inf\{d(x, y); y \in Y\},\ x \in X.$$

Given the couple α, β in $R^0_+ \cup \{\infty\} =]0, \infty]$ with $\alpha < \beta$, we say that φ has an *error bound* (modulo (α, β)) property, when

(a01) $S := [\varphi \leq \alpha]$ is nonempty

(a02) $\text{dist}(u, S) \leq \varphi(u) - \alpha$, for each $u \in [\alpha \leq \varphi < \beta]$.

When $\beta = \infty$, this will be referred to as a *global* (modulo α) error bound property; and, if $\beta \in]\alpha, \infty[$ is generic, we talk about a *local* (modulo α) error bound property.

The study of such problems has important applications to *sensitivity* analysis of mathematical programming and *convergence* analysis of algorithms in numerical methods. The relevant achievements in the area for the continuous/convex case were discussed in the survey paper by Lewis and Pang [19]. Concerning the discontinuous/nonconvex case, we must note the basic contribution due to Ng and Zheng [22]; further extensions of these were obtained by Wu and Ye [32]. The following result obtained in the quoted paper is our starting point.

Theorem 1.1. *Suppose that (in addition),* (X, d) *is complete and*

(11-i) φ *is proper:* $\mathrm{Dom}(\varphi) := \{x \in X : \varphi(x) < \infty\} \neq \emptyset$

(11-ii) φ *is d-lsc over* X*:* $\liminf_n \varphi(x_n) \geq \varphi(x)$*, whenever* $x_n \xrightarrow{d} x$.

Further, take some couple α, β *in* $R^0_+ \cup \{\infty\}$ *with*

(11-iii) $\alpha < \beta$ *and* $[\varphi < \beta] \neq \emptyset$

(11-iv) for each $x \in [\alpha < \varphi < \beta]$ *there exists* $y \in [\alpha \leq \varphi < \beta]$ *with* $0 < d(x, y) \leq \varphi(x) - \varphi(y)$.

Then, φ *has an error bound (modulo* (α, β)*) property.*

The original proof of this result is based on the Nonconvex Minimization Theorem in Takahashi [24]; see also Hamel [15]. Further, in his survey paper, Aze [1] proposed a different proof of the same, by means of Ekeland's variational principle [13]. It is our aim in the following to provide a new proof of the statement in question, with precise localizing formulae. The basic tool for this is a *threshold* variant of the Brezis-Browder ordering principle [4]; and the framework to be considered is the *almost metrical* one. Further aspects will be delineated elsewhere.

2 Dependent Choice Principles

Throughout this exposition, the axiomatic system in use is Zermelo-Fraenkel's (abbreviated: ZF); cf. Cohen [9, Ch 2]. The notations and basic facts to be considered are standard; some important ones are described below.

(A) Let X be a nonempty set. By a *relation* over X, we mean any (nonempty) part $\mathcal{R} \subseteq X \times X$; then, $(X, \mathcal{R})$ will be referred to as a *relational structure.* For simplicity, we sometimes write $(x, y) \in \mathcal{R}$ as $x\mathcal{R}y$. Note that $\mathcal{R}$ may be regarded as a mapping between X and $\exp[X]$ (=the class of all subsets in X). In fact, denote

$X(x, \mathcal{R}) = \{y \in X; x\mathcal{R}y\}$ (the *section* of $\mathcal{R}$ through x), $x \in X$;

then, the desired mapping representation is $(\mathcal{R}(x) = X(x, \mathcal{R}); x \in X)$. A basic example of such object is

$\mathcal{I} = \{(x, x); x \in X\}$ [the *identical relation* over X].

Given the relations $\mathcal{R}$, $\mathcal{S}$ over X, define their *product* $\mathcal{R} \circ \mathcal{S}$ as

$(x, z) \in \mathcal{R} \circ \mathcal{S}$, if there exists $y \in X$ with $(x, y) \in \mathcal{R}$, $(y, z) \in \mathcal{S}$.

Also, for each relation $\mathcal{R}$ over X, denote

$\mathcal{R}^{-1} = \{(x, y) \in X \times X; (y, x) \in \mathcal{R}\}$ (the *inverse* of $\mathcal{R}$).

Finally, given the relations $\mathcal{R}$ and $\mathcal{S}$ on X, let us say that $\mathcal{R}$ is *coarser* than $\mathcal{S}$ (or, equivalently: $\mathcal{S}$ is *finer* than $\mathcal{R}$), provided

$\mathcal{R} \subseteq \mathcal{S}$; i.e.: $x\mathcal{R}y$ implies $x\mathcal{S}y$.

Given a relation $\mathcal{R}$ on X, the following properties are to be discussed here:

(P1) $\mathcal{R}$ is *reflexive*: $\mathcal{I} \subseteq \mathcal{R}$

(P2) $\mathcal{R}$ is *irreflexive*: $\mathcal{R} \cap \mathcal{I} = \emptyset$

(P3) $\mathcal{R}$ is *transitive*: $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$

(P4) $\mathcal{R}$ is *symmetric*: $\mathcal{R}^{-1} = \mathcal{R}$

(P5) $\mathcal{R}$ is *antisymmetric*: $\mathcal{R}^{-1} \cap \mathcal{R} \subseteq \mathcal{I}$.

This yields the classes of relations to be used; the following ones are important for our developments:

(C0) $\mathcal{R}$ is *amorphous* (i.e.: it has no specific properties)

(C1) $\mathcal{R}$ is a *quasi-order* (reflexive and transitive)

(C2) $\mathcal{R}$ is a *strict order* (irreflexive and transitive)

(C3) $\mathcal{R}$ is an *equivalence* (reflexive, transitive, symmetric)

(C4) $\mathcal{R}$ is a *(partial) order* (reflexive, transitive, antisymmetric)

(C5) $\mathcal{R}$ is the *trivial relation* (i.e.: $\mathcal{R} = X \times X$).

(B) A basic example of relational structure is to be constructed as below. Let $N := \{0, 1, ...\}$ denote the set of *natural* numbers. Technically speaking, the basic (algebraic and order) structures over N may be obtained by means of the *(immediate) successor* function suc : $N \to N$, and the following Peano properties (deductible in our axiomatic system (ZF)):

(pea-1) $(0 \in N$ and$)$ $0 \notin \text{suc}(N)$

(pea-2) suc(.) is injective ($\text{suc}(n) = \text{suc}(m)$ implies $n = m$)

(pea-3) if $M \subseteq N$ fulfills $[0 \in M]$ and $[\text{suc}(M) \subseteq M]$, then $M = N$.

[Note that, in the absence of our axiomatic setting, these properties become the well known *Peano axioms*, as described in Halmos [14, Ch 12]; we do not give details]. In fact, starting from these properties, one may construct, in a recurrent way, an *addition* $(a, b) \mapsto a + b$ over N, according to

$(\forall m \in N)$: $m + 0 = m$; $m + \text{suc}(n) = \text{suc}(m + n)$.

This, in turn, makes possible the introduction of a relation $(\leq)$ over N, as

$(m, n \in N)$: $m \leq n$ iff $m + p = n$, for some $p \in N$.

Concerning the properties of this structure, the most important one writes

$(N, \leq)$ is well ordered:
any (nonempty) subset of N has a first element;

hence, in particular, $(N, \leq)$ is (partially) ordered. Denote, for simplicity

$N(r, \leq) = \{n \in N; r \leq n\} = \{r, r + 1, ..., \}$, $r \in N$,
$N(r, >) = \{n \in N; r > n\} = \{0, ..., r - 1\}$, $r \in N(1, \leq)$;

the latter one is referred to as the *initial interval* (in N) induced by r. Any set P with $P \sim N$ (in the sense: there exists a *bijection* from P to N) will be referred to as *effectively denumerable*. In addition, given some natural number $n \geq 1$, any set Q with $Q \sim N(n, >)$ will be said to be *n-finite*; when n is generic here, we say that Q is *finite*. Finally, the (nonempty) set Y is called (at most) *denumerable* iff it is either effectively denumerable or finite.

Let X be a nonempty set. By a *sequence* in X, we mean any mapping $x \in \mathcal{F}(N, X)$; where, as already precise, $N := \{0, 1, ...\}$ is the set of *natural* numbers. [Here, for each couple A, B of nonempty sets, $\mathcal{F}(A, B)$ denotes the class of all functions from A to B; when $A = B$, we write $\mathcal{F}(A)$ in place of $\mathcal{F}(A, A)$]. For simplicity reasons, it will be useful to denote it as $(x(n); n \geq 0)$, or $(x_n; n \geq 0)$; moreover, when no confusion can arise, we further simplify this notation as $(x(n))$ or (x_n), respectively. Given such an object, $(x_n; n \geq 0)$, any sequence $(y_n := x_{i(n)}; n \geq 0)$ with $(i(n); n \geq 0)$ being strictly ascending [hence, $i(n) \to \infty$ as $n \to \infty$] will be referred to as a *subsequence* of $(x_n; n \geq 0)$. Note that, under such a convention, the relation "subsequence of" is transitive; i.e.:

(z_n)=subsequence of (y_n) and (y_n)=subsequence of (x_n)
imply (z_n)=subsequence of (x_n).

(C) Remember that, an outstanding part of (ZF) is the *Axiom of Choice* (abbreviated: AC); which, in a convenient manner, may be written as

(AC) For each couple (J, X) of nonempty sets and each function
$F : J \to \exp(X)$, there exists a (selective) function
$f : J \to X$, with $f(\nu) \in F(\nu)$, for each $\nu \in J$.

(Here, $\exp(X)$ stands for the class of all nonempty elements in $\exp[X]$). Sometimes, when the ambient set X is endowed with denumerable type structures, the case of $J = N$ will suffice for all choice reasonings; and the existence of such a selective function may be determined by using a weaker form of (AC), called: *Dependent Choice* principle (in short: DC). Call the relation $\mathcal{R}$ over X, *proper* when

$(X(x, \mathcal{R}) =)\mathcal{R}(x)$ is nonempty, for each $x \in X$.

Then, $\mathcal{R}$ is to be viewed as a mapping between X and $\exp(X)$; and the couple $(X, \mathcal{R})$ will be referred to as a *proper relational structure.* Further, given $a \in X$, let us say that the sequence $(x_n; n \geq 0)$ in X is $(a; \mathcal{R})$*-iterative*, provided

$x_0 = a$ and $x_n \mathcal{R} x_{n+1}$ (i.e.: $x_{n+1} \in \mathcal{R}(x_n)$), for all n.

Proposition 2.1. *Let the relational structure $(X, \mathcal{R})$ be proper. Then, for each $a \in X$ there is at least an $(a, \mathcal{R})$-iterative sequence in X.*

This principle – proposed, independently, by Bernays [2] and Tarski [25] – is deductible from (AC), but not conversely; cf. Wolk [30]. Moreover, by the developments in Moskhovakis [21, Ch 8], and Schechter [23, Ch 6], the *reduced system* (ZF-AC+DC) it comprehensive enough so as to cover the "usual" mathematics; see also Moore [20, Appendix 2].

Let $(\mathcal{R}_n; n \geq 0)$ be a sequence of relations on X. Given $a \in X$, let us say that the sequence $(x_n; n \geq 0)$ in X is $(a; (\mathcal{R}_n; n \geq 0))$*-iterative*, provided

$x_0 = a$ and $x_n \mathcal{R}_n x_{n+1}$ (i.e.: $x_{n+1} \in \mathcal{R}_n(x_n)$), for all n.

The following *Diagonal Dependent Choice* principle (in short: DDC) is available.

Proposition 2.2. *Let $(\mathcal{R}_n; n \geq 0)$ be a sequence of proper relations on X. Then, for each $a \in X$ there exists at least one $(a; (\mathcal{R}_n; n \geq 0))$-iterative sequence in X.*

Clearly, (DDC) includes (DC); to which it reduces when $(\mathcal{R}_n; n \geq 0)$ is constant. The reciprocal of this is also true. In fact, letting the premises of (DDC) hold, put $P = N \times X$; and let $\mathcal{S}$ be the relation over P introduced as

(b01) $\mathcal{S}(i, x) = \{i + 1\} \times \mathcal{R}_i(x), \ (i, x) \in P.$

It will suffice applying (DC) to $(P, \mathcal{S})$ and $b := (0, a) \in P$ to get the conclusion in our statement; we do not give details.

Summing up, (DDC) is provable in (ZF-AC+DC). This is valid as well for its variant, referred to as: *Selected Dependent Choice* principle (in short: SDC).

Proposition 2.3. *Let the map $F : N \to \exp(X)$ and the relation $\mathcal{R}$ over X fulfill*

(b02) $(\forall n \in N)$: $\mathcal{R}(x) \cap F(n + 1) \neq \emptyset$, for all $x \in F(n)$.

Then, for each $a \in F(0)$ there exists a sequence $(x(n); n \geq 0)$ in X, with

$$x(0) = a, \ x(n) \in F(n), \ x(n + 1) \in \mathcal{R}(x(n)), \ \ \forall n.$$

As before, (SDC) $\Longrightarrow$ (DC) ($\Longleftrightarrow$ (DDC)); just take $(F(n) = X; n \geq 0)$. But, the reciprocal is also true, in the sense: (DDC) $\Longrightarrow$ (SDC). This follows from

Proof. **(Proposition 2.3)** Let the premises of (SDC) be true. Define a sequence of relations $(\mathcal{R}_n; n \geq 0)$ over X as: for each $n \geq 0$,

(b03) $\mathcal{R}_n(x) = \mathcal{R}(x) \cap F(n + 1)$, if $x \in F(n)$,
$\mathcal{R}_n(x) = \{x\}$, otherwise $(x \in X \setminus F(n))$.

Clearly, $\mathcal{R}_n$ is proper, for all $n \geq 0$. So, by (DDC), it follows that, for the starting $a \in F(0)$, there exists an $(a, (R_n; n \geq 0))$-iterative sequence $(x(n); n \geq 0)$ in X. This, along with the very definition above, gives all desired conclusions. □

In particular, when $\mathcal{R} = X \times X$, the regularity condition imposed in (SDC) holds. The corresponding variant of the underlying statement is just (AC(N)) (=the *Denumerable Axiom of Choice*). Precisely, we have

Proposition 2.4. *Let $F : N \to \exp(X)$ be a function. Then, for each $a \in F(0)$ there exists a function $f : N \to X$ with $f(0) = a$ and $f(n) \in F(n)$, $\forall n \in N$.*

As a consequence of the above facts, (DC) $\Longrightarrow$ (AC(N)) in (ZF-AC). A direct verification of this is obtainable by taking $Q = N \times X$ and introducing the relation $\mathcal{R}$ over it, according to:

(b04) $\mathcal{R}(n, x) = \{n + 1\} \times F(n + 1), \ n \geq 0, \, x \in X;$

we do not give details. The reciprocal of the written inclusion is not true; see Moskhovakis [21, Ch 8, Sect 8.25] for details.

3 Conv-Cauchy structures

Let X be a nonempty set; and $\mathcal{S}(X)$, stand for the class of all *sequences* (x_n) in X. By a (sequential) *convergence structure* on X we mean any part $\mathcal{C}$ of $\mathcal{S}(X) \times X$, with the properties (cf. Kasahara [18]):

(conv-1) $\mathcal{C}$ is *hereditary*:
$((x_n); x) \in \mathcal{C} \Longrightarrow ((y_n); x) \in \mathcal{C}$, for each subsequence (y_n) of (x_n)

(conv-2) $\mathcal{C}$ is *reflexive*: for each $u \in X$,
the constant sequence $(x_n = u; n \geq 0)$ fulfills $((x_n); u) \in \mathcal{C}$.

For each sequence (x_n) in $\mathcal{S}(X)$ and each $x \in X$, we write $((x_n); x) \in \mathcal{C}$ as $x_n \xrightarrow{\mathcal{C}} x$; this reads:

(x_n), $\mathcal{C}$-converges to x (also referred to as: x is the *$\mathcal{C}$-limit* of (x_n)).

The set of all such x is denoted $\lim_n(x_n)$; when it is nonempty, we say that (x_n) is *$\mathcal{C}$-convergent*. The following condition is to be optionally considered here:

(conv-3) $\mathcal{C}$ is *separated*: $\lim_n(x_n)$ is an *asingleton*, for each sequence (x_n);

when it holds, $x_n \xrightarrow{\mathcal{C}} z$ also writes $\lim_n(x_n) = z$. [Here, $Y \in \exp[X]$ is called *almost-singleton* (in short: *asingleton*) provided $y_1, y_2 \in Y$ implies $y_1 = y_2$; and *singleton* if, in addition, $Y \in \exp(X)$; note that in this case $Y = \{y\}$, for some $y \in X$].

Further, by a (sequential) *Cauchy structure* on X we shall mean any part $\mathcal{H}$ of $\mathcal{S}(X)$ with (cf. Turinici [28])

(Cauchy-1) $\mathcal{H}$ is *hereditary*:
$(x_n) \in \mathcal{H} \Longrightarrow (y_n) \in \mathcal{H}$, for each subsequence (y_n) of (x_n)

(Cauchy-2) $\mathcal{H}$ is *reflexive*: for each $u \in X$,
the constant sequence $(x_n = u; n \geq 0)$ fulfills $(x_n) \in \mathcal{H}$.

Each element of $\mathcal{H}$ will be referred to as a *$\mathcal{H}$-Cauchy* sequence in X.

Finally, given the couple $(\mathcal{C}, \mathcal{H})$ as before, we shall say that it is a *conv-Cauchy structure* on X. The natural conditions about the conv-Cauchy structure $(\mathcal{C}, \mathcal{H})$ to be considered here are

(CC-1) $(\mathcal{C}, \mathcal{H})$ is *regular*: each $\mathcal{C}$-convergent sequence in X is $\mathcal{H}$-Cauchy

(CC-2) $(\mathcal{C}, \mathcal{H})$ is *complete*: each $\mathcal{H}$-Cauchy sequence in X is $\mathcal{C}$-convergent.

A standard way of introducing such structures is the *(pseudo) metrical* one. By a *pseudometric* over X we mean any mapping $d : X \times X \to R_+$. Fix such an object; endowed with the properties (am-1) d is *reflexive*: $d(x,x) = 0, \ \forall x \in X$

(am-2) d is *sufficient*: $d(x,y) = 0$ implies $x = y$

(am-3) d is *triangular*: $d(x,z) \le d(x,y) + d(y,z)$, $\forall x,y,z \in X$.

Note that, if in addition,

(am-4) d is *symmetric*: $d(x,y) = d(y,x), \ \forall x,y \in X$,

then d is just a *metric* on X; and (X,d) is referred to as a *metric space*. But, in the following, the symmetry condition is not considered. In this case, the map $d(.,.)$ is referred to as an *almost metric*; and (X,d), as an *almost metric space*.

Given the sequence (x_n) in X and the point $x \in X$, we say that (x_n), *d-converges* to x (written as: $x_n \xrightarrow{d} x$) provided $d(x_n,x) \to 0$ as $n \to \infty$; i.e.,

(c01) $\forall \varepsilon > 0$, $\exists i = i(\varepsilon)$: $i \le n \Longrightarrow d(x_n,x) < \varepsilon$.

By this very definition, we have the hereditary and reflexive properties:

(d-conv-1) ($(\xrightarrow{d})$ is *hereditary*)
$x_n \xrightarrow{d} x$ implies $y_n \xrightarrow{d} x$, for each subsequence (y_n) of (x_n)

(d-conv-2) ($(\xrightarrow{d})$ is *reflexive*)
$(\forall u \in X)$: the constant sequence $(x_n = u; n \ge 0)$ fulfills $x_n \xrightarrow{d} u$.

As a consequence, $(\xrightarrow{d})$ is a sequential convergence on X. The set of all such limit points of (x_n) will be denoted $\lim_n(x_n)$; if it is nonempty, then (x_n) is called *d-convergent*. The following condition about this structure is to be considered

(d-conv-3) $(\xrightarrow{d})$ is *separated* (referred to as d is *separated*):
$\lim_n(x_n)$ is an asingleton, for each sequence (x_n) in X.

By the conditions imposed upon d, this requirement is not in general fulfilled. However, when d is symmetric (hence, a *metric* on X), the separated property holds.

Further, call the sequence (x_n), *d-Cauchy* when

(c02) $\lim_n \sup\{d(x_n,x_{n+m}); m \ge 1\} = 0$; that is:
$\forall \varepsilon > 0$, $\exists j = j(\varepsilon)$: $j \le n < p \Longrightarrow d(x_n,x_p) < \varepsilon$;

the class of all these will be denoted as $Cauchy(X,d)$. As before, we have the hereditary and reflexive properties

(d-Cauchy-1) ($Cauchy(X, d)$ is *hereditary*)
(x_n) is d-Cauchy implies (y_n) is d-Cauchy,
for each subsequence (y_n) of (x_n)

(d-Cauchy-2) ($Cauchy(X, d)$ is *reflexive*)
$(\forall u \in X)$: the constant sequence $(x_n = u; n \geq 0)$ is d-Cauchy;

hence, $Cauchy)(X, d)$ is a Cauchy structure on X.

Now – according to the general setting – call the couple $((\xrightarrow{d}), Cauchy(X, d))$, a *conv-Cauchy structure* induced by d. The following regularity conditions about this structure are to be considered

(CC-1) d is *regular*: each d-convergent sequence in X is d-Cauchy

(CC-2) d is *complete*: each d-Cauchy sequence in X is d-convergent.

Generally, none of these is holding under our setting; however, the former one is retainable if (in addition) d is *symmetric* (hence, a metric on X).

Let again (X, d) be an almost metric space. In the following, some classes of sequences (related to the d-Cauchy ones) are introduced.

I) Given the sequence (x_n), call it

(c03) *d-asymptotic*, provided $\lim_n d(x_n, x_{n+1}) = 0$

(c04) *d-global-asymptotic*, if $\lim_n d(x_n, x_{n+i}) = 0$, $\forall i \in N(1, \leq)$.

Clearly, we have (for each sequence (x_n) in X):

d-Cauchy $\Longrightarrow$ d-global-asymptotic $\Longrightarrow$ d-asymptotic.

Moreover, in our context, the second inclusion may be reversed; as results from

Proposition 3.1. *We have, for each sequence (x_n) in X,*

(x_n) is d-asymptotic iff (x_n) is d-global-asymptotic.

Proof. Let $i \in N(1, \leq)$ be arbitrary fixed. By the triangular inequality,

$d(x_n, x_{n+i}) \leq \rho_n + ... + \rho_{n+i-1}$, $\forall n$, where $(\rho_n = d(x_n, x_{n+1}); n \geq 0)$.

By the imposed hypothesis, the right member of this relation tends to zero as $n \to \infty$; wherefrom, all is clear. □

II) Let us say that the sequence (x_n) is *d-strongly-asymptotic*, when

$\sum_n d(x_n, x_{n+1}) (= d(x_0, x_1) + d(x_1, x_2) + ...) < \infty$.

The relationship with the d-Cauchy property is to be clarified in

Proposition 3.2. *For each sequence* (x_n) *in* X*, one has*

(x_n) *is* d*-strongly-asymptotic implies* (x_n) *is* d*-Cauchy.*

The reciprocal is not in general true.

Proof. By hypothesis, we have under the notation $(\rho_n := d(x_n, x_{n+1}); n \geq 0)$,

$$\sigma_n := \sum_{k \geq n} \rho_k \to 0 \text{ as } n \to \infty.$$

In this case, for each $n, m \geq 0$ with $n < m$,

$$d(x_n, x_m) \leq \rho_n + \rho_{n+1} + ... + \rho_{m-1} \leq \sigma_n;$$

and this, along with the property of (σ_n), gives the needed conclusion. □

Note that if the triangular property is not accepted, the above inclusion is not in general true. This is very well observed at the level of generalized metrical structures, introduced as in Branciari [3]; we do not give details.

4 Threshold maximal principles

Let M be some nonempty set, and $(\leq)$ be a *quasi-order* over it; the couple $(M, \leq)$ will be referred to as a *quasi-ordered structure.*

(A) Take a function $\psi : M \to R \cup \{-\infty, \infty\}$. Call the point $z \in M$, $(\leq, \psi)$-*maximal* when: $w \in M$ and $z \leq w$ imply $\psi(z) = \psi(w)$; or, equivalently: ψ is constant on $M(z, \leq) := \{x \in M; z \leq x\}$; the class of all these will be denoted $\max(\leq, \psi; M)$. In the following, sufficient conditions are given under which

$\max(\leq, \psi; M)$ is (nonempty and) *cofinal* in M:
for each $u \in M$, there exists $v \in \max(\leq, \psi; M)$ with $u \leq v$.

The basic one in this series writes

(d01) ψ is $(\leq)$-decreasing $(x \leq y \Longrightarrow \psi(x) \geq \psi(y))$.

For the remaining one, we need a convention. Let (x_n) be an *ascending* sequence in M $(x_i \leq x_j$ when $i \leq j)$. We say that $u \in M$ is an *upper bound* of it, provided

$x_n \leq u$, for all n (written as: $(x_n) \leq u$).

Denote $\text{ubd}(x_n) = \{u \in M; (x_n) \leq u\}$; when this set is nonempty, we say that (x_n) is *bounded above.* The announced condition may now be written as

(d02) $(M, \leq)$ is sequentially inductive:
each ascending sequence has an upper bound (modulo $(\leq)$).

We are now in position to state an appropriate answer to our question (referred to as: Cârjă-Ursescu ordering principle; in short: CU).

Theorem 4.1. *Let the quasi-ordered structure* $(M, \leq)$ *and the function* $\psi : M \to R \cup \{-\infty, \infty\}$ *be such that*

ψ *is* $(\leq)$*-decreasing and* $(M, \leq)$ *is sequentially inductive.*

Then, $\max(\leq, \psi; M)$ *is (nonempty and) cofinal in* M*.*

For a direct argument, we refer to the 2007 monograph by Cârjă et al [7, Ch 2, Sect 2.1]. However, the idea of proof goes back to Cârjă and Ursescu [8]; this, among others, motivated our convention.

In particular, when

(d03) $\psi(M) \subseteq R$ and $\inf[\psi(M)] > -\infty$

the Cârjă-Ursescu maximal principle (CU) reduces to the 1976 ordering principle in Brezis and Browder [4] (in short: BB).

Theorem 4.2. *Let the quasi-ordered structure* $(M, \leq)$ *and the function* $\psi : M \to R$ *be such that*

(42-i) ψ *is* $(\leq)$*-decreasing and bounded from below (*$\inf \psi(M) > -\infty$*)*

(42-ii) $(M, \leq)$ *is sequentially inductive.*

Then, $\max(\leq, \psi; M)$ *is (nonempty and) cofinal in* M*.*

Note that, from a logical perspective, there is no distinction between (CU) and (BB); because (cf. our last section)

(d04) (BB) $\Longrightarrow$ (CU); hence; (BB) $\Longleftrightarrow$ (CU).

Moreover, (cf. Zhu and Li [33]), $(R \cup \{-\infty, \infty\}, \geq)$ may be substituted by a separable ordered structure $(S, \leq)$ without altering the conclusion of (CU); see also the 2006 paper by Turinici [27].

(B) In the following, we shall establish global/local reciprocals of the maximal principle (CU) (or, equivalently, the maximal principle (BB)). The former of these (referred to as Global Converse of (CU); in short: (CU-c-glo)) writes

Theorem 4.3. *Let the quasi-ordered structure* $(M, \leq)$ *and the* $(\leq)$*-decreasing function* $\psi : M \to R \cup \{-\infty, \infty\}$ *be such that*

$\max(\leq, \psi; M)$ *is empty:*
for each $x \in M$, *there exists* $y \in M$ *with* $x \leq y$, $\psi(x) > \psi(y)$.

Then, necessarily,

$(M, \leq)$ *is not sequentially inductive: there exists*
an ascending sequence in M *without upper bound (in* M*).*

Proof. Evident, by the very definitions involved. □

We are now passing to the local version of the same principles. This will necessitate some conventions.

Let P be some nonempty part of M. Call $z \in P$, $(\leq, \psi)$-*maximal* (in P) when

$w \in P, z \leq w$ imply $\psi(z) = \psi(w)$; or, equivalently:
ψ is constant over $P(z, \leq) := M(z, \leq) \cap P$.

The class of all these will be denoted $\max(\leq, \psi; P)$. Note that, by the very definition of the involved elements,

$P \cap \max(\leq, \psi; M) \subseteq \max(\leq, \psi; P)$;
but the converse inclusion is not in general true.

Further, let (x_n) be an ascending sequence in P. Denote

$\mathrm{ubd}((x_n); P) = \mathrm{ubd}(x_n) \cap P$ (the upper bounds of (x_n) in P);

if such points exist, we say that (x_n) is bounded above in P. An alternate situation is represented by

(d05) $\mathrm{ubd}(x_n) \neq \emptyset$ and $\mathrm{ubd}((x_n); P) = \emptyset$; hence $\emptyset \neq \mathrm{ubd}(x_n) \subseteq M \setminus P$;

we then say that (x_n) is a *threshold* (modulo P) ascending sequence in P (with origin at x_0). The announced answer to the posed question (referred to as Local Converse of (CU); in short: (CU-c-loc)) may be written as follows.

Theorem 4.4. *Let the quasi-ordered structure* $(M, \leq)$, *the* $(\leq)$*-decreasing function* $\psi : M \to R \cup \{-\infty, \infty\}$, *and the nonempty subset* P *of* M *be such that*

(44-i) $\max(\leq, \psi; P)$ *is empty:*
for each $x \in P$ *there exists* $y \in P$ *with* $x \leq y$, $\psi(x) > \psi(y)$

(44-ii) $(P, \leq)$ *is sequentially inductive in* M*:*
each ascending sequence in P *is bounded above in* M.

Then, for each $u \in P$, there exists a threshold (modulo P) ascending sequence (x_n) in P with $x_0 = u$.

Proof. Let $u \in P$ be arbitrary fixed. For the moment, we have that

(44-a) $\max(\leq, \psi; P(u, \leq)) \subseteq \max(\leq, \psi; P)$.

In fact, letting $y \in \max(\leq, \psi; P(u, \leq))$ be arbitrary fixed, we have

$$y \leq w \in P \Longrightarrow y \leq w \in P(u, \leq) \Longrightarrow \psi(y) = \psi(w);$$

whence $y \in \max(\leq; \psi; P)$. Combining with (44-i), one gets

(44-b) $\max(\leq, \psi; P(u, \leq)) = \emptyset$;
so, (43-i) holds with $P(u, \leq)$ in place of M.

This, along with (CU-c-glo) (with respect to $P(u, \leq)$, $(\leq)$, and ψ), tells us that

$P(u, \leq)$ is not sequentially inductive: there exists an ascending sequence (y_n) in $P(u, \leq)$ with $\text{ubd}((y_n); P(u, \leq)) = \emptyset$; and this, via $\text{ubd}((y_n); P) \subseteq \text{ubd}((y_n); P(u, \leq))$, gives $\text{ubd}((y_n); P) = \emptyset$.

Combining with condition (44-ii), gives

$\text{ubd}((y_n); M \setminus P)$ is nonempty;
wherefrom, (y_n) is a threshold (modulo P) sequence in $P(u, \leq)$.

It suffices now putting $(x_0 = u, x_n = y_{n-1}, n \geq 1)$ to get the desired sequence. □

Note that further extensions of this statement are available if $(R \cup \{-\infty, \infty\}, \geq)$ is substituted by an "abstract" separable ordered structure $(S, \leq)$; cf. Zhu and Li [33]. We shall discuss these facts elsewhere.

5 Main results

Let X be some nonempty set. By a *(generalized) pseudometric* over X we shall mean any map $e : X \times X \to R_+ \cup \{\infty\}$. Fix such an object; endowed with

(am-1) e is *reflexive* $[e(x, x) = 0, \forall x \in X]$,

(am-2) e is *sufficient* $[e(x, y) = 0 \Longrightarrow x = y]$,

(am-3) e is *triangular* $[e(x, z) \leq e(x, y) + e(y, z), \forall x, y, z \in X]$.

Note that, if in addition,

(am-4) e is *symmetric*: $e(x,y) = e(y,x), \ \forall x, y \in X$,

then $e(.,.)$ is just a *(generalized) metric* on X; and (X, e) is referred to as a *(generalized) metric space*. But, in the following, the symmetry condition is not considered. In this case, the map $e(.,.)$ is referred to as an *almost metric* on X; and (X, d), as an *almost metric space*.

Further, let $\varphi : X \to R \cup \{\infty\}$ be some function with

(e01) φ is proper: $\mathrm{Dom}(\varphi) := [\varphi < \infty]$ is nonempty.

In the following, we shall discuss a basic regularity condition involving our data. Some conventions and auxiliary facts are in order.

Let us introduce an *e-convergence* structure over X as

$x_n \xrightarrow{e} x$ iff $e(x_n, x) \to 0$ as $n \to \infty$; that is
$\forall \delta > 0$, $\exists n(\delta)$, such that $n(\delta) \leq p$ implies $e(x_p, x) < \delta$.

This will be also referred to as: x is the e-limit of (x_n); or, equivalently: (x_n), e-converges towards x. The class of all these points will be denoted as $\lim_n(x_n)$; when it is nonempty, we say that (x_n) is *e-convergent*.

Further, let the *e-Cauchy* property for (x_n) be the usual one:

$e(x_n, x_m) \to 0$ as $n, m \to \infty$ with $n \leq m$; that is
$\forall \delta > 0, \exists n(\delta)$, such that $n(\delta) \leq p \leq q \Longrightarrow e(x_p, x_q) < \delta$;

the class of all these will be denoted $Cauchy(X, e)$.

Finally, call (x_n), *e-strongly-asymptotic* (in short: *e-strasy*) when:

the series $\sum_n e(x_n, x_{n+1})$ converges (in R);

the class of all these will be denoted $strasy(X, e)$.

By the triangular property of e, we have (see above):

(for each (ascending) sequence (x_n) in X):
(x_n) is e-strasy implies (x_n) is e-Cauchy.

The converse is not in general true. Nevertheless, in many conditions involving *all* such objects, a substitution between these is possible. A concrete example is to be constructed below.

Let us introduce the regularity condition

(e02) (e, φ) is *weakly db-complete*: for each e-strasy sequence
(x_n) in $\mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$=descending bounded,
there exists $x \in X$ with $x_n \xrightarrow{e} x$ and $\lim_n \varphi(x_n) \geq \varphi(x)$.

By the generic property above, it is implied by its (stronger) counterpart

(e03) (e, φ) is *db-complete*: for each e-Cauchy sequence
(x_n) in $\text{Dom}(\varphi)$ with $(\varphi(x_n))$=descending bounded,
there exists $x \in X$ with $x_n \xrightarrow{e} x$ and $\lim_n \varphi(x_n) \geq \varphi(x)$.

The remarkable fact to be added is that the reciprocal inclusion also holds.

Proposition 5.1. *Under the precise conventions, we have*

(e02) $\Longrightarrow$ *(e03); hence, (e02)* $\Longleftrightarrow$ *(e03).*

Proof. Assume that (e02) holds; and let (x_n) be an e-Cauchy sequence in $\text{Dom}(\varphi)$ with $(\varphi(x_n))$=descending bounded. By the very definition of this property, there must be an e-strasy subsequence $(y_n = x_{i(n)})$ of (x_n) with $(\varphi(y_n))$=descending bounded. This, along with (e02), yields an element $y \in X$ fulfilling $y_n \xrightarrow{e} y$ and $\lim_n \varphi(y_n) \geq \varphi(y)$. It is now clear that (by the choice of (x_n)) the point y has all desired in (e03) properties. □

(B) Further, let $(\leq)$ stand for the quasi-order on X

(e04) $(x, y \in X)$: $x \leq y$ iff $e(x, y) + \varphi(y) \leq \varphi(x)$.

Its restriction to $M := \text{Dom}(\varphi)$ (also denoted as $(\leq)$) is antisymmetric; hence, an order on M; as usual, denote by $(<)$ its associated strict order. Let again φ stand for the restriction of the initial function φ to M. By the reflexive-sufficient properties (of e), φ is $(\leq)$-strictly-decreasing; i.e.,

(e05) $x, y \in M, x < y \Longrightarrow \varphi(x) > \varphi(y)$.

An interesting property of $(M, \leq)$ (to be used further) is that involving the boundedness of certain ascending sequences in M. Precisely, we have:

Proposition 5.2. *Suppose that [φ is proper and] (e02)/(e03) holds. Then*

$(M, \leq; \varphi)$ *is limit sequentially admissible; i.e.,*
each ascending sequence (x_n) *in* M *with* $(\varphi(x_n))$*=bounded (below)*
is e-convergent and $\lim_n(x_n) \cap \text{ubd}(x_n; M) \neq \emptyset$*; hence,* $\text{ubd}(x_n; M) \neq \emptyset$.

Proof. Let (x_n) be an ascending sequence in M with $(\varphi(x_n))$=bounded (below). In particular, this yields

(e06) $e(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m)$, whenever $n \leq m$.

The sequence $(\varphi(x_n))$ is therefore descending bounded (below); hence a Cauchy one; wherefrom (again combining with (e06)) (x_n) is an e-Cauchy sequence in M. Putting these together, it follows via (e02)/(e03), that there exists $y \in X$ with

(e07) $x_n \xrightarrow{e} y$ and $\varphi(x_n) \geq \varphi(y)$, for all n.

This firstly gives $y \in M$; since (x_n) is a sequence in M. Secondly, fix some rank n. By (e06) (and the triangular property of e)

$$e(x_n, y) \leq e(x_n, x_m) + e(x_m, y) \leq \varphi(x_n) - \varphi(x_m) + e(x_m, y),\ \forall m \geq n.$$

Passing to limit as $m \to \infty$ yields (by (e07))

$e(x_n, y) \leq \varphi(x_n) - \varphi(y)$ (i.e.: $x_n \leq y$) for all n;

and the conclusion follows. □

(C) We may now state our main results of this exposition. Letting some nonempty part P of $M := \mathrm{Dom}(\varphi)$ and some $\alpha \in R$, define the property

(P, α) is *normal*: $P \subseteq [\varphi \geq \alpha]$ (that is: $\varphi(x) \geq \alpha$, $\forall x \in P$)

Denote for simplicity

$$P_0 := P \cap [\varphi > \alpha],\ P_\infty := P \cap [\varphi = \alpha].$$

By the normality condition, $P_0 \cup P_\infty = P$, $P_0 \cap P_\infty = \emptyset$. However, (P_0, P_∞) is not a partition of P; because, e.g., the alternative of

$P_0 = \emptyset$ (i.e., $P_\infty = P$) or $P_\infty = \emptyset$ (i.e., $P_0 = P$)

cannot be excluded.

Theorem 5.3. *Let the almost metrical structure (X, e) and the proper function $\varphi : X \to R \cup \{\infty\}$ be such that (e02)/(e03) holds. Further, take some normal couple (P, α) (where $\emptyset \neq P \subseteq M := \mathrm{Dom}(\varphi)$ and $\alpha \in R$), according to*

(53-i) $P_\infty := P \cap [\varphi = \alpha] = \emptyset$; *hence,* $P_0 := P \cap [\varphi > \alpha] = P$

(53-ii) for each $x \in P$ there exists $y \in P$ with $x < y$.

Then, $M \setminus P \neq \emptyset$ and the following conclusion is retainable

$\forall u \in P$, $\exists v \in M \setminus P$, *such that* $u < v$, $e(u, v) \leq \varphi(u) - \alpha$.

Proof. As φ is $(\leq)$-strictly-decreasing on M, one gets via (53-ii)

max$(\leq, \varphi; P)$ is empty:
for each $x \in P$ there exists $y \in P$ with $x < y$, $\varphi(x) > \varphi(y)$.

On the other hand, Proposition 5.2 tells us (via (P, α)=normal) that

$(P, \leq)$ is sequentially inductive in M:
each ascending sequence in P is bounded above in M.

By the Local Converse of (CU) it then follows that, for the arbitrary fixed $u \in P$, there exists a threshold (modulo P) ascending sequence (x_n) in P, with

$x_0 = u$, $\emptyset \neq \mathrm{ubd}(x_n) \subseteq M \setminus P$, and (e06) holds.

This, again by Proposition 5.2, yields $\emptyset \neq \lim(x_n) \cap \mathrm{ubd}(x_n)$; wherefrom

$\emptyset \neq \lim(x_n) \cap \mathrm{ubd}(x_n) \subseteq M \setminus P$ and (e06) holds.

Let v be any point in this intersection; hence $u < v$. For the arbitrary fixed index n we get, [via (e06)]

$e(x_n, x_m) \leq \varphi(x_n) - \alpha$, for al $m \geq n$.

This, along with the triangular property of e, gives

$e(x_n, v) \leq e(x_n, x_m) + e(x_m, v) \leq \varphi(x_n) - \alpha + e(x_m, v)$, $\forall m \geq n$.

So, by simply passing to limit as $m \to \infty$, we derive

$e(x_n, v) \leq \varphi(x_n) - \alpha$, for all n;

wherefrom, conclusion of our statement follows (for $n = 0$). □

Note that (53-ii) (taken as itself) implies (53-i). For, if $P_\infty \neq \emptyset$, then,

$u \in P_\infty$, $v \in P$ and $u \leq v$ imply $v \in P_\infty$, $u = v$;

in contradiction with (53-ii); hence the claim. So, we may ask of what happens under $P_\infty \neq \emptyset$. The answer to this is contained in

Theorem 5.4. *Let the almost metrical structure* (X, e) *and the proper function* φ *be as in (e02)/(e03). Further, take some normal couple* (P, α) *(where* $\emptyset \neq P \subseteq M :=$ Dom(φ) *and* $\alpha \in R$*), according to*

(54-i) $P_\infty := P \cap [\varphi = \alpha] \neq \emptyset$*; hence,* $P_0 := P \cap [\varphi > \alpha] \neq P$

(54-ii) for each $x \in P_0$ *there exists* $y \in P$ *with* $x < y$.

Then, $S := (M \setminus P) \cup P_\infty$ *is nonempty and the following conclusion holds*

$$\forall u \in P_0,\ \exists v \in S,\ \text{such that } u < v,\ e(u,v) \le \varphi(u) - \alpha.$$

Proof. If $P_0 = \emptyset$, the conclusion is vacuously holding; so, we may assume that $P_0 \neq \emptyset$. Fix $u \in P_0$. If $P(u, \le) \cap P_\infty \neq \emptyset$, we are done (by simply taking v as some point in this intersection; hence, $v \in S$); so, it remains to discuss the case of

(e08) $Q := P(u, \le) \subseteq P_0$; wherefrom, $P(x, \le) \subseteq Q \subseteq P_0, \forall x \in Q$.

Clearly,

(Q, α) is a normal couple, because $Q \subseteq P \subseteq [\varphi \ge \alpha]$.

Moreover, by the second working hypothesis, it results (via (e08)) that

for each $x \in Q$ there exists $y \in Q$ with $x < y$.

Summing up, Theorem 5.3 applies to the normal couple (Q, α). This (and $u \in Q$), tells us that there must be some $v \in M \setminus Q$, with

$u < v$ and $e(u, v) \le \varphi(u) - \alpha$.

The former of these gives $v \in M \setminus P$ (hence, $v \in S$); for, otherwise, $v \in P(u, \le) = Q$, contradiction. Combining with the later gives our desired conclusion. □

6 Error bound properties

With this information at hand, we may now return to the questions of our introductory section.

Take an almost metric structure (X, d) and a proper function $\varphi : X \to R \cup \{\infty\}$ in accordance with (e02)/(e03). Let again $(\le)$ stand for the relation

(f01) $(x, y \in X)$: $x \le y$ iff $e(x, y) + \varphi(y) \le \varphi(x)$;

remember that it is a quasi-order on X and an order on $M := \mathrm{Dom}(\varphi)$.

(A) We first establish a global error bound property for these data.

Theorem 6.1. *Let the almost metric structure* (X, d) *and the proper function* $\varphi : X \to R \cup \{\infty\}$ *be as in (e02)/(e03). Further, let the number* $\alpha \in R$ *be such that*

$$\forall x \in [\alpha < \varphi < \infty],\ \exists y \in [\alpha \le \varphi < \infty] \text{ with } x < y.$$

Then, $S := [\varphi \le \alpha]$ *is nonempty and*

$$\forall u \in [\alpha < \varphi < \infty],\ \exists v \in S \text{ with } u < v,\ d(u, v) \le \varphi(u) - \alpha.$$

Proof. Denote $P = [\alpha \le \varphi < \infty]$. If $P = \emptyset$, then $M \setminus P = M = [\varphi < \alpha]$; whence, $S = M \neq \emptyset$; and conclusion is vacuously holding. If $P \neq \emptyset$, let us put

$$P_0 = [\alpha < \varphi < \infty],\ P_\infty = [\varphi = \alpha];\ \text{hence, } P_0 \cup P_\infty = P,\ P_0 \cap P_\infty = \emptyset.$$

There are two cases to be considered.

Case 1. Assume that

$$P_\infty = \emptyset;\ \text{so that (by definition) } P_0 = [\alpha < \varphi < \infty] = P.$$

The working conditions of Theorem 6.1 are then identical with the ones in Theorem 5.3. By the quoted result, we therefore have that $M \setminus P$ is nonempty, and

$$\forall u \in P,\ \exists v \in M \setminus P,\ \text{such that } u < v,\ e(u,v) \le \varphi(u) - \alpha.$$

This, added to $M \setminus P = S = [\varphi \le \alpha]$, completes the argument.

Case 2. Assume that

$$P_\infty = [\varphi = \alpha] \neq \emptyset;\ \text{hence, } P_0 = [\alpha < \varphi < \infty] \neq P.$$

The working conditions of Theorem 6.1 are then identical with the ones in Theorem 5.4. By the quoted result, we therefore have that $S := (M \setminus P) \cup P_\infty$ is nonempty, and the conclusion below holds

$$\forall u \in P_0,\ \exists v \in S,\ \text{such that } u < v,\ e(u,v) \le \varphi(u) - \alpha.$$

This, along with

$$S = [\varphi < \alpha] \cup [\varphi = \alpha] = [\varphi \le \alpha],$$

completes the argument. □

Note that by the conclusions of this statement one derives the ones of global error bound theorem of our introductory part. However, the converse implication is not true in general.

(B) We are now passing to the local version of the result above.

Theorem 6.2. *Let the almost metric structure (X, d) and the proper function $\varphi : X \to R \cup \{\infty\}$ be as in (e02)/(e03). Further, let $\alpha, \beta \in R$ be such that*

(62-i) $\alpha < \beta$ *and* $[\varphi < \beta] \neq \emptyset$

(62-ii) for each $x \in [\alpha < \varphi < \beta]$ there exists $y \in [\alpha \le \varphi < \beta]$ with $x < y$.

Then, $S := [\varphi \le \alpha]$ is nonempty and

$$\forall u \in [\alpha < \varphi < \beta],\ \exists v \in S \text{ with } u < v,\ d(u,v) \le \varphi(u) - \alpha.$$

A reduction of this statement to the ones in our preceding section is immediate. Here, we show that a reduction to Theorem 6.1 is also possible.

Proof. (**Theorem 6.2**) Let $\varphi_\beta : X \to R \cup \{\infty\}$ stand for the function

(f02) $\varphi_\beta(x) = \varphi(x)$, if $x \in [\varphi < \beta]$; $\varphi_\beta(x) = \infty$, otherwise.

By the former condition of Theorem 6.2,

$\mathrm{Dom}(\varphi_\beta) = [\varphi < \beta] \neq \emptyset$; hence, φ_β is proper.

On the other hand (e02)/(e03) remains true for $(X, d; \varphi_\beta)$, as it can be directly seen. Finally, by the very definition of φ_β (and $\alpha < \beta$)

$[\alpha < \varphi < \beta] = [\alpha < \varphi_\beta < \infty]$, $[\alpha \leq \varphi < \beta] = [\alpha \leq \varphi_\beta < \infty]$.

Summing up, Theorem 6.1 is applicable to (X, d), φ_β and α; whence, by its conclusion, $S := [\varphi_\beta \leq \alpha]$ is nonempty and

$\forall u \in [\alpha < \varphi_\beta < \infty]$, $\exists v \in S$ with $u < v$, $d(u, v) \leq \varphi(u) - \alpha$.

This, along with $[\varphi \leq \alpha] = [\varphi_\beta \leq \alpha]$ ends the argument. □

In particular, when $d(.,.)$ is a complete metric over X, the regularity conditions (e02)/(e03) are fulfilled under

(f03) φ is proper and d-lsc over M;

so, Theorems 6.1 and Theorem 6.2 are nothing else than global/local versions of Theorem 1.1. Further aspects may be found in Wu [31]; see also Durea [11].

7 Nonconvex Minimization Theorem

An interesting particular case of the above developments may be described along the lines below. Let (X, e) be an almost metrical structure. Take the function $\varphi : X \to R \cup \{\infty\}$ according to

(g01) φ is proper ($\mathrm{Dom}(\varphi) := [\varphi < \infty]$ is nonempty)

(g02) φ is bounded below: $\varphi_* := \inf[\varphi(X)] > -\infty$.

Note that, in such a context, the regularity condition (e02) may be written as

(g03) (e, φ) is weakly descending complete: for each e-strasy sequence $(x_n) \subseteq \mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$=descending,
there exists $x \in X$ with $x_n \xrightarrow{e} x$ and $\lim_n \varphi(x_n) \geq \varphi(x)$.

Likewise, the regularity condition (e03) becomes

(g04) (e, φ) is descending complete: for each e-Cauchy sequence (x_n) in $\mathrm{Dom}(\varphi)$ with $(\varphi(x_n))$=descending,
there exists $x \in X$ with $x_n \xrightarrow{e} x$ and $\lim_n \varphi(x_n) \geq \varphi(x)$.

Since Proposition 5.1 is also true in this setting, it follows that

(g03) $\Longrightarrow$ (g04); hence, (g03) $\Longleftrightarrow$ (g04).

Assume further that (g03)/(g04) holds. Let again $(\leq)$ stand for the quasi-order

$(x, y \in X)$: $x \leq y$ iff $e(x, y) + \varphi(y) \leq \varphi(x)$;

remember that it is an order on $M := \mathrm{Dom}(\varphi)$. Denote also

$Z := [\varphi = \varphi_*]$; clearly, $Z \subseteq \max(\leq, \varphi; X)$.

(A) As a direct consequence of Theorem 6.1 (under $\alpha = \varphi_*$), we get the global type maximal principle:

Theorem 7.1. *Let the an almost metrical structure (X, e) and the proper bounded from below function $\varphi : X \to R \cup \{\infty\}$ be taken according to (g03)/(g04). In addition, assume that*

(71-i) for each $x \in \mathrm{Dom}(\varphi) \setminus Z$ there exists $y \in \mathrm{Dom}(\varphi)$ with $x < y$.

Then, Z is nonempty and

(71-a) for each $u \in \mathrm{Dom}(\varphi) \setminus Z$ there exists $v \in Z$ with $u < v$; hence,

71-b) for each $u \in X$ there exists $v \in Z$ with $u \leq v$.

In particular, when $e : X \times X \to R_+$ is symmetric (hence, a metric on X), the regularity conditions (g03)/(g04) hold when (in addition)

e is complete and φ is d-lsc;

note that in such a case, Theorem 7.1 is just the Nonconvex minimization theorem in Takahashi [24] (in short: (T-nmt)). However, our proof (based, as above said, on the Brezis-Browder ordering principle [4]) is rather different from the one proposed by the quoted author. Further aspects may be found in Hamel [15].

(B) A local version of these facts may be given as follows. Let $\beta > \varphi_*$ be fixed; note that (according to definition)

(g05) $X_\beta := [\varphi < \beta]$ is nonempty.

As a direct consequence of Theorem 6.2, we have

Theorem 7.2. *Let the an almost metrical structure* (X, e) *and the proper bounded from below function* $\varphi : X \to R \cup \{\infty\}$ *be taken as to (g03)/(g04). In addition, suppose that*

(72-i) for each $x \in X_\beta \setminus Z$ *there exists* $y \in X_\beta$ *with* $x < y$.

Then, Z *is nonempty and*

(72-a) for each $u \in X_\beta \setminus Z$ *there exists* $v \in Z$ *with* $u < v$*; hence,*

(72-b) for each $u \in X_\beta$ *there exists* $v \in Z$ *with* $u \leq v$.

As before, when $e : X \times X \to R_+$ is symmetric (hence, a metric on X), the regularity conditions (g03)/(g04) hold when (in addition)

e is complete and φ is d-lsc;

note that, in such a case, Theorem 7.2 is just the main result in Wu [31]. Further aspects may be found in Aze [1] and the references therein.

8 Equivalence results

In the following, the relationships between our maximal principles used above and the Dependent Choice Principle (DC) are being clarified. Further aspects involving these facts will be also discussed.

(I) Let M be a nonempty set. For the generic function $\psi : M \to R \cup \{-\infty, \infty\}$, let us introduce the boundedness properties

(P0) $\psi(.)$ is arbitrary (no additional boundedness condition holds)

(P1) $\psi(.)$ is R-valued bounded below ($\inf \psi(M) > -\infty$ and $\psi(M) \subseteq R$)

(P2) $\psi(.)$ is bounded ($-\infty < \inf \psi(M) \leq \sup \psi(M) < \infty$).

Further, given a *quasi-order* $(\leq)$ over M, define the sequential properties

(Q0) $(M, \leq; \psi)$ is strictly sequentially inductive:
each ascending (modulo $(\leq)$) sequence (x_n) in M with
$(\psi(x_n))$=strictly descending, has an upper bound (modulo $(\leq)$)

(Q1) $(M, \leq; \psi)$ is sequentially inductive:
each ascending (modulo $(\leq)$) sequence (x_n) in M
has an upper bound (modulo $(\leq)$).

Finally, define the $(\leq, \psi)$-maximal property of some $z \in M$ as before:

$$z \leq w \in M \Longrightarrow \psi(z) = \psi(w).$$

The following *combined* maximal principle (referred to as: (Ph;Qk)) is our starting point for these developments.

Theorem 8.1. *Assume that the quasi-ordered structure $(M, \leq)$ and the function $\varphi : M \to R \cup \{-\infty, \infty\}$ are such that*

(81-i) φ has the property (Ph) (for some $h \in \{0, 1, 2\}$)

(81-ii) $(M, \leq; \varphi)$ has the property (Qk) (for some $k \in \{0, 1\}$).

Then, the following conclusions hold

(81-a) *If, in addition, φ is $(\leq)$-decreasing, then $(M, \leq; \varphi)$ is a Zorn-Bourbaki structure: for each $u \in M$ there exists a $(\leq, \varphi)$-maximal $v \in M$ with $u \leq v$.*

(81-b) *If, in addition, φ is $(\leq)$-strictly-decreasing, then $(M, \leq)$ is a Zorn-Bourbaki structure: for each $u \in M$ there exists a $(\leq)$-maximal $v \in M$ with $u \leq v$.*

Note that (P0;Q1) is just the Cârjă-Ursescu maximal principle [8] (in short: CU). On the other hand, (P1;Q1) is the Brezis-Browder ordering principle [4] (in short: BB); The relationships between the principles are being clarified in

Proposition 8.2. *Under these conventions, we have*

(82-1) (Pi;Q0) $\Longrightarrow$ (Pi;Q1), $\forall i \in \{0, 1, 2\}$

(82-2) (P0;Qj) $\Longrightarrow$ (P1,Qj) $\Longrightarrow$ (P2;Qj) $\Longrightarrow$ (P0;Qj), $\forall j \in \{0, 1\}$.

Proof. i): Evident, by definition.

ii): For the moment, it is clear that (P0;Qj) $\Longrightarrow$ (P1,Qj) $\Longrightarrow$ (P2;Qj), $\forall j \in \{0, 1\}$. Further, suppose that (P2;Qj) is holding (for $(\leq)$-decreasing functions); and let the premises of (P0;Qj) (for $(\leq)$-decreasing functions) hold: $(M, \leq)$ is a quasi-ordered structure and $\varphi : M \to R \cup \{-\infty, \infty\}$ is a $(\leq)$-decreasing function with

φ fulfills (P0), and $(M, \leq; \varphi)$ fulfills (Qj).

Define the real valued function $\chi : M \to [0, \pi]$ as

$$\chi(x) = A(\varphi(x)), x \in M \text{ (i.e., } \chi = A \circ \varphi);$$

where the function $A : R \cup \{-\infty, \infty\} \to [0, \pi]$ is introduced as

$$A(t) = \pi/2 + \text{arctg}(t) \text{ if } t \in R;\ A(-\infty) = 0;\ A(\infty) = \pi.$$

Clearly, $A(.)$ is an order isomorphism between $(R \cup \{-\infty, \infty\}, \leq)$ to $([0, \pi], \leq)$; whence, χ is $(\leq)$-decreasing and bounded on M. Therefore, by (P2;Qj), for each $u \in M$ there exists a $(\leq, \chi)$-maximal $v \in M$ with $u \leq v$. This, along with

$$\text{(for each } v, w \in M\text{): } \chi(v) = \chi(w) \Longleftrightarrow \varphi(v) = \varphi(w)$$

tells us that v is $(\leq, \varphi)$-maximal too; and we are done. □

The extremal principles in this series are therefore

(P0;Q0)=the strict version of (CU) (denoted as (CU-str))

(P2;Q1)=the bounded version of (BB) (denoted as (BB-bd)).

As a consequence of this, the deduction of all these principles is equivalent with the deduction of the weakest one, (P0;Q0) – or, equivalently, (P2;Q0) – from the Principle of Dependent Choices (DC). This is obtainable as follows.

Proposition 8.3. *The following inclusions hold*

(83-1) (DC) $\Longrightarrow$ (P2;Q0), in (ZF-AC); so that,

(83-2) (Pi;Qj) is deductible in (ZF-AC+DC), $\forall i \in \{0, 1, 2\}$, $\forall j \in \{0, 1\}$.

A direct argument for establishing this (in the context of (CU)) was provided in the 2007 monograph by Cârjă et al [7, Ch 2, Sect 2.1]; however, the idea of proof goes back to Cârjă and Ursescu [8]. For completeness reasons, we shall describe it, with some modifications.

Proof. Let the premises of (P2;Q0) hold. In particular, as $\varphi : M \to R$ is bounded, the function

$$(\beta : M \to R)\text{: } \beta(v) := \inf \varphi(M(v, \leq)),\ v \in M$$

is well defined; with, in addition,

β=increasing and $\varphi(v) \geq \beta(v)$, for all $v \in M$.

This, added to the decreasing property of φ, gives at once a characterization like

v is $(\leq, \varphi)$-maximal iff $\varphi(v) = \beta(v)$.

Now, assume by contradiction that the conclusion in this statement is false; i.e. [in combination with the above] there must be some $u \in M$ such that:

for each $v \in M_u := M(u, \leq)$, one has $\varphi(v) > \beta(v)$.

Consequently (for all such v), $\varphi(v) > (1/2)(\varphi(v) + \beta(v)) > \beta(v)$; hence

(h01) $(\exists w \in M_u)$: $v \le w$ and $\varphi(v) > (1/2)(\varphi(v) + \beta(v)) > \varphi(w)$.

The relation $\mathcal{R}$ over M_u introduced via (h01) fulfills

$M_u(v, \mathcal{R}) \neq \emptyset$, for all $v \in M_u$; whence, $\mathcal{R}$ is proper.

So, by the Dependent Choice Principle (DC), there must be a sequence (u_n) in M_u, with $u_0 = u$ and

(h02) $u_n \le u_{n+1}$, $\varphi(u_n) > (1/2)(\varphi(u_n) + \beta(u_n)) > \varphi(u_{n+1})$, for all n.

We have thus constructed an ascending (modulo $(\le)$) sequence (u_n) in M_u for which the real sequence $(\varphi(u_n))$ is strictly descending and bounded below; hence $\lambda := \lim_n \varphi(u_n)$ exists in R. As $(M, \le; \varphi)$ is strictly sequentially inductive,

(u_n) is bounded from above in M:
there exists $v \in M$ such that $u_n \le v$, $\forall n$ [hence, $v \in M_u$].

From φ=decreasing, β=increasing, and the properties of (u_n)

$\varphi(u_n) > \varphi(v) > \beta(v) \ge \beta(u_n)$, $\forall n$.

The first part of this relation gives $\lambda \ge \varphi(v)$ (passing to limit as $n \to \infty$). On the other hand, the second of this relations yields (via (h02))

$(1/2)(\varphi(u_n) + \beta(v)) > \varphi(u_{n+1})$, $\forall n$.

Passing to limit as $n \to \infty$ gives $(\varphi(v) \ge)\beta(v) \ge \lambda$; so, combining with the preceding one, $\varphi(v) = \beta(v)(= \lambda)$, contradiction. Hence, our working assumption cannot be accepted; and the conclusion follows. □

(II) A useful application of these facts is to variational principles. Let M be a nonempty set; and $d : M \times M \to R_+$ be a *metric* over it (in the usual sense); the couple (M, d) will be then referred to as a *metric space*. Given some function $\varphi : M \to R \cup \{-\infty, \infty\}$, the following condition is to be used here

$(M, d; \varphi)$ is *descending complete*: each d-Cauchy sequence (x_n) in M
with $(\varphi(x_n))$=descending, is d-convergent.

We are now in position to state the announced result (referred to as: Descending Ekeland variational principle; in short: (EVP-des)).

Theorem 8.4. *Let the metric space* (M, d) *and the function* $\varphi : M \to R \cup \{\infty\}$ *be such that*

(84-i) $(M, d; \varphi)$ *is descending complete*

(84-ii) φ *is inf-proper (*$\mathrm{Dom}(\varphi) \neq \emptyset$ *and* $\varphi_* := \inf[\varphi(M)] > -\infty$*)*

(84-iii) φ *is d-lsc over M:* $\liminf_n \varphi(x_n) \geq \varphi(x)$*, whenever* $x_n \xrightarrow{d} x$*.*

Then, for each $u \in \mathrm{Dom}(\varphi)$ *there exists* $v \in \mathrm{Dom}(\varphi)$*, with*

(84-a) $d(u, v) \leq \varphi(u) - \varphi(v)$ *(hence* $\varphi(u) \geq \varphi(v)$*)*

(84-b) $d(v, x) > \varphi(v) - \varphi(x)$*, for each* $x \in M \setminus \{v\}$*.*

A basic particular case of our developments corresponds to the descending completeness replaced by the usual completeness condition. The corresponding version of (EVP-des) under this setting is just the 1974 Ekeland's variational principle [12] (in short: EVP).

Theorem 8.5. *Let the metric space* (M, d) *be complete and the function* $\varphi : M \to R \cup \{\infty\}$ *be inf-proper, d-lsc. Then, for each* $u \in \mathrm{Dom}(\varphi)$ *there exists* $v \in \mathrm{Dom}(\varphi)$*, with the properties*

(85-a) $d(u, v) \leq \varphi(u) - \varphi(v)$ *(hence* $\varphi(u) \geq \varphi(v)$*)*

(85-b) $d(v, x) > \varphi(v) - \varphi(x)$*, for all* $x \in M \setminus \{v\}$*.*

Concerning the relationships between these, we have for the moment

(EVP-des) $\Longrightarrow$ (EVP), in (ZF-AC).

As we shall see, the reciprocal inclusion is also true. Our main tool in establishing this is a deduction of (EVP-des) in the minimal axiomatic system (ZF-AC+DC). Precisely, the following relative type answer is holding.

Proposition 8.6. *Under the accepted setting, one has*

(86-1) (DC) $\Longrightarrow$ *(BB-bd)* $\Longrightarrow$ *(EVP-des), in (ZF-AC); so that,*

(86-2) (EVP-des) and (EVP) are deductible in (ZF-AC+DC).

Proof. Denote for simplicity $M[u] = \{x \in M; \varphi(u) \geq \varphi(x)\}$. Clearly, $\emptyset \neq M[u] \subseteq \mathrm{Dom}(\varphi)$; moreover, by the posed hypotheses,

(h03) $M[u]$ is closed; hence $(M[u], d)$ is complete.

Let $(\preceq)$ stand for the quasi-order on M

$(x, y \in M)$: $x \preceq y$ iff $d(x, y) + \varphi(y) \leq \varphi(x)$.

Clearly, $(\preceq)$ acts as an *order* (antisymmetric quasi-order) on $\text{Dom}(\varphi)$; so, it remains as such on $M[u]$. We claim that conditions of (BB-bd) are fulfilled on $(M[u], \preceq)$ and φ. In fact, by this very definition, φ is bounded and $(\preceq)$-decreasing on $M[u]$. On the other hand, let (x_n) be a $(\preceq)$-ascending sequence in $M[u]$:

(h04) $d(x_n, x_m) \le \varphi(x_n) - \varphi(x_m)$, if $n \le m$.

The sequence $(\varphi(x_n))$ is descending and bounded from below; hence a Cauchy one. This, along with (h03), tells us that (x_n) is a d-Cauchy sequence in $M[u]$ with $(\varphi(x_n))$=descending; wherefrom (by descending completeness condition and (h03)), there must be some $y \in M[u]$ with $x_n \xrightarrow{d} y$. Passing to limit as $m \to \infty$ in (h04) one derives

$$x_n \le y, d(x_n, y) \le \varphi(x_n) - \varphi(y), \text{ (i.e.: } x_n \preceq y\text{), for all } n.$$

In other words, $y \in M[u]$ is an upper bound (modulo $(\preceq)$) of (x_n); and this shows that $(M[u], \preceq)$ is sequentially inductive. From (BB-bd) it then follows that, for the starting $u \in M[u]$ there exists $v \in M[u]$ with

$$u \preceq v \text{ and } [v \preceq x \in M[u] \text{ implies } \varphi(v) = \varphi(x)].$$

The former of these is just our first conclusion. And the latter one gives the second conclusion. In fact, let $y \in M$ be such that $v \le y$, $d(v, y) \le \varphi(v) - \varphi(y)$. This yields $y \in M[u]$ and $v \preceq y$; so that (by maximality), $\varphi(v) = \varphi(y)$. Combining with the previous relation gives $d(v, y) = 0$ (hence $v = y$); and we are done. □

(III) By the developments above, we have the implications:

$$\text{(DC)} \implies \text{(CU-str)} \implies \text{(BB-bd)} \implies \text{(EVP-des)} \implies \text{(EVP)}.$$

So, we may ask whether these may be reversed. As we shall see, the natural setting of this problem is (ZF-AC+DC) (=the *reduced* Zermelo-Fraenkel system).

Let $(X, \le)$ be a partially ordered structure. We say that $(\le)$ has the *inf-lattice* property, provided:

$$x \wedge y := \inf(x, y) \text{ exists, for all } x, y \in X.$$

Further, call $z \in X$, $(\le)$-*maximal* if $X(z, \le) = \{z\}$; the class of all these points will be denoted as $\max(X, \le)$. In this case, $(\le)$ is termed a *Zorn order* when

$\max(X, \le)$ is nonempty and *cofinal* in X
(for each $u \in X$ there exists a $(\le)$-maximal $v \in X$ with $u \le v$).

Further aspects are to be described in a metrical setting. Let $d : X \times X \to R_+$ be a metric over X; and $\varphi : X \to R_+$ be some function. Then, the natural choice for $(\le)$ above is

$x \leq_{(d,\varphi)} y$ iff $d(x,y) \leq \varphi(x) - \varphi(y)$;

referred to as the Brøndsted order [5] attached to (d,φ). Denote

$X(x,\rho) = \{u \in X; d(x,u) < \rho\}$, $x \in X$, $\rho > 0$
(the *open sphere* with center x and radius ρ).

Call the ambient metric space (X,d), *discrete* when

for each $x \in X$ there exists $\rho = \rho(x) > 0$ such that $X(x,\rho) = \{x\}$.

Note that, under such a hypothesis, any function $\psi : X \to R$ is continuous over X. However, the *d-Lipschitz* property

$|\psi(x) - \psi(y)| \leq Ld(x,y)$, $x,y \in X$, for some $L > 0$

cannot be assured, in general. In particular, this remain valid for the *d-nonexpansive* property (corresponding to $L = 1$).

The following maximal/variational statement (referred to as: discrete Lipschitz countable version of (EVP) (in short: (EVP-dLc)) is now coming into discussion.

Theorem 8.7. *Let the metric space* (X,d) *and* $\varphi : X \to R_+$ *satisfy*

(87-i) (X,d) *is discrete bounded and complete*

(87-ii) $(\leq_{(d,\varphi)})$ *has the inf-lattice property*

(87-iii) φ *is d-nonexpansive and* $\varphi(X)$ *is countable.*

Then, $(\leq_{(d,\varphi)})$ *is a Zorn order.*

Clearly, (EVP) $\Longrightarrow$ (EVP-dLc). The remarkable fact to be added is that this last principle yields (DC); so, it completes the circle between all these.

Proposition 8.8. *The following inclusions hold*

(88-1) (EVP-dLc) $\Longrightarrow$ *(DC), in (ZF-AC)*

(88-2) The maximal/variational principles (CU-str), (BB-bd), (EVP-des) and (EVP) are all equivalent with (DC); hence, mutually equivalent

(88-3) Each maximal principle (MP) with (DC) $\Longrightarrow$ *(MP)* $\Longrightarrow$ *(BB) or (BB)* $\Longrightarrow$ *(MP)* $\Longrightarrow$ *(EVP) is equivalent with both (BB) and (EVP).*

For a complete version of this proof, we refer to the 2014 survey paper by Turinici [29]. Its basic lines are as below.

Proof. **Step 1.** Let M be a nonempty set; and $\mathcal{R}$ stand for some proper relation over it. Fix in the following $a \in M$, $b \in M(a, \mathcal{R})$. For each $p \in N(2, \leq)$, denote by X_p the class of all finite sequences $x : N(p, >) \to M$ with: $x(0) = a$, $x(1) = b$, and $x(n)\mathcal{R}x(n+1)$ for $0 \leq n \leq p-2$. (Here, $N(p, >) := \{0, ..., p-1\}$ stand for the *initial segment* determined by p). In this case, $N(p, >)$ is just $\mathrm{Dom}(x)$ (the *domain* of x); and $p = \mathrm{card}(N(p, >))$ will be referred to as the *order* of x [denoted as $\omega(x)$].

Step 2. Denote $X = \cup\{X_n; n \geq 2\}$; then, let $(\preceq)$ stand for the partial order

$$(x, y \in X)\text{: } x \preceq y \text{ iff } \mathrm{Dom}(x) \subseteq \mathrm{Dom}(y) \text{ and } x = y|_{\mathrm{Dom}(x)};$$

and $(\prec)$ be its associated strict order ($x \prec y$ iff $x \preceq y$ and $x \neq y$).

Let $x, y \in X$ be arbitrary fixed. Denote

$$K(x, y) := \{n \in \mathrm{Dom}(x) \cap \mathrm{Dom}(y); x(n) \neq y(n)\}.$$

If x and y are *comparable* (i.e.: either $x \preceq y$ or $y \preceq x$; written as: $x <> y$) then $K(x, y) = \emptyset$. Conversely, if $K(x, y) = \emptyset$, then $x \preceq y$ if $\mathrm{Dom}(x) \subseteq \mathrm{Dom}(y)$ and $y \preceq x$ if $\mathrm{Dom}(y) \subseteq \mathrm{Dom}(x)$; hence $x <> y$. Summing up,

$$(x, y \in X)\text{: } \ x <> y \text{ if and only if } K(x, y) = \emptyset.$$

The negation of this property means: x and y are *not comparable* (denoted as: $x||y$). By the characterization above, it is equivalent with $K(x, y) \neq \emptyset$. Note that, in such a case, $k(x, y) := \min(K(x, y))$ is well defined as an element of $N(2, \leq)$; and $N(k(x, y), >)$ is the largest initial interval of $\mathrm{Dom}(x) \cap \mathrm{Dom}(y)$ where x and y are identical.

Step 3. Let the function $\varphi : X \to R^0_+$ be introduced as

(h05) $\varphi(x) = 3^{-\omega(x)}$, $x \in X$;

note that $\varphi(X) = \{3^{-n}; n \geq 2\}$ (hence, φ has countable many strictly positive values). Then, define

(h06) $d(x, y) = |\varphi(x) - \varphi(y)|$, if $x <> y$; $d(x, y) = \varphi(x \wedge y)$, when $x||y$.

The mapping $d(.,.)$ is a (standard) metric on X. Moreover

(h07) $m \geq 2$, $x, y \in X$, $\omega(x) \leq m$, $d(x, y) < 2 \cdot 3^{-m-1} \Longrightarrow x = y$;
so that, the metric space (X, d) is discrete,

(h08) $|\varphi(x) - \varphi(y)| \leq d(x, y)$, $\forall x, y \in X$; whence, φ is d-nonexpansive.

Step 4. Given the couple (d, φ) as before, let $(\leq_{(d,\varphi)})$ stand for the Brøndsted order on X; denote it as $(\leq)$, for simplicity. We necessarily have

$x \preceq y$ if and only if $x \leq y$; that is: these partial orders coincide over X.

Step 5. As $\max(X, \leq) = \emptyset$, the property below holds

for each $x \in X$ there exists $y \in X$ with $x < y$.

This, along with (EVP-dLc) tells us that (X, d) is not complete; i.e., there exists at least one d-Cauchy sequence (x_n) in X which is not d-convergent. This sequence gives us (by means of some direct constructions) the desired conclusion. □

In particular, when the boundedness and Lipschitz properties are ignored, this result is just the one in Brunner [6]. Further aspects may be found in Turinici [29].

Summing up, all variational principles in this exposition (derived from (DC)) – as well as the ones described in Kang and Park [17] or Turinici [26] – are nothing but logical equivalents of (EVP). Note that the list of these is rather comprehensive; see the 1997 monograph by Hyers et al [16, Ch 5] for details. Finally, it is natural to ask whether the remaining (sequential) maximality principles – such as the Perturbed minimization statement in Deville and Ghoussoub [10] – are also reducible to (DC). The answer to this is affirmative; further aspects will be delineated elsewhere.

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