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On the solution set of a second-order integro-differential inclusion

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Abstract: We consider a nonconvex and nonclosed second-order integro-differential inclusion and we prove the arcwise connectedness of the set of its mild solutions.

Keywords: Set-valued map, differential inclusion, mild solution, fixed-point.

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Dedicated to Professor Vasile Staicu on his 60th anniversary

1 Introduction

In this note we study integro-differential inclusions of the form

$$x''(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s),H(s,x(s)))ds,\ x(0) = x_0, x'(0) = y_0, \quad (1.1)$$

where X is a real separable Banach space, $\mathcal{P}(X)$ is the family of all subsets of X, $I = [0,T]$, $F(.,.,.) : I \times X^2 \to \mathcal{P}(X)$, $H(.,.) : I \times X \to \mathcal{P}(X)$, $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that generates an evolution system of operators $\{G(t,s)\}_{t,s\in[0,T]}$, $\Delta = \{(t,s) \in [0,T] \times [0,T]; t \geq s\}$, $K(.,.) : \Delta \to \mathbf{R}$ is continuous and $x_0, y_0 \in X$. The general framework of evolution operators $\{A(t)\}_{t\geq 0}$ that define problem (1.1) has been developed by Kozak ([10]) and improved by Henriquez ([8]).

Existence results and some qualitative properties of the mild solutions of problem

$$x''(t) \in A(t)x(t) + \int_0^t K(t,s)F(s,x(s))ds,\ x(0) = x_0, x'(0) = y_0, \quad (1.2)$$

may be found in [7]. In [7] the set-valued map F is assumed to be closed-valued and Lipschitzian in the state variable. Such an assumption is quite natural in order to obtain good properties of the solution set, but it is interesting to investigate the

problem when the right-hand side of the multivalued equation may have nonclosed values.

Following the approach in [12] we consider the problem (1.1), where F and H are closed-valued multifunctions Lipschitzian with respect to the second variable and F is contractive in the third variable. Obviously, the values of the set-valued map $(t,x) \to F(t,x,H(t,x))$ are, in general, neither convex nor closed. Therefore, the right-hand side of the differential inclusion in (1.1) is also neither convex nor closed. We prove the arcwise connectedness of the solution set of problem (1.1). The main tool is a result ([11, 12]) concerning the arcwise connectedness of the fixed point set of a class of nonconvex nonclosed set-valued contractions.

In several recent papers ([1, 2, 3, 5, 6]) existence results and qualitative properties of mild solutions have been obtained for the following problem

$$x'' \in A(t)x + F(t,x), \quad x(0) = x_0, \quad x'(0) = y_0,$$

with $A(.)$ and $F(.,.)$ as above. The result in the present paper extends to the integro-differential framework (1.1) the result in [4] obtained for problem

$$x'' \in A(t)x + F(t,x,H(t,x)), \quad x(0) = x_0, \quad x'(0) = y_0,$$

with $A(.)$, $F(.,.,.)$ and $H(.,.)$ as above.

The paper is organized as follows: in Section 2 we recall some preliminary results that we use in the sequel and in Section 3 we prove our main result.

2 Preliminaries

Let Z be a metric space with the distance d_Z and let 2^Z be the family of all nonempty closed subsets of Z. For $a \in Z$ and $A, B \in 2^Z$ set $d_Z(a,B) = \inf_{b\in B} d_Z(a,b)$ and $d^*_Z(A,B) = \sup_{a\in A} d_Z(a,B)$. Denote by D_Z the Pompeiu-Hausdorff generalized metric on 2^Z defined by

$$D_Z(A,B) = \max\{d^*_Z(A,B), d^*_Z(B,A)\}, \quad A, B \in 2^Z.$$

In what follows, when the product $Z = Z_1 \times Z_2$ of metric spaces $Z_i, i = 1,2$, is considered, it is assumed that Z is equipped with the distance
$d_Z((z_1,z_2),(z_1',z_2')) = \sum_{i=1}^2 d_{Z_i}(z_i,z_i')$.

Let X be a nonempty set and let $F : X \to 2^Z$ be a set-valued map from X to Z. The range of F is the set $F(X) = \cup_{x\in X} F(x)$. Let $(X,\mathcal{F})$ be a measurable space. The multifunction $F : X \to 2^Z$ is called measurable if $F^{-1}(\Omega) \in \mathcal{F}$ for any open set $\Omega \subset Z$, where $F^{-1}(\Omega) = \{x \in X; F(x) \cap \Omega \neq \emptyset\}$. Let (X, d_X) be a metric space. The

multifunction F is called Hausdorff continuous if for any $x_0 \in X$ and every $\epsilon > 0$ there exists $\delta > 0$ such that $x \in X, d_X(x, x_0) < \delta$ implies $D_Z(F(x), F(x_0)) < \epsilon$.

Let $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space and let $(X, |.|_X)$ be a Banach space. We denote by $L^1(T, X)$ the Banach space of all (equivalence classes of) Bochner integrable functions $u : T \to X$ endowed with the norm

$$|u|_{L^1(T,X)} = \int_T |u(t)|_X d\mu$$

A nonempty set $K \subset L^1(T, X)$ is called decomposable if, for every $u, v \in K$ and every $A \in \mathcal{F}$, one has

$$\chi_A.u + \chi_{T\setminus A}.v \in K$$

where $\chi_B, B \in \mathcal{F}$ indicates the characteristic function of B.

A metric space Z is called an absolute retract if, for any metric space X and any nonempty closed set $X_0 \subset X$, every continuous function $g : X_0 \to Z$ has a continuous extension $g : X \to Z$ over X. It is obvious that every continuous image of an absolute retract is an arcwise connected space.

In what follows we recall some preliminary results that are the main tools in the proof of our result.

Let $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space, S a separable Banach space and let $(X, |.|_X)$ be a real Banach space. To simplify the notation we write E in place of $L^1(T, X)$. The proofs of the next two lemmas may be found in [12].

Lemma 2.1. *Assume that $\phi : S \times E \to 2^E$ and $\psi : S \times E \times E \to 2^E$ are Hausdorff continuous multifunctions with nonempty, closed, decomposable values, satisfying the following conditions*

a) There exists $L \in [0, 1)$ such that, for every $s \in S$ and every $u, u' \in E$,

$$D_E(\phi(s, u), \phi(s, u')) \leq L|u - u'|_E.$$

b) There exists $M \in [0, 1)$ such that $L + M < 1$ and for every $s \in S$ and every $(u, v), (u', v') \in E \times E$,

$$D_E(\psi(s, u, v), \psi(s, u', v')) \leq M(|u - u'|_E + |v - v'|_E).$$

Set $Fix(\Gamma(s, .)) = \{u \in E; u \in \Gamma(s, u)\}$, where $\Gamma(s, u) = \psi(s, u, \phi(s, u))$, $(s, u) \in S \times E$. Then

1) For every $s \in S$ the set $Fix(\Gamma(s, .))$ is nonempty and arcwise connected.

2) For any $s_i \in S$, and any $u_i \in Fix(\Gamma(s, .)), i = 1, ..., p$ there exists a continuous function $\gamma : S \to E$ such that $\gamma(s) \in Fix(\Gamma(s, .))$ for all $s \in S$ and $\gamma(s_i) = u_i, i = 1, ..., p$.

Lemma 2.2. *Let $U : T \to 2^X$ and $V : T \times X \to 2^X$ be two nonempty closed-valued multifunctions satisfying the following conditions*

a) U is measurable and there exists $r \in L^1(T)$ such that $D_X(U(t), \{0\}) \leq r(t)$ for almost all $t \in T$.

b) The multifunction $t \to V(t, x)$ is measurable for every $x \in X$.

c) The multifunction $x \to V(t, x)$ is Hausdorff continuous for all $t \in T$.

Let $v : T \to X$ be a measurable selection from $t \to V(t, U(t))$.

Then there exists a selection $u \in L^1(T, X)$ such that $v(t) \in V(t, u(t)), t \in T$.

In what follows $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X that genearates an evolution system of operators $\{G(t, s)\}_{t,s\in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in X and is independent of t.

Definition 2.3. ([8, 10]) A family of bounded linear operators $G(t, s) : X \to X$, $(t, s) \in \Delta := \{(t, s) \in I \times I; s \leq t\}$ is called an evolution operator of the equation

$$x''(t) = A(t)x(t) \tag{2.1}$$

if

i) For any $x \in X$, the map $(t, s) \to G(t, s)x$ is continuously differentiable and

a) $G(t, t) = 0$, $t \in I$.

b) If $t \in I, x \in X$ then $\frac{\partial}{\partial t}G(t, s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}G(t, s)x|_{t=s} = -x$.

ii) If $(t, s) \in \Delta$, then $\frac{\partial}{\partial s}G(t, s)x \in D(A(t))$, the map $(t, s) \to G(t, s)x$ is of class C^2 and

a) $\frac{\partial^2}{\partial t^2}G(t, s)x \equiv A(t)G(t, s)x$.

b) $\frac{\partial^2}{\partial s^2}G(t, s)x \equiv G(t, s)A(t)x$.

c) $\frac{\partial^2}{\partial s\partial t}G(t, s)x|_{t=s} = 0$.

iii) If $(t, s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2\partial s}G(t, s)x$, $\frac{\partial^3}{\partial s^2\partial t}G(t, s)x$ and

a) $\frac{\partial^3}{\partial t^2\partial s}G(t, s)x \equiv A(t)\frac{\partial}{\partial s}G(t, s)x$ and the map $(t, s) \to A(t)\frac{\partial}{\partial s}G(t, s)x$ is continuous.

b) $\frac{\partial^3}{\partial s^2\partial t}G(t, s)x \equiv \frac{\partial}{\partial t}G(t, s)A(s)x$.

As an example for equation (2.1) one may consider the problem (e.g., [8])

$$\frac{\partial^2 z}{\partial t^2}(t, \tau) = \frac{\partial^2 z}{\partial \tau^2}(t, \tau) + a(t)\frac{\partial z}{\partial t}(t, \tau), \quad t \in [0, T], \tau \in [0, 2\pi],$$

$$z(t, 0) = z(t, \pi) = 0, \quad \frac{\partial z}{\partial \tau}(t, 0) = \frac{\partial z}{\partial \tau}(t, 2\pi),\ t \in [0, T],$$

where $a(.) : I \to \mathbf{R}$ is a continuous function. This problem is modeled in the space $X = L^2(\mathbf{R}, \mathbf{C})$ of 2π-periodic 2-integrable functions from $\mathbf{R}$ to $\mathbf{C}$, $A_1 z = \frac{d^2 z(\tau)}{d\tau^2}$

with domain $H^2(\mathbf{R},\mathbf{C})$ the Sobolev space of 2π-periodic functions whose derivatives belong to $L^2(\mathbf{R},\mathbf{C})$. It is well known that$A_1$ is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X. Moreover, A_1 has discrete spectrum; namely the spectrum of A_1 consists of eigenvalues $-n^2$, $n \in \mathbf{Z}$ with associated eigenvectors $z_n(\tau) = \frac{1}{\sqrt{2\pi}}e^{in\tau}$, $n \in \mathbf{N}$. The set z_n, $n \in \mathbf{N}$ is an orthonormal basis of X. In particular, $A_1 z = \sum_{n\in\mathbf{Z}} -n^2 < z, z_n > z_n$, $z \in D(A_1)$. The cosine function is given by $C(t)z = \sum_{n\in\mathbf{Z}} \cos(nt) < z, z_n > z_n$ with the associated sine function $S(t)z = t < z, z_0 > z_0 + \sum_{n\in\mathbf{Z}^*} \frac{\sin(nt)}{n} < z, z_n > z_n$.

For $t \in I$ define the operator $A_2(t)z = a(t)\frac{dz(\tau)}{d\tau}$ with domain $D(A_2(t)) = H^1(\mathbf{R},\mathbf{C})$. Set $A(t) = A_1 + A_2(t)$. It has been proved in [8] that this family generates an evolution operator as in Definition 2.1.

Definition 2.4. A continuous mapping $x(.) \in C(I,X)$ is called a mild solution of problem (1.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I,X)$ such that

$$f(t) \in F(t, x(t), H(t, x(t))) \quad a.e.\,(I), \tag{2.2}$$

$$x(t) = -\frac{\partial}{\partial s}G(t,0)x_0 + G(t,0)y_0 + \int_0^t G(t,s)\int_0^s K(s,\tau)f(\tau)d\tau,\ t \in I. \tag{2.3}$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1.1) if $f(.)$ verifies (2.2) and $x(.)$ is defined by (2.3).

We shall use the following notations for the solution sets of (1.1).

$$\mathcal{S}(x_0, y_0) = \{(x(.), f(.));\ (x(.), f(.)) \text{ is a trajectory-selection pair of (1.1)}\}. \tag{2.4}$$

In what follows we assume the following hypothesis.

Hypothesis 2.5. i) There exists an evolution operator $\{G(t,s)\}_{t,s\in I}$ associated to the family $\{A(t)\}_{t\geq 0}$.

ii) There exist $M, M_0 \geq 0$ such that $|G(t,s)|_{B(X)} \leq M$, $|\frac{\partial}{\partial s}G(t,s)| \leq M_0$, for all $(t,s) \in \Delta$.

$F : I \times X \times X \to \mathcal{P}(X)$ and $H : I \times X \to \mathcal{P}(X)$ are two set-valued maps with nonempty closed values, satisfying

iii) The set-valued maps $t \to F(t,u,v)$ and $t \to H(t,u)$ are measurable for all $u, v \in X$.

iv) There exist $l(.) \in L^1(I,\mathbf{R})$ such that, for every $u, u' \in X$,

$$D(H(t,u), H(t,u')) \leq l(t)|u - u'| \quad a.e.\ (I).$$

v) There exist $m(.) \in L^1(I,\mathbf{R})$ and $\theta \in [0,1)$ such that, for every $u, v, u', v' \in X$,

$$D(F(t,u,v), F(t,u',v')) \leq m(t)|u - u'| + \theta|v - v'| \quad a.e.\ (I).$$

vi) There exist $f, g \in L^1(I, \mathbf{R})$ such that

$$d(0, F(t,0,0)) \leq f(t), \quad d(0, H(t,0)) \leq g(t) \quad a.e.\ (I).$$

In what follows $N(t) = \max\{l(t), m(t)\}$, $t \in I$, $N^*(t) = \int_0^t N(s)ds$.

Given $\alpha \in \mathbf{R}$ we denote by L^1 the Banach space of all (equivalence classes of) Lebesgue measurable functions $\sigma : I \to X$ endowed with the norm

$$|\sigma|_1 = \int_0^T e^{-\alpha N^*(t)} |\sigma(t)| dt.$$

We note that condition (2.3) can be rewritten as

$$x(t) = -\frac{\partial}{\partial s} G(t,0)x_0 + G(t,0)y_0 + \int_0^t U(t,s)f(s)ds \quad \forall t \in I, \tag{2.5}$$

where $U(t,s) = \int_s^t G(t,\tau)K(\tau,s)d\tau$.

Denote $K_0 := \sup_{(t,s)\in\Delta} |K(t,s)|$ and remark that $|U(t,s)| \leq MK_0(t-s) \leq MK_0T$.

3 Main result

Even if the multifunction from the right-hand side of (1.1) has, in general, nonclosed nonconvex values, its solution set $\mathcal{S}(x_0, y_0)$ defined in (2.4) has some meaningful properties, stated in theorem below.

Theorem 3.1. *Assume that Hypothesis 2.5 is satisfied and let* $\alpha > \frac{2MK_0T}{1-\theta}$. *Then*

1) For every $(x_0, y_0) \in X \times X$, *the solution set* $\mathcal{S}(x_0, y_0)$ *is nonempty and arcwise connected in the space C(I,X).*

2) For any $(\xi_i, \mu_i) \in X \times X$ *and any* $x_i \in \mathcal{S}(\xi_i, \mu_i)$, $i = 1, ..., p$, *there exists a continuous function* $s : X \times X \to C(I, X)$ *such that* $s(\xi, \mu) \in \mathcal{S}(\xi, \mu)$ *for any* $(\xi, \mu) \in X \times X$ *and* $s(\xi_i, \mu_i) = x_i, i = 1, ..., p$.

3) The set $\mathcal{S} = \cup_{(\xi,\mu)\in X\times X} \mathcal{S}(\xi, \mu)$ *is arcwise connected in C(I,X).*

Proof. 1) For $(\xi, \mu) \in X \times X$ and $f \in L^1$, set

$$x_{\xi,\mu}(t) = -\frac{\partial}{\partial s} G(t,0)\xi + G(t,0)\mu + \int_0^t U(t,s)f(s)ds \tag{3.1}$$

and consider $\lambda : X \times X \to C(I, X)$ defined by $\lambda(\xi, \mu)(t) = -\frac{\partial}{\partial s} G(t,0)\xi + G(t,0)\mu$.

We prove that the multifunctions $\phi : X \times X \times L^1 \to 2^{L^1}$ and $\psi : X \times X \times L^1 \times L^1 \to 2^{L^1}$ given by

$$\phi((\xi, \mu), u) = \{v \in L^1; \quad v(t) \in H(t, x_{\xi,\mu}(t)) \quad a.e.(I)\},$$

$$\psi((\xi,\mu),u,v)=\{w\in L^1;\quad w(t)\in F(t,x_{\xi,\mu}(t),v(t))\quad a.e.(I)\},$$

$(\xi,\mu)\in X\times X$, $u,v\in L^1$ satisfy the hypotheses of Lemma 2.1.

Since $x_{\xi,\mu}(.)$ is measurable and H satisfies Hypothesis 2.5 iii) and iv), the multifunction $t\to H(t,x_{\xi,\mu}(t))$ is measurable and nonempty closed-valued, it has a measurable selection. Therefore due to Hypothesis 2.5 vi), the set $\phi((\xi,\mu),u)$ is nonempty. The fact that the set $\phi((\xi,\mu),u)$ is closed and decomposable follows by a simple computation. In the same way we obtain that $\psi((\xi,\mu),u,v)$ is a nonempty closed decomposable set.

Pick $((\xi,\mu),u),((\xi_1,\mu_1),u_1)\in X\times X\times L^1$ and choose $v\in\phi((\xi,\mu),u)$. For each $\varepsilon>0$ there exists $v_1\in\phi((\xi_1,\mu_1),u_1)$ such that, for every $t\in I$, one has

$$\begin{gathered}|v(t)-v_1(t)|\le D(H(t,x_{\xi,\mu}(t)),H(t,x_{\xi_1,\mu_1}(t)))+\varepsilon\le\\ l(t)[M_0|\xi-\xi_1|+M|\mu-\mu_1|+MK_0T\textstyle\int_0^t|u(s)-u_1(s)|ds]+\varepsilon.\end{gathered}$$

Hence

$$\begin{gathered}|v-v_1|_1\le[M_0|\xi-\xi_1|+M|\mu-\mu_1|]\textstyle\int_0^T e^{-\alpha N^*(t)}l(t)dt+\\ MK_0T\textstyle\int_0^T e^{-\alpha N^*(t)}l(t)(\int_0^t|u(s)-u_1(s)|ds)dt+\varepsilon T\le\\ \frac{1}{\alpha}[M_0|\xi-\xi_1|+M|\mu-\mu_1|]+\frac{MK_TT}{\alpha}|u-u_1|_1+\varepsilon T\end{gathered}$$

for any $\varepsilon>0$.

This implies

$$d_{L^1}(v,\phi((\xi_1,\mu_1),u_1))\le\frac{1}{\alpha}[M_0|\xi-\xi_1|+M|\mu-\mu_1|]+\frac{MK_0T}{\alpha}|u-u_1|_1$$

for all $v\in\phi((\xi,\mu),u)$. Therefore,

$$d^*_{L^1}(\phi((\xi,\mu),u),\phi((\xi_1,\mu_1),u_1))\le\frac{1}{\alpha}[M_0|\xi-\xi_1|+M|\mu-\mu_1|]+\frac{MK_0T}{\alpha}|u-u_1|_1$$

Consequently,

$$D_{L^1}(\phi((\xi,\mu),u),\phi((\xi_1,\mu_1),u_1))\le\frac{1}{\alpha}[M_0|\xi-\xi_1|+M|\mu-\mu_1|]+\frac{MK_0T}{\alpha}|u-u_1|_1$$

which shows that ϕ is Hausdorff continuous and satisfies the assumptions of Lemma 2.1.

Pick $((\xi,\mu),u,v),((\xi_1,\mu_1),u_1,v_1)\in X\times X\times L^1\times L^1$ and choose $w\in\psi((\xi,\mu),u,v)$. Then, as before, for each $\varepsilon>0$ there exists $w_1\in\psi((\xi_1,\mu_1),u_1,$
$v_1)$ such that for every $t\in I$

$$\begin{gathered}|w(t)-w_1(t)|\le D(F(t,x_{\xi,\mu}(t),v(t)),F(t,x_{\xi_1,\mu_1}(t),v_1(t)))+\varepsilon\le m(t)[M_0|\xi\\ -\xi_1|+M|\mu-\mu_1|+MK_0T\textstyle\int_0^t|u(s)-u_1(s)|ds]+\theta|v(t)-v_1(t)|+\varepsilon.\end{gathered}$$

Hence

$$\begin{aligned}|w-w_1|_1 &\leq \tfrac{1}{\alpha}[M_0|\xi-\xi_1|+M|\mu-\mu_1|]+\tfrac{MK_0T}{\alpha}|u-u_1|_1+\theta|v-v_1|_1+\varepsilon T\\ &\leq \tfrac{1}{\alpha}[M_0|\xi-\xi_1|+M|\mu-\mu_1|]+(\tfrac{MK_0T}{\alpha}+\theta)(|u-u_1|_1+|v-v_1|_1)+\varepsilon T\\ &\leq \tfrac{1}{\alpha}[M_0|\xi-\xi_1|+M|\mu-\mu_1|]+(\tfrac{MK_0T}{\alpha}+\theta)d_{L^1\times L^1}((u,v),(u_1,v_1))+\varepsilon T.\end{aligned}$$

As above, we deduce that

$$\begin{gathered}D_{L^1}(\psi((\xi,\mu),u,v),\psi((\xi_1,\mu_1),u_1,v_1))\leq\\ \tfrac{1}{\alpha}[M_0|\xi-\xi_1|+M|\mu-\mu_1|]+(\tfrac{MK_0T}{\alpha}+\theta)d_{L^1\times L^1}((u,v),(u_1,v_1)).\end{gathered}$$

namely, the multifunction ψ is Hausdorff continuous and satisfies the hypothesis of Lemma 2.1.

Define $\Gamma((\xi,\mu),u)=\psi((\xi,\mu),u,\phi((\xi,\mu),u)),((\xi,\mu),u)\in X^2\times L^1$. According to Lemma 2.1, the set $Fix(\Gamma((\xi,\mu),.))=\{u\in L^1; u\in\Gamma((\xi,\mu),u)\}$ is nonempty and arcwise connected in $L^1(I,X)$. Moreover, for fixed $(\xi_i,\mu_i)\in X^2$ and $u_i\in Fix(\Gamma((\xi_i,\mu_i),.)), i=1,...,p$, there exists a continuous function $\gamma: X^2\to L^1$ such that

$$\gamma((\xi,\mu))\in Fix(\Gamma((\xi,\mu),.)),\quad \forall(\xi,\mu)\in X^2, \tag{3.2}$$

$$\gamma((\xi_i,\mu_i))=u_i,\quad i=1,...,p. \tag{3.3}$$

We shall prove that

$$Fix(\Gamma((\xi,\mu),.))=\{u\in L^1;\quad u(t)\in F(t,x_{\xi,\mu}(t),H(t,x_{\xi,\mu}(t)))\quad a.e.\ (I)\}. \tag{3.4}$$

Denote by $A(\xi,\mu)$ the right-hand side of (3.4). If $u\in Fix(\Gamma((\xi,\mu),.))$ then there is $v\in\phi((\xi,\mu),v)$ such that $u\in\psi((\xi,\mu),u,v)$. Therefore, $v(t)\in H(t,x_{\xi,\mu}(t))$ and

$$u(t)\in F(t,x_{\xi,\mu}(t),v(t))\subset F(t,x_{\xi,\mu}(t),H(t,x_{\xi,\mu}(t)))\quad a.e.\ (I),$$

so that $Fix(\Gamma((\xi,\mu),.))\subset A(\xi,\mu)$.

Let now $u\in A(\xi,\mu)$. By Lemma 2.2, there exists a selection $v\in L^1$ of the multifunction $t\to H(t,x_{\xi,\mu}(t))$ satisfying

$$u(t)\in F(t,x_{\xi,\mu}(t),v(t))\quad a.e.\ (I).$$

Hence, $v\in\phi((\xi,\mu),v)$, $u\in\psi((\xi,\mu),u,v)$ and thus $u\in\Gamma((\xi,\mu),u)$, which completes the proof of (3.4).

We next note that the function $T: L^1\to C(I,X)$,

$$T(u)(t):=\int_0^t U(t,s)u(s)ds$$

is continuous and one has

$$\mathcal{S}(\xi,\mu) = \lambda(\xi,\mu) + T(Fix(\Gamma((\xi,\mu),.))), \quad (\xi,\mu) \in X^2. \tag{3.5}$$

Since $Fix(\Gamma((\xi,\mu),.))$ is nonempty and arcwise connected in L^1, the set $\mathcal{S}(\xi,\mu)$ has the same properties in $C(I,X)$.

2) Let $(\xi_i,\mu_i) \in X^2$ and let $x_i \in \mathcal{S}(\xi_i,\mu_i), i = 1,...,p$ be fixed. By (3.5) there exists $v_i \in Fix(\Gamma((\xi_i,\mu_i),.))$ such that

$$x_i = \lambda(\xi_i,\mu_i) + T(v_i), \quad i = 1,...,p.$$

If $\gamma : X^2 \to L^1$ is a continuous function satisfying (3.2) and (3.3) we define, for every $(\xi,\mu) \in X^2$,

$$s(\xi,\mu) = \lambda(\xi,\mu) + T(\gamma(\xi,\mu)).$$

Obviously, the function $s : X \to C(I,X)$ is continuous, $s(\xi,\mu) \in \mathcal{S}(\xi,\mu)$ for all $(\xi,\mu) \in X^2$ and

$$s(\xi_i,\mu_i) = \lambda(\xi_i,\mu_i) + T(\gamma(\xi_i,\mu_i)) = \lambda(\xi_i,\mu_i) + T(v_i) = x_i, \quad i = 1,...,p.$$

3) Let $x_1, x_2 \in \mathcal{S} = \cup_{(\xi,\mu)\in X^2}\mathcal{S}(\xi,\mu)$ and choose $(\xi_i,\mu_i) \in X^2$, $i = 1,2$ such that $x_i \in S(\xi_i,\mu_i)$, $i = 1,2$. From the conclusion of 2) we deduce the existence of a continuous function $s : X^2 \to C(I,X)$ satisfying $s(\xi_i,\mu_i) = x_i$, $i = 1,2$ and $s(\xi,\mu) \in \mathcal{S}(\xi,\mu)$, $(\xi,\mu) \in X^2$. Let $h : [0,1] \to X^2$ be a continuous mapping such that $h(0) = (\xi_1,\mu_1)$ and $h(1) = (\xi_2,\mu_2)$. Then the function $s \circ h : [0,1] \to C(I,X)$ is continuous and verifies

$$s \circ h(0) = x_1, \quad s \circ h(1) = x_2, \quad s \circ h(\tau) \in \mathcal{S}(h(\tau)) \subset \mathcal{S}, \quad \tau \in [0,1],$$

which completes the proof. □

References

[1] A. Baliki, M. Benchohra and J.R. Graef, Global existence and stability of second order functional evolution equations with infinite delay, *Electronic J. Qual. Theory Differ. Equations*, **2016**, no. 23, (2016), 1–10.

[2] A. Baliki, M. Benchohra and J.J. Nieto, Qualitative analysis of second-order functional evolution equations, *Dynamic Syst. Appl.*, **24** (2015), 559–572.

[3] M. Benchohra and I. Medjadj, Global existence results for second order neutral functional differential equations with state-dependent delay, *Comment. Math. Univ. Carolin.* **57** (2016), 169–183.

[4] A. Cernea, On the solution set of a nonconvex nonclosed second-order evolution inclusion, *Proceedings of EQUADIFF 2017 Conf.*, Ed. K. Mikula, D. Ševčovič, J. Urbán, Slovak Univ. Techn. Publ. House 2017, 21–28.

[5] A. Cernea, A note on the solutions of a second-order evolution inclusion in non separable Banach spaces, *Comment. Math. Univ. Carolin.* **58** (2017), 307–314.

[6] A. Cernea, Some remarks on the solutions of a second-order evolution inclusion, *Dynamic Syst. Appl.* **27** (2018), 319–330.

[7] A. Cernea, On the mild solutions of a class of second-order integro-differential inclusions, *J. Nonlinear Var. Anal.*, **3** (2019), 247–256.

[8] H.R. Henriquez, Existence of solutions of nonautonomous second order functional differential equations with infinite delay, *Nonlinear Anal.*, **74** (2011), 3333–3352.

[9] H.R. Henriquez, V. Poblete and J.C. Pozo, Mild solutions of non-autonomous second order problems with nonlocal initial conditions, *J. Math. Anal. Appl.*, **412** (2014), 1064–1083.

[10] M. Kozak, A fundamental solution of a second-order differential equation in a Banach space, *Univ. Iagel. Acta. Math.*, **32** (1995), 275–289.

[11] S. Marano, Fixed points of multivalued contractions with nonclosed, *Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, **5** (1994), 203–212.

[12] S. Marano and V. Staicu, On the set of solutions to a class of nonconvex nonclosed differential inclusions, *Acta Math. Hungar.*, **76** (1997), 287–301.

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