Libertas Mathematica (new series)
Volume 38(2018), No. 2, 111-124

S-asymptotically ω-periodic mild solutions to some fractional integro-differential equations with infinite delay

Enock R. Oueama-Guengai and Gaston M. N'Guérékata

Abstract: Under appropriate conditions and using the Krasnosel'skii's fixed point theorem, we prove that the semilinear fractional integro-differential equation in a Banach space X $u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} Au(s)ds + F(t, u_t)$, $t \geq 0$, and $u_0 = \phi$, possesses S-asymptotically ω-periodic mild solutions where $1 < \alpha < 2$, $\phi \in \mathcal{B}$ an abstract space, $A : D(A) \subset X \to X$ a closed (not necessarily bounded) linear operator and $F : \mathbb{R}^+ \times \mathcal{B} \to X$ a continuous function, $u_t : (-\infty, 0] \to X$ with $u_t(\theta) = u(t + \theta)$ is an associated history function to the function $u : \mathbb{R} \to X$.

Keywords: S-asymptotically ω-periodic functions; fractional integro-differential equations; mild solutions

MSC2010: 34K13, 34G20

Dedicated to Professor Constantin Corduneanu on the occasion of his 90th birthday

1 Introduction

This paper is devoted to the study of the existence of S-asymptotically ω-periodic mild solutions of the following semilinear fractional integro-differential equation in a Banach space X

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} Au(s)ds + F(t, u_t), \ \ t \geq 0, \quad u_0 = \phi \tag{1.1}$$

where $1 < \alpha < 2$, $\phi \in \mathcal{B}$ an abstract space to be defined later, $A : D(A) \subset X \to X$ a closed (not necessarily bounded) linear operator and $F : \mathbb{R}^+ \times X \to X$ a continuous function.
For any function $u : \mathbb{R} \to X$, we define the associated history function $t \to u_t$ for $t \geq 0$ as $u_t : (-\infty, 0] \to X$ with $u_t(\theta) = u(t + \theta)$.

The concept of S-asymptotically ω-periodic functions is relatively new. It was introduced in the literature by Henriquez et al. ([5, 6]) and turns out to include the class of asymptotically periodic functions in the sense of M. Fréchet. Qualitative properties of such functions were discussed in [14]. Several papers dealing with the existence of S-asymptotically ω-periodic solutions to fractional differential equations and evolution equations with or without delay have been published recently (cf. for instance [1, 3, 5, 8, 13, 14, 16]).

The paper is organized as follows. In Section 1, we introduce the problem and recall some preliminary facts in Section 2. Section 3 is devoted to information about S-asymptotically ω-periodic functions. The main result is Theorem 4.4 which we present in Section 4.

2 Preliminaries

In this paper $(X, \|\cdot\|)$ will denote a complex Banach space, $L(X)$ the space of all bounded linear operators $X \to X$, $BC((-\infty,0],X)$, (resp. $BUC((-\infty,0],X)$) the space of all bounded continuous (resp. bounded uniformly continuous) functions $(-\infty,0] \to X$, and $C_0(I,X)$ the space of all continuous functions $h : I \to X$ such that $\lim_{t\to\infty} \|h(t)\| = 0$ where $I = \mathbb{R}^+$, or $\mathbb{R}$.

Definition 2.1. A closed linear operator $A : D(A) \subset X \to X$ with a dense domain $D(A)$ is called a sectorial operator of type $\overline{\omega}$ and angle θ if there exists constants $M > 0$, $\overline{\omega}$ and an angle $\theta \in]0, \frac{\pi}{2}[$ such that its resolvent is outside the sector

$$\overline{\omega} + \sum_{\theta} := \{\lambda + \overline{\omega} : \lambda \in \mathbb{C},\ |arg(-\lambda)| < \theta\}$$

and

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \overline{\omega}|} \quad \lambda \notin \overline{\omega} + \sum_{\theta}$$

Definition 2.2. ([10]) Let $\alpha > 0$ and A be a closed linear operator densely defined in X with resolvent $\rho(A)$. A will be called the generator of a solution operator if there exists $\overline{\omega} \in \mathbb{R}$ and a strongly continuous function $E_\alpha : \mathbb{R}^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \overline{\omega}\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1} = \int_0^\infty e^{-\lambda t} E_\alpha(t)x dt, \quad Re\lambda > \overline{\omega}, \quad x \in X.$$

In this case E_α is called a solution operator generated by A and $E_\alpha(0) = I$.

Let's assume that A is a sectorial operator with $0 \leq \theta \leq \pi(1-\frac{\alpha}{2})$, then A is the generator of a solution operator given by

$$E_\alpha(t) = \int_\Gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}dt, \ \ t \geq 0$$

with Γ a suitable path lying outside the sectorial $\overline{\omega} + \sum_\theta$.

Lemma 2.3. *Let $1 < \alpha < 2$ and $A : D(A) \subset X \to X$ be a sectorial operator with $M > 0$, $\overline{\omega} < 0$ and $0 \leq \theta \leq \pi(1-\frac{\alpha}{2})$.*
Then there exists a constant $C_{(\theta,\alpha)} > 0$ depending on θ and α, such that

$$\|E_\alpha(t)\| \leq \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|t^\alpha}, \ \ t \geq 0. \tag{2.1}$$

From the above, it is easy to verify that E_α is integrable. In the border cases $\alpha = 1, 2$, the family $E_\alpha(t)$ corresponds respectively to a C_0-semigroup and a cosine family.

In what follows, $(\mathcal{B}, \|\cdot\|_\mathcal{B})$ will denote a linear seminormed space of functions $(-\infty, 0] \to \mathbb{X}$ satisfying the fundamental axioms of Kato and Hale below:
($\mathbf{A_0}$) If the function $x :]-\infty, T] \to \mathbb{X}$ is continuous on $I = [0, T]$ and $x_0 \in \mathcal{B}$, then for every $t \in I$, the following conditions hold:
(1) $x_t \in \mathcal{B}$
(ii) $\|x(t)\| \leq H\|x_t\|_\mathcal{B}$
(iii) $\|x_t\|_\mathcal{B} \leq C_1(t)\sup_{0\leq s\leq t}\|x(s)\| + C_2(t)\|x_0\|_\mathcal{B}$
where H is a constant $H \geq 0$, $C_1 : [0,\infty) \to [0,\infty)$ is a continuous fucntion and $C_2 : [0,\infty) \to [0,\infty)$ is a locally bounded function, H, C_1, C_2 are independent of $x(\cdot)$
($\mathbf{A_1}$) For the function x_0 in ($\mathbf{A_0}$), x_t is a $\mathcal{B}$-valued continuous function on I.
($\mathbf{A_2}$) The space $\mathcal{B}$ is complete.

Let's recall some examples of phase spaces.

Example 2.4. $BUC((-\infty, 0], X)$ the Banach space of all bounded and uniformly continuous functions $\nu : (-\infty, 0] \to X$ equipped with the supnorm is a phase space.

Example 2.5. $C_0((-\infty, 0], X)$ the Banach space of all bounded and continuous functions $\nu : (-\infty, 0] \to X$ such that $\nu(s) \to 0$ as $s \to -\infty$ equipped with the norm $|\nu| := \sup_{s\leq 0}\|\nu(s)\|$ is also a phase space.

3 S-asymptotically ω-periodic functions

Definition 3.1. (M. Fréchet) A function $f \in BC(\mathbb{R}^+; \mathbb{X})$ is said to be asymptotically ω-periodic for some $\omega > 0$ if there exists $g \in P_\omega(\mathbb{R}; \mathbb{X})$ and $h \in C_0(\mathbb{R}^+; \mathbb{X})$ such that

$$f(t) = g(t) + h(t), \ \ t \in \mathbb{R}^+.$$

The space of all asymptotically ω-periodic $f : \mathbb{R}^+ \to \mathbb{X}$ will be denoted $AP_\omega(\mathbb{X})$.

Definition 3.2. ([5]) A function $f \in BC(\mathbb{R}^+; \mathbb{X})$ is said to be S-asymptotically ω-periodic for some $\omega > 0$ if

$$\lim_{t\to\infty} (f(t+\omega) - f(t)) = 0.$$

In this case, ω will be called an asymptotic period of f. It is clear that if ω is an asymptotic period for f, then every $k\omega, \quad k = n = 1, 2, ...$ is also an asymptotic period for f.

We denote by $SAP_\omega(\mathbb{X})$ the space of such functions. It is known that $SAP_\omega(\mathbb{X})$ is a Banach space under the supnorm and that the inclusion $AP_\omega(\mathbb{X}) \subset SAP_\omega(\mathbb{X})$ is strict.

Example 3.3. ([5]). Let c_0 be space of all sequences $x = (x_n)_n$ where $x_n \in \mathbb{R}$ and $\lim_{n\to\infty} x_n = 0$ equipped with the norm $\|x\| = \sup_{n\in\mathbb{N}} |x_n|$, and define $f : \mathbb{R}^+ \to \mathbb{X}$ by

$$f(t) = (\frac{2nt}{t^2+n^2})_n.$$

Then for every $\mu > 0$, f is S-asymptotically ω-periodic, but not asymptotically ω-periodic.

Note that other examples of S-asymptotically ω-periodic functions are given in our previous work [14].

Proposition 3.4. *Let $f \in SAP_\omega(\mathbb{X})$. Then we have*
(i) $f_a(t) := f(t+a)$ is in $SAP_\omega(\mathbb{X})$ for any $a > 0$.
(ii) $h(t) := \frac{1}{f(t)}$ is $SAP_\omega(\mathbb{X})$ if $\inf_{t\in\mathbb{R}^+} \|f(t)\| > 0$.
(iii), If $f \in SAP_\omega(\mathbb{X})$ and $A : \mathbb{X} \to \mathbb{X}$ is a bounded linear operator, then $Af(t) \in SAP_\omega(\mathbb{X})$.

Proof. (i) and (ii) are proved in ([14]).
(iii) is straightforward. □

$SAP_\omega(\mathbb{X})$ turns out to be a Banach space when equipped with the norm $\|\cdot\|_\infty$.

Lemma 3.5. *([14]) Let f be S-asymptotically ω-periodic $f : \mathbb{R}^+ \to \mathbb{X}$ then there exists $T > 0$ such that $\|f(t+\omega) - f(t)\| < \frac{\epsilon}{a}$, for all $t > T$. Then the function $F_a(t)$ defined by*

$$F_a(t) := \int_t^{t+a} f(s)ds \tag{3.1}$$

belongs to $SAP_\omega(\mathbb{X})$ for $a > 0$ fixed.

Theorem 3.6. *([1]) Let $\mathbb{X}, \mathbb{Y}$ be two Banach spaces and $\phi : \mathbb{X} \to \mathbb{Y}$ a function which is uniformly continuous on bounded sets of $\mathbb{X}$ and such that ϕ maps bounded sets of $\mathbb{X}$ into bounded sets of $\mathbb{Y}$. Then for all $f \in SAP_\omega(\mathbb{X})$, the composition $\phi o f := [t \to \phi(f(t))]$ is in $SAP_\omega(\mathbb{Y})$*

Corollary 3.7. *Let the map $\phi : \mathbb{X} \to \mathbb{Y}$ be such that there exists $L > 0$ such that $\|\phi(u) - \phi(v)\| < L\|u - v\|$ for any u, v in a bounded set of $\mathbb{X}$, then if $f \in SAP_\omega(\mathbb{X})$, the composition $\phi o f := [t \to \phi(f(t))]$ is in $SAP_\omega(\mathbb{Y})$.*

Proof. Uniform continuity of ϕ is obvious. Now let K be a bounded set in $\mathbb{X}$ and $u \in K$. Then we have

$$\|\phi(u)\| = \|\phi(u) - \phi(0) + \phi(0)\| \leq L\|u\| + \|\phi(0)\| < \infty.$$

Which means $\phi(K)$ is bounded.
The result follows from the theorem above.

$\square$

Definition 3.8. ([5]) A continuous function $f : [0, \infty) \times \mathbb{X} \to \mathbb{X}$ is said to be uniformly S-asymptotically ω-periodic on bounded sets if for every bounded set $K \subset \mathbb{X}$, the set $\{f(t, x) : t \in [0, \infty), x \in K\}$ is bounded and

$$\lim_{t\to\infty} (f(t + \omega, x) - f(t, x)) = 0$$

uniformly in $x \in K$.

Definition 3.9. A continuous function $f : [0, \infty) \times \mathbb{X} \to \mathbb{X}$ is said to be asymptotically uniformly continuous on bounded sets if for every $\epsilon > 0$ and every bounded set $K \subset \mathbb{X}$, there exists $L_{\epsilon,K} > 0$ and $\delta_{\epsilon,K} > 0$ such that

$$\|f(t, x) - f(t, y)\| < \epsilon, \ \ \forall x, y \in K, \|x - y\| < \delta_{\epsilon,K}.$$

Theorem 3.10. *([5]) Let $f : [0, \infty) \times \mathbb{X} \to \mathbb{X}$ be a function which is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Let $u : [0, \infty) \to \mathbb{X}$ be S-asymptotically ω-periodic. The Nemytskii operator*

$$\phi(\cdot) := f(\cdot, u(\cdot))$$

is S-asymptotically ω-periodic.

Corollary 3.11. *Let $f : [0, \infty) \times \mathbb{X} \to \mathbb{X}$ be a function with the property that there exists $L > 0$ such that*

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \ \ \forall t \geq 0, \ \forall x, y \in K$$

for any bounded set $K \subset \mathbb{X}$. Suppose also that f is uniformly S-asymptotically ω-periodic on bounded sets and $u : [0, \infty) \to \mathbb{X}$ is a S-asymptotically ω-periodic function. Then

$$\phi(\cdot) := f(\cdot, u(\cdot))$$

is S-asymptotically ω-periodic.

Proof. Consider $K = \overline{R(u)}$, where $R(u)$ is the range of u. Since $R(u)$ of $u(\cdot)$ is a bounded set, it follows that ϕ is a bounded function. Moreover, by assumption, there exists L such that

$$\forall t > L \ \ \|f(t+\omega, u(t+\omega)) - f(t, u(t+\omega))\| \leq \frac{\epsilon}{2}$$

and from Theorem 3.10,
$\forall \epsilon > 0$, $\exists \delta_{\epsilon,K} > 0$, $L_{\epsilon,K} > 0$ such that $\forall t > L_{\epsilon,K}$, $\|f(t, u(t+\omega)) - f(t, u(t))\| \leq \frac{\epsilon}{2}$ and $\|u(t+\omega) - u(t)\| < \delta$.

Therefore $\forall t > max\,\{L_{\epsilon,K}, L\}$
$\|f(t+\omega, u(t+\omega)) - f(t, u(t))\| \leq \|f(t+\omega, u(t+\omega)) - f(t, u(t+\omega))\| +$

$$\|f(t, u(t+\omega)) - f(t, u(t))\|$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

.
The proof is now completes. □

4 fractional integro-differential equation

From now on, we denote by $\mathcal{M}$, the following space of admissible functions:

$$\mathcal{M} := \{f \in BC(\mathbb{R}, X) : f|_{\mathbb{R}^+} \in SAP_\omega(X)\}.$$

This is a Banach space under the sup norm

$$\|f\|_{\mathcal{M}} := \sup_{t \in \mathbb{R}} \|f(t)\|.$$

Let's consider the semilinear fractional integro-differential equation (SFIDE)

$$\begin{cases} u'(t) &= \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}Au(s)ds + F(t,u_t), \quad t \geq 0, \\ u_0 &= \phi \end{cases}$$

where $1 < \alpha < 2$, $\phi \in \mathcal{B}$ a phase space, $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ a sectorial operator which is the generator of a solution operator $E_\alpha(t)$, and $F : [0,\infty) \times \mathcal{B} \to \mathbb{X}$ a continuous function.

Definition 4.1. ([11]) A bounded continuous function $u : \mathbb{R} \to \mathbb{X}$ is said to be a mild solution to (SFIDE) if it satisfies the following

$$\begin{cases} u(t) &= E_\alpha(t)\phi(0) + \int_0^t E_\alpha(t-s)F(s,u_s)ds, \quad t > 0, \\ u(t) &= \phi(t), \quad t \leq 0 \end{cases}$$

Lemma 4.2. *([14]) Let $f \in SAP_\omega(\mathbb{X})$. Then the function defined by*

$$G(t) = \int_0^t E_\alpha(t-s)f(s)ds$$

is also in $SAP_\omega(\mathbb{X})$.

Theorem 4.3. *Consider equation (SFIDE) and suppose*
(H1) The function $F : [0,\infty) \times \mathcal{B} \to \mathbb{X}$ is a continuous function and there exists $L > 0$ such that

$$\|F(t,u) - F(t,v)\| \leq L\|u - v\|_{\mathcal{B}}, \;\; \forall t \geq 0, \; \forall u, v \in K$$

for any bounded set $K \subset \mathcal{B}$.
(H2) F is uniformly S-asymptotically ω-periodic on bounded sets
(H3) $C^\star := \sup_{t\geq 0} C_1(t) < \infty$.
Then equation (SFIDE) has a unique solution in $\mathcal{M}$ provided $L < \frac{\alpha sin(\frac{\pi}{2})}{C^\star MC_{(\theta,\alpha)}|\varpi|^{-\frac{1}{2}}\pi}$

Proof. Let $u \in \mathcal{M}$; then it is easy to see that $u_s \in \mathcal{M}$. Then by (H2) $F(s,u_s) \in SAP_\omega(\mathbb{X})$. So in view of the lemma above,

$$\int_0^t E_\alpha(t-s)F(s,u_s)ds \in SAP_\omega(\mathbb{X}).$$

Consequently $E_\alpha(t)\phi(0) + \int_0^t E_\alpha(t-s)F(s,u_s)ds \in SAP_\omega(\mathbb{X})$. Which means that the operator $\Omega : \mathcal{M} \to \mathcal{M}$ such that

$$(\Omega u)(t) := E_\alpha(t)\phi(0) + \int_0^t E_\alpha(t-s)F(s,u_s)ds$$

is well-defined.

Now let $u, v \in \mathcal{M}$ be solutions to (SFIDE). Then we have

$$\|(\Omega u)(t) - (\Omega v)(t)\| = \|\int_0^t E_\alpha(t-s)[F(s,u_s) - F(s,v_s)]ds\|$$

$$\leq L\int_0^t \|E_\alpha(t-s)\|\|u_s - v_s\|_{\mathcal{B}} ds$$

$$\leq L\int_0^t \|E_\alpha(t-s)\|C_1(s)\sup_{0\leq\sigma\leq s}\|u(\sigma)-v(\sigma)\|ds$$

$$\leq L\|u-v\|_\infty \int_0^t C_1(s)\|E_\alpha(t-s)\|ds$$

$$\leq LC^\star\|u-v\|_\infty \int_0^t \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|(t-s)^\alpha}ds$$

$$\leq LC^\star\|u-v\|_\infty \int_0^t \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|(s)^\alpha}ds$$

$$= LC^\star C_{(\theta,\alpha)}M\frac{|\overline{\omega}|^{\frac{-1}{\pi}}}{\alpha sin(\frac{\pi}{\alpha})}\|u-v\|_\infty$$

Therefore

$$\|(\Omega u) - (\Omega v)\|_\infty \leq LC^\star C_{(\theta,\alpha)}M\frac{|\overline{\omega}|^{\frac{-1}{\pi}}}{\alpha sin(\frac{\pi}{\alpha})}\|u-v\|_\infty$$

Now we use the Banach's fixed point principle to complete the proof. □

Now we will use the Krasnosel'skii fixed point theorem to prove our second result.

Theorem 4.4. *(Krasnosel'skii) Let M be a closed convex and non-empty subset of a Banach space and P,Q are two operators such that*
(i) $Pu + Qv \in M$, whenever $u, v \in M$
(ii) P is compact and continuous
(iii) Q is a contraction mapping
Then there exists $z \in M$ such that $z = Pz + Qz$

Now we make the following assumptions
(T1) $F : [0,\infty)\times\mathcal{B} \to \mathbb{X}$ is a function which is uniformly S-asymptotically ω-periodic and asymptotically uniformly continuous on bounded sets.
(T2) There exists $C_F > 0$ such that
$\|F(t,x)\| < C_F(1 + \|x\|_{\mathcal{B}})$, for all $t \geq 0$, $x \in \mathcal{B}$.

(T3) There exists $L_g > 0$ such that for all $u, v \in BC([0,\infty), \mathbb{X}) \to \mathbb{X}$,
$\|g(u) - g(v)\| < L_g \|u - v\|_\infty$
We assume $C_{(\theta,\alpha)} M L_g < 1$.

Now we state and prove our second result.

Theorem 4.5. *Assume (T1)-(T2)-(T3). Then Eq(1) has at least one mild solution $u(t) \in \mathcal{M}$ if we assume that $E_\alpha(t)$ is compact for any $t > 0$.*

Proof. Note that(T3) implies that there exists a constant $C_g > 0$ such that $\|g(u)\| \leq C_g(1 + \|u\|)$, for any $u \in BC([0,\infty), \mathbb{X})$.
We consider the same operator Ω as in the proof of Theorem 4.3 and use several steps to achieve our conclusion.
Step 1: Let B_γ be defined by
$B_\gamma := \{u \in \mathcal{M} : \|u\|_\infty \leq \gamma\}$, where $\gamma > max\left\{\frac{C_F}{1-(MC_{(\theta,\alpha)}+C_F)}, 0\right\}$
Now define the operators

$$P, Q : \mathcal{M} \to \mathcal{M}$$

by

$$(Pv)(t) := E_\alpha(t)v_0 = E_\alpha(t)\phi(0)$$

and

$$(Qu)(t) := \int_0^t E_\alpha(t-s)F(s,u_s)ds$$

Using (T2) we get

$$\|(Pv)(t) + (Qu)(t)\| \leq \|E_\alpha(t)\|\|\phi(0)\| + \int_0^t \|E_\alpha(t-s)F(s,u_s)\|ds$$

$$\leq \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|t^\alpha}\left\{\|\phi(0)\| + \int_0^t \|F(s,u_s)\|ds\right\}$$

$$\leq \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|t^\alpha}\left\{\|\phi(0)\| + C_F(1+\|u\|)t\right\}$$

$$\leq \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|t^{\alpha}}\left\{\gamma + C_F(1+\gamma)t\right\}$$

$$\leq C_{(\theta,\alpha)}M\left\{\gamma + C_F(1+\gamma)\right\}$$

$$\leq \gamma$$

We conclude that for all $u, v \in B_\gamma$, $Pu + Qv \in B_\gamma$.
Step 2 : Show that the operator P is contractive.
Indeed , let $u, v \in SAP_\omega(\mathbb{X})$
Then we have

$$\|(Pu)(t) + (Pv)(t)\| \leq \|E_\alpha(t)\|\|g(u) - g(v)\|$$

$$\leq \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|t^{\alpha}}L_g\|u-v\|_\infty$$

Thus we obtain

$$\|Pu - Pv\|_\infty \leq \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|t^{\alpha}}L_g\|u-v\|_\infty$$

$$\leq C_{(\theta,\alpha)}ML_g\|u-v\|_\infty$$

We conclude in using the assumption $C_{(\theta,\alpha)}ML_g < 1$. Finally P is contractive.
Step 3: Show that the operator Q is continuous on B_γ.
Let $(u_n) \subset B_\gamma$ such that $u_n \to u$ in B_γ. Then in view of Definition 3.9 $F(s, u_n(s)) \to F(s, u(s))$ as $n \to \infty$ for all $s \in [0, \infty)$.
Now we have

$$\|(Qu_n)(t) - (Qu)(t)\| = \|\int_0^t E_\alpha(t-s)[F(s,u_n(s)) - F(s,u(s))]ds\|$$

$$\leq \int_0^t \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|(t-s)^\alpha}\|F(s,u_n(s))-F(s,u(s))\|ds$$

$$\leq \int_0^t \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|(t-s)^\alpha}C_F(2+\|u_n(s)\|+\|u(s)\|)ds$$

$$\leq 2C_F(1+\gamma)\int_0^t \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|(t-s)^\alpha}ds$$

$$= 2C_F(1+\gamma)C_{(\theta,\alpha)}M\frac{|\overline{\omega}|^{\frac{-1}{\pi}}}{\alpha sin(\frac{\pi}{\alpha})} < \infty$$

Therefore $Qu_n \to Qu$ as $n \to \infty$ by the Lebesgues's Dominated convergence theorem.

Step 4: The set (Qu_n) where $(Qu_n) \subset B_\gamma$ is uniformly bounded.
Indeed for all n, we have

$$\|(Qu_n)(t)\| = \|\int_0^t E_\alpha(t-s)F(s,u_n(s))ds\|$$

$$\leq \int_0^t \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|(t-s)^\alpha}\|F(s,u_n(s))\|ds$$

$$\leq \int_0^t \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|(t-s)^\alpha}C_F(1+\|u_n(s)\|)ds$$

$$\leq C_F(1+\gamma)\int_0^t \frac{C_{(\theta,\alpha)}M}{1+|\overline{\omega}|(t-s)^\alpha}ds$$

$$= C_F(1+\gamma)C_{(\theta,\alpha)}M\frac{|\overline{\omega}|^{\frac{-1}{\pi}}}{\alpha sin(\frac{\pi}{\alpha})}$$

This shows that (Qu_n) is uniformly bounded.

Step 5: Q is compact.
First, let's show that the set $\{(Qu)(t) : u(t) \in B_\gamma\}$ is relatively compact in $\mathbb{X}$ for each $t > 0$.
To this end, fixed $t > 0$ and ε_0 such that $0 < \varepsilon_0 < t$. We have $\left\{(Q_{\varepsilon_0}u)(t) := \int_0^{t-\varepsilon_0} E_\alpha(t-\varepsilon_0-s)F(s,u(s))ds\right\}$ is uniformly bounded for $u \in B_\gamma$. This with the assumption that $E_\alpha(\varepsilon_0)$ is compact yield the set $\{E_\alpha(\varepsilon_0)(Q_{\varepsilon_0}u)(t) : u \in B_\gamma\}$ is relatively compact.
Since from Definition 2.2, $E_\alpha(0) = I$ and $E_\alpha(t)x$ is continuous for any $x \in \mathbb{X}$, we obtain

$$E_\alpha(\varepsilon_0)(Q_{\varepsilon_0}u)(t) = E_\alpha(\varepsilon_0)\int_0^{t-\varepsilon_0} E_\alpha(t-\varepsilon_0-s)F(s,u(s))ds$$

which shows that

$$\lim_{\varepsilon_0 \to 0} E_\alpha(\varepsilon_0)(Q_{\varepsilon_0}u)(t) = (Qu)(t)$$

We conclude that $\{(Qu)(t) : u(t) \in B_\gamma\}$ is relatively compact in $\mathbb{X}$.
Finally, Q is compact as claimed.
From all of the above, we conclude that Eq(1) has at least one mild solution $u(t) \in \mathcal{M}$, using the Krasnosel'ski's fixed point theorem. □

Remark 4.6. These results are new even in the context of asymptotically ω-periodic functions.

References

[1] J. Blot, P. Cieutat, G.M. N'Guérékata, *S-asymptoticallyω-periodic functions and applications to evolution equations*, African Diaspora J. Math. **12 (2)** (2011), 113-121.

[2] A. Caicedo, C. Cuevas, G.M. Mophou, G.M. N'Guérékata, *Asymptotic behavior of solutions of some semilinear functional differential and integro-differential equations with infinite delay in Banach spaces*, J. Franklin Institute 349 (2012), 1.24.

[3] W. Dimbour, G. Mophou, G.M. N'Guérékata, *S-asymptotically periodic solutions for partial differential equations with finite delay*, Electr. J. Diff. Equ., Vol.2011(2011), No.117, 1-12.

[4] R. Gorenflo, F. Mainardi, *On Mittag-Leffler-type functions in fractional evolution provesses*, J. Comput. Appl. Math. **118** (2000) 283-299.

[5] H.R. Henriquez, M. Pierri, P. Tábos, *On S-asymptotically ω-periodic functions on Banach spaces and applications*, J. Math. Anal. Appl. 343 (2008) 1119-1130

[6] H. R. Henríquez, M.Pierre, P.Táboas *Existence of S-asymptotically ω-periodic solutions for abstract neutral equations*, Bull. Austr. Math. Soc 78(2008), 365-382.

[7] N. Heymans, I. Podlubny, *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives*, Rheol. Acta **45(5)**(2006), 765-771.

[8] C. Lizama, G.M. N'Guérékata, *Bounded mild solutions for semilinear integro-differential equations in Banach spaces*, Oper. Theory **68** (2010), 207-227.

[9] A. Lunardi, *Analytic and optimal regularity in parabolic problems*, Birkhauser, Basel, Boston-Berlin, 1995

[10] A. Méril, G. Mophou, G.M. N'Guérékata, *Asymptotic behavior of bounded mild solutions of some functional differential and fractional differential equations*, 24(5-6)(2011), 401-416.

[11] G. Mophou, G.M. N'Guérékata, V. Valmorin, *Asymptotic behavior of mild solutions of some fractional functional integro-differential equations*, African Disapora J. Math.**16** (2013), 70-81.

[12] Gaston M. N'Guérékata, *Spectral theory for bounded functions and applications to evolution equations*, Nova Science Publishers, New York 2017.

[13] S. Nicola, M. Pierri, *A note on S-asymptotically ω-periodic functions*, Nonlinear Analysis, R.W.A. **10** (2009), 2937-2938.

[14] E. R. Oueama, G.M. N'Guérékata, *On S-asymptotically ω-periodic and Bloch periodic mild solutions to some fractional differential equations in abstract spaces*, Math. Meth. Appl. Sci. (2018),1-8, DOI.org/10.1002/mmas5082.

[15] J. Pruss, *Evolutionary Integral Equations and Applications*, Monographs Math, vol. 87, Birkhauser-Verlag, 1993.

[16] J.Zhao, H. Ding, G.M. N'Guérékata, *S-asymptotically periodic solutions for an epidemic model with superlinear perturbation*, Adv. Difference Equ., 2016, paper No. 221, 7 pp.

E. R. Oueama
Département de Mathématiques et Informatique,
Faculté des Sciences, Université de Bangui,
B.P. 908 Bangui,
Central African Republic
E-mail: oueama@yahoo.fru

G. M. N'Guerekata
Department of Mathematics, Morgan State University,
1700 East Cold Spring Lane,
Baltimore, MD 21251,
USA
E-mail: Gaston.N'Guerekata@morgan.edu