Libertas Mathematica (new series)
Volume 41(2021), No. 1, 23-36

Inexact infinite products of nonexpansive mappings with nonsummable errors

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Abstract: Given a sequence of nonexpansive mappings which map a closed subset of a complete metric space into the space, we study the convergence of its inexact infinite products to its common fixed point set in the case where the errors are nonsummable. Previous results in this direction concerned nonexpansive self-mappings of a complete metric space and inexact iterates with summable errors.

Keywords: Complete metric space, fixed point, inexact infinite product, nonexpansive mapping.

MSC2010: 47H09, 47H10, 54E50

1 Introduction and Preliminaries

For almost six decades now, there has been a lot of research activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [2, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 21, 22] and the references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers, in particular, the convergence of (inexact) orbits of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this area including, for example, studies of feasibility and common fixed point problems, which find important and diverse applications in the physical, medical and engineering sciences [4, 6, 17, 19, 20, 21, 22].

For instance, in [3] it was shown that if every exact orbit of a nonexpansive mapping converges to one of its fixed points, then this convergence property also holds for all its inexact orbits with summable errors. This result was established for a nonexpansive self-mapping of a complete metric space.

In the present paper we are concerned with a sequence of nonexpansive mappings which map a closed subset of a complete metric space into the space. We study the convergence of its inexact infinite products to its common fixed point set in the case where the errors are *nonsummable.* Our paper contains four results. Prototypes of the first two of them can be found in [18], where we were concerned with inexact orbits of nonexpansive mappings.

2 First result

Let (X, ρ) be a complete metric space. For each $x \in X$ and each $r > 0$, set

$$B(x, r) := \{y \in X : \ \rho(x, y) \leq r\}.$$

For each $x \in X$ and each nonempty set $A \subset X$, put

$$\rho(x, A) := \inf\{\rho(x, y) : \ y \in A\}.$$

Let $K \subset X$ be a nonempty closed set and let the mappings $T_i : K \to X$, $i = 1, 2, \ldots,$ satisfy

$$\rho(T_i(x), T_i(y)) \leq \rho(x, y) \text{ for all } x, y \in K \text{ and all integers } i \geq 1. \tag{2.1}$$

Fix a point $\theta \in K$ and suppose that $F \subset K$ is a nonempty closed set such that

$$T_i(x) = x \text{ for all } x \in F \text{ and all integers } i \geq 1 \tag{2.2}$$

and that the following property holds:

(P1) for every positive number ϵ and every positive number M, there exists an integer $n(M, \epsilon) \geq 1$ such that if $x \in B(\theta, M)$, $p \geq 1$ is an integer and $\prod_{i=1}^{n(M,\epsilon)} T_{p+i}(x)$ exists, then

$$\rho(\prod_{i=1}^{n(M,\epsilon)} T_{p+i}(x), F) \leq \epsilon.$$

Note that property (P1) indeed holds for a sequence of strict contractions and for many infinite products of nonexpansive mappings of contractive type [16].

The following theorem is our first main result.

Theorem 2.1. *Assume that a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ is bounded,*

$$\lim_{i \to \infty} \rho(x_{i+1}, T_{i+1}(x_i)) = 0 \tag{2.3}$$

and that there exists a positive number r such that

$$B(x_i, r) \subset K$$

for all sufficiently large natural numbers i. Then

$$\lim_{i \to \infty} \rho(x_i, F) = 0.$$

Proof. By assumption, there exists a number $M > 0$ for which

$$x_i \in B(\theta, M) \text{ for every integer } i \geq 0. \tag{2.4}$$

There also exists a natural number p_0 such that

$$B(x_i, r) \subset K \text{ for all integers } i \geq p_0. \tag{2.5}$$

Let ϵ be a positive number. By property (P1), there exists an integer $n_0 \geq 1$ such that the following property holds:

(P2) for every point $x \in B(\theta, M + 4)$ and every integer $p \geq 1$ such that $\prod_{i=1}^{n_0} T_{p+i}(x)$ exists, we have

$$\rho(\prod_{i=1}^{n_0} T_{p+i}(x), F) \leq \epsilon/4.$$

Choose a positive number

$$r_0 \leq n_0^{-1} \min\{2^{-1}r, 4^{-1}\epsilon\}. \tag{2.6}$$

Equation (2.3) implies that there exists an integer $n_1 \geq 1$ such that

$$\rho(x_{i+1}, T_{i+1}(x_i)) \leq r_0 \text{ for every integer } i \geq n_1. \tag{2.7}$$

Assume that

$$n \geq n_0 + n_1 + p_0 \tag{2.8}$$

is an integer. We claim that

$$\rho(x_n, F) \leq \epsilon.$$

To see this, consider first the point x_{n-n_0}. Equations (2.5) and (2.8) imply the inclusion

$$B(x_{n-n_0}, r) \subset K. \tag{2.9}$$

It follows from (2.7) and (2.8) that

$$n - n_0 \geq n_1 + p_0 \tag{2.10}$$

and that

$$\rho(x_{n-n_0+1}, T_{n-n_0+1}(x_{n-n_0})) \leq r_0. \tag{2.11}$$

In view of (2.5), (2.6) and (2.11), we have

$$B(T_{n-n_0+1}(x_{n-n_0}), r - r_0) \subset B(x_{n-n_0+1}, r) \subset K. \tag{2.12}$$

Assume that

$$k \in \{1, \dots, n_0\} \setminus \{n_0\}$$

and that for every integer $i \in \{1, \dots, k\}$,

$$\prod_{j=1}^{i} T_{j+n-n_0}(x_{n-n_0}) \in K \text{ is defined,}$$

$$\rho(x_{i+n-n_0}, \prod_{j=1}^{i} T_{j+n-n_0}(x_{n-n_0})) \leq ir_0 \leq 2^{-1}r \tag{2.13}$$

and

$$B(\prod_{j=1}^{i} T_{j+n-n_0}(x_{n-n_0}), r - ir_0) \subset B(x_{n-n_0+i}, r) \subset K. \tag{2.14}$$

(Note that by (2.11)–(2.13), our assumption does hold for $k = 1$.) Equations (2.1), (2.5)–(2.8) and (2.13) imply that

$$\rho(x_{k+n-n_0+1}, \prod_{j=1}^{k+1} T_{j+n-n_0}(x_{n-n_0}))$$

$$\leq \rho(x_{k+n-n_0+1}, T_{k+n-n_0+1}(x_{k+n-n_0}))$$

$$+\rho(T_{k+n-n_0+1}(x_{k+n-n_0}), \prod_{j=1}^{k+1} T_{j+n-n_0}(x_{n-n_0}))$$

$$\leq r_0 + \rho(x_{k+n-n_0}, \prod_{j=1}^{k} T_{j+n-n_0}(x_{n-n_0}))$$

$$\leq r_0 + kr_0 \leq n_0 r_0 \leq 2^{-1}r$$

and

$$B(\prod_{j=1}^{k+1} T_{j+n-n_0}(x_{n-n_0}), r - (k+1)r_0) \subset B(x_{k+1+n-n_0}, r) \subset K.$$

This means that the assumption made for k also holds for $k+1$. Therefore, by using induction, we have shown that our assumption holds for $k = n_0$,

$$\prod_{j=1}^{n_0} T_{j+n-n_0}(x_{n-n_0}) \in K \text{ is defined,}$$

$$\rho(x_n, \prod_{j=1}^{n_0} T_{j+n-n_0}(x_{n-n_0})) \le n_0 r_0 \le 2^{-1} r \tag{2.15}$$

and

$$B(\prod_{j=1}^{n_0} T_{j+n-n_0}(x_{n-n_0}), 2^{-1}r) \subset K. \tag{2.16}$$

By property (P2), (2.4) and (2.16), we have

$$\rho(\prod_{j=1}^{n_0} T_{j+n-n_0}(x_{n-n_0}), F) \le \epsilon/4.$$

When combined with (2.6) and (2.15), this inequality implies that

$$\rho(x_n, F) \le n_0 r_0 + \epsilon/4 \le \epsilon.$$

Thus for all integers $n \ge n_0 + n_1 + p_0$, we have

$$\rho(x_n, F) \le \epsilon,$$

as claimed. Since ϵ is an arbitrary positive number, this completes the proof of Theorem 2.1. □

3 Second result

In this section we continue to use the assumptions and notation introduced in Section 2.

Theorem 3.1. *Assume that a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ satisfies*

$$\lim_{i\to\infty} \rho(x_{i+1}, T_{i+1}(x_i)) = 0, \tag{3.1}$$

$$\liminf_{i\to\infty} \rho(x_i, X \setminus K) > 0 \tag{3.2}$$

and that it has a bounded subsequence $\{x_{i_p}\}_{p=1}^{\infty}$. Then

$$\liminf_{i\to\infty} \rho(x_i, F) = 0.$$

Proof. There exists a number $M > 0$ for which

$$x_{i_p} \in B(\theta, M) \text{ for every integer } p \ge 1. \tag{3.3}$$

In view of (3.2), there exist $r > 0$ and a natural number p_0 such that

$$B(x_i, r) \subset K \text{ for every integer } i \geq p_0 \tag{3.4}$$

Let ϵ be a positive number. It follows from property (P1) that there exists a natural number n_0 such that the following property holds:

(P3) for every point $x \in B(\theta, M + 4)$ and every integer $p \geq 1$ such that $\prod_{i=1}^{n_0} T_{p+i}(x)$ exists, we have

$$\rho(\prod_{i=1}^{n_0} T_{p+i}(x), F) \leq \epsilon/4.$$

Choose a positive number

$$r_0 \leq n_0^{-1} \min\{2^{-1}r, 4^{-1}\epsilon\}. \tag{3.5}$$

Equation (3.1) implies that there exists an integer $n_1 \geq 1$ for which

$$\rho(x_{i+1}, T_{i+1}(x_i)) \leq r_0 \text{ for every integer } i \geq n_1. \tag{3.6}$$

Assume now that an integer $p \geq 1$ satisfies

$$i_p \geq n_1 + p_0. \tag{3.7}$$

We claim that

$$\rho(x_{i_p+n_0}, F) \leq \epsilon.$$

Indeed, in view of (3.4) and (3.7),

$$B(x_{i_p}, r) \subset K. \tag{3.8}$$

By (3.6) and (3.7), we have

$$\rho(x_{i_p+1}, T_{i_p+1}(x_{i_p})) \leq r_0 \tag{3.9}$$

and

$$B(T_{i_p+1}(x_{i_p}), r - r_0) \subset B(x_{i_p+1}, r) \subset K. \tag{3.10}$$

Assume that

$$k \in \{1, \dots, n_0\} \setminus \{n_0\}$$

and that for each $j \in \{1, \dots, k\}$,

$$\prod_{i=1}^{j} T_{i+i_p}(x_{i_p}) \in K \text{ is defined,}$$

$$\rho(x_{i_p+j}, \prod_{i=1}^{j} T_{i+i_p}(x_{i_p})) \le jr_0 \le 2^{-1}r \tag{3.11}$$

and

$$B(\prod_{i=1}^{j} T_{i+i_p}(x_{i_p}), r - jr_0) \subset B(x_{i_p+j}, r) \subset K. \tag{3.12}$$

(Note that in view of (3.9) and (3.10) our assumption does hold for $k = 1$.) It follows from (2.1), (3.5)–(3.7) and (3.11) that

$$\rho(x_{i_p+k+1}, \prod_{i=1}^{k+1} T_{i_p+i}(x_{i_p}))$$

$$\le \rho(x_{i_p+k+1}, T_{i_p+k+1}(x_{i_p+k})) + \rho(T_{i_p+k+1}(x_{i_p+k}), \prod_{i=1}^{k+1} T_{i_p+i}(x_{i_p}))$$

$$\le r_0 + kr_0 = (k+1)r_0$$

and

$$B(\prod_{i=1}^{k+1} T_{i_p+i}(x_{i_p}), r - (k+1)r_0) \subset B(x_{i_p+k+1}, r) \subset K.$$

In other words, the assumption made for k also holds for $k+1$. Therefore, by using induction, we have shown that our assumption holds for $k = n_0$,

$$\prod_{i=1}^{n_0} T_{i_p+i}(x_{i_p}) \in K \text{ is defined,}$$

$$\rho(x_{i_p+n_0}, \prod_{i=1}^{n_0} T_{i_p+i}(x_{i_p})) \le n_0 r_0 \tag{3.13}$$

and

$$B(\prod_{i=1}^{n_0} T_{i_p+i}(x_{i_p}), 2^{-1}r) \subset K. \tag{3.14}$$

It follows from property (P3), (3.3) and (3.14) that

$$\rho(\prod_{i=1}^{n_0} T_{i_p+i}(x_{i_p}), F) \le \epsilon/4.$$

When combined with equations (3.5) and (3.13), the above inequality implies that

$$\rho(x_{i_p+n_0}, F) \le n_0 r_0 + \epsilon/4 \le \epsilon,$$

as claimed. Since the above inequality holds for all integers $p \geq 1$ such that $i_p \geq n_1 + p_0$, we conclude that

$$\liminf_{i \to \infty} \rho(x_i, F) \leq \epsilon.$$

Since ϵ is an arbitrary positive number, this completes the proof of Theorem 3.1. □

4 Third result

Theorem 4.1. *Assume that a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ satisfies*

$$\lim_{i \to \infty} \rho(x_{i+1}, T_{i+1}(x_i)) = 0, \tag{4.1}$$

$r > 0$,

$$B(x_i, r) \subset K \tag{4.2}$$

for all sufficiently large natural numbers i and that there exists a bounded subsequence $\{x_{i_p}\}_{p=1}^{\infty}$ such that

$$\sup\{i_{p+1} - i_p : \ p = 1, 2, \dots\} < \infty. \tag{4.3}$$

Then

$$\lim_{i \to \infty} \rho(x_i, F) = 0.$$

Proof. By Theorem 2.1, it suffices to show that the sequence $\{x_i\}_{i=0}^{\infty}$ is bounded.

To see this, we first note that, by assumption, there exists a number $M > 0$ such that

$$x_{i_p} \in B(\theta, M), \ p = 1, 2, \dots. \tag{4.4}$$

In view of (4.1), there exists $\Delta > 0$ such that

$$\rho(x_{i+1}, T_{i+1}(x_i)) \leq \Delta. \tag{4.5}$$

Equation (4.3) implies that there exists a natural number q such that

$$i_{p+1} < i_p + q, \ p = 1, 2 \dots. \tag{4.6}$$

Fix

$$\theta_0 \in F. \tag{4.7}$$

Let $j \geq i_1$ be an integer. By (4.6), there exists a natural number p such that

$$i_p \leq j < i_{p+1} < i_p + q. \tag{4.8}$$

Equation (4.4) implies that

$$\rho(\theta_0, x_{i_p}) \leq \rho(\theta_0, \theta) + \rho(\theta, x_{i_p}) \leq \rho(\theta_0, \theta) + M. \tag{4.9}$$

It follows from (2.1), (2.2), (4.5) and (4.7) that for each $i \in \{i_p, \dots, i_{p+1} - 1\}$, we have

$$\begin{aligned}\rho(\theta_0, x_{i+1}) &\leq \rho(\theta_0, T_{i+1}(x_i)) + \rho(T_{i+1}(x_i), x_{i+1}) \\ &\leq \rho(\theta_0, x_i) + \Delta.\end{aligned}$$

When combined with (4.8) and (4.9), this implies that

$$\rho(\theta_0, x_j) \leq q\Delta + \rho(\theta_0, x_{i_p}) \leq q\Delta + M + \rho(\theta_0, \theta)$$

and

$$\rho(\theta_0, x_j) \leq q\Delta + M + \rho(\theta_0, \theta)$$

for all integers $j \geq i_1$. Therefore the sequence $\{x_i\}_{i=0}^{\infty}$ is indeed bounded. This completes the proof of Theorem 4.1. □

5 Fourth result

Theorem 5.1. *Assume that the set F is bounded, a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ satisfies*

$$\lim_{i \to \infty} \rho(x_{i+1}, T_{i+1}(x_i)) = 0, \tag{5.1}$$

$r > 0$,

$$B(x_i, r) \subset K \tag{5.2}$$

for all sufficiently large natural numbers i and that there exists a bounded subsequence $\{x_{i_p}\}_{p=1}^{\infty}$. Then

$$\lim_{i \to \infty} \rho(x_i, F) = 0.$$

Proof. By assumption, there exists a number $M > 0$ such that

$$F \subset B(\theta, M) \tag{5.3}$$

and

$$x_{i_p} \in B(\theta, M), \ p = 1, 2, \dots. \tag{5.4}$$

In view of (5.2), there exists a natural number q_0 such that

$$B(x_i, r) \subset K \text{ for every integer } i \geq q_0. \tag{5.5}$$

It follows from property (P1) that there also exists a natural number n_0 such that the following property holds:

(P4) for every point $x \in B(\theta, 2M+4)$ and every integer $p \geq 1$ such that $\prod_{i=1}^{n_0} T_{p+i}(x)$ exists, we have

$$\rho(\prod_{i=1}^{n_0} T_{p+i}(x), F) \leq 1.$$

Choose a positive number

$$r_0 \leq n_0^{-1} \min\{2^{-1}r, 2^{-1}\}. \tag{5.6}$$

Equation (5.1) implies that there exists an integer $n_1 \geq 1$ for which

$$\rho(x_{i+1}, T_{i+1}(x_i)) \leq r_0 \text{ for every integer } i \geq n_1. \tag{5.7}$$

Next, assume that an integer m satisfies

$$m \geq n_1 + q_0, \tag{5.8}$$

and

$$\rho(x_m, \theta) \leq 2M + 4. \tag{5.9}$$

In view of (5.5) and (5.8),

$$B(x_m, r) \subset K.$$

By (5.7) and (5.8), we have

$$\rho(x_{m+1}, T_{m+1}(x_m)) \leq r_0 \tag{5.10}$$

and

$$B(T_{m+1}(x_m), r - r_0) \subset B(x_{m+1}, r) \subset K. \tag{5.11}$$

Assume that

$$k \in \{1, \dots, n_0\} \setminus \{n_0\}$$

and that for each $j \in \{1, \dots, k\}$,

$$\prod_{i=1}^{j} T_{i+m}(x_m) \in K \text{ is defined,}$$

$$\rho(x_{m+j}, \prod_{i=1}^{j} T_{m+i}(x_m)) \leq jr_0 \leq 2^{-1}r \tag{5.12}$$

and

$$B(\prod_{i=1}^{j} T_{i+m}(x_m), r - jr_0) \subset B(x_{m+j}, r) \subset K. \tag{5.13}$$

(Note that in view of (5.10) and (5.11) our assumption does hold for $k = 1$.) It follows from (2.1), (5.7), (5.8) and (5.12) that

$$\rho(x_{m+k+1}, \prod_{i=1}^{k+1} T_{m+i}(x_m))$$

$$\leq \rho(x_{m+k+1}, T_{m+k+1}(x_{m+k})) + \rho(T_{m+k+1}(x_{m+k}), \prod_{i=1}^{k+1} T_{m+i}(x_m))$$

$$\leq r_0 + kr_0 = (k+1)r_0$$

and

$$B(\prod_{i=1}^{k+1} T_{m+i}(x_m), r - (k+1)r_0) \subset B(x_{m+k+1}, r) \subset K.$$

In other words, the assumption made for k also holds for $k + 1$. Therefore, by using induction, we have shown that our assumption holds for $k = n_0$,

$$\prod_{i=1}^{n_0} T_{m+i}(x_m) \in K \text{ is defined,}$$

$$\rho(x_{m+n_0}, \prod_{i=1}^{n_0} T_{m+i}(x_m)) \leq n_0 r_0 \tag{5.14}$$

and

$$B(\prod_{i=1}^{n_0} T_{m+i}(x_m), 2^{-1}r) \subset K. \tag{5.15}$$

It now follows from property (P4), (5.9) and (5.15) that

$$\rho(\prod_{i=1}^{n_0} T_{m+i}(x_m), F) \leq 1. \tag{5.16}$$

By (5.3), (5.6), (5.14) and (5.16), we have

$$\rho(x_{m+n_0}, F) \leq \rho(x_{m+n_0}, \prod_{i=1}^{n_0} T_{m+i}(x_m)) + \rho(\prod_{i=1}^{n_0} T_{m+i}(x_m), F)$$

$$\leq n_0 r_0 + 1 \leq 2.$$

It follows from the inequality above and (5.3) that

$$\rho(x_{m+n_0}, \theta) \leq \rho(x_{m+n_0}, F) + \sup\{\rho(z, \theta) : z \in F\} \leq 2 + M.$$

Thus we have shown that the following property holds:

(a) if an integer $m \geq n_1 + q_0$ and $\rho(x_m, \theta) \leq 2M + 4$, then

$$\rho(x_{m+n_0}, \theta) \leq M + 2.$$

There exists an integer $p_0 \geq 1$ such that

$$i_{p_0} \geq n_1 + q_0.$$

By this inequality, (5.4) and property (a),

$$\rho(x_{i_{p_0}+kn_0}, \theta) \leq M + 2$$

for all integers $k \geq 1$. Now the assertion of the theorem follows from Theorem 4.1. □

Acknowledgments. The first author was partially supported by the Israel Science Foundation (Grant No. 820/17), by the Fund for the Promotion of Research at the Technion and by the Technion General Research Fund.

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