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Bounds for global solutions of nonlinear heat equations with nonlinear boundary conditions

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Abstract: In this paper, we consider the initial-boundary value problem for nonlinear heat equations with nonlinear boundary conditions of radiation type. The local well-posedness of this problem is shown by applying an abstract theory for the evolution equation governed by subdifferential operators. Moreover results on the uniform boundedness for time global solutions are obtained.

Keywords: Nonlinear boundary conditions, Bounds for global solutions.

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1 Introduction

In this paper, we are concerned with the asymptotic behavior of global solutions of the following initial boundary value problem for the nonlinear heat equation:

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-2}u & t > 0,\ x \in \Omega, \\ \partial_\nu u + |u|^{q-2}u = 0 & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega. \end{cases} \tag{P}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$; $T > 0$ is a given constant; $p \in (2, \infty)$, $q \in (1, p)$ are given numbers; $u : [0, T] \times \Omega \to \mathbb{R}$ is a real-valued unknown function. This problem (P) is a prototype of nonlinear heat equations with nonlinear boundary conditions of radiation type.

When one tries to set up mathematical models for describing actual nonlinear phenomena, it is crucial to determine right ruling nonlinear structures in domains where the phenomena occur, but it is also very important to pay careful attention to the choice of the boundary conditions. In other words, when we are concerned with the diffusion equations, it should be noted that the standard boundary condition such as Dirichlet or Neumann boundary condition is realistic only for the case where there is some artificial control of the flux on the boundary. For a large scale system, however, it is impossible to give such a control on the boundary. If there is no control of heat flux on the boundary, there is a prototype model in physics well known as Stefan-Boltzmann's law, which says that the heat energy radiation from the surface of the body is proportional to the fourth power of the difference of temperatures between the inside and outside of the body. In this sense, from a physical point of view, it could be more natural to consider nonlinear boundary conditions rather than the linear boundary conditions such as the homogeneous Dirichlet or Neumann boundary condition.

There are large amounts of works concerning the asymptotic behavior of solutions of the following nonlinear heat equation with the homogeneous Dirichlet boundary condition:

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-2}u & t > 0,\ x \in \Omega, \\ u = 0 & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega. \end{cases} \tag{1.1}$$

Uniform bounds of global solutions of (1.1) was first studied by [14] as an abstract equation of the form $u_t + \partial\psi^1(u) - \partial\psi^2(u) = 0$ in $L^2(\Omega)$. Here $\partial\psi^i$ are subdifferentials of lower semi-continuous convex and homogeneous functionals ψ^i $(i = 1, 2)$ on $L^2(\Omega)$, where it is shown that every global solution of (1.1) is uniformly bounded in $H_0^1(\Omega)$ with respect to time for $p \in (2, p_S)$. Here p_S is the Sobolev critical exponent defined by $p_S = \infty$ for $N = 1, 2$; $p_S = \frac{2N}{N-2}$ for $N \geq 3$. Cazenave-Lions [5] showed that every global solution (allowing sing-changing) solution is bounded in $L^\infty(\Omega)$ uniformly in time provided that $p \in (2, p_{CL})$, where $p_{CL} = \infty$ when $N = 1$; $p_{CL} = 2 + \frac{12}{3N-4}$ when $N \geq 2$. (Note that $p_{CL} \leq p_S$ for any $N \in \mathbb{N}$). Giga [6] removed this restriction on p for positive global solutions. Namely the uniform boundedness of every positive global solution of (1.1) in $L^\infty(\Omega)$ was shown for any $p \in (2, p_S)$. Quittner [16] extended this result for sign-changing solutions. The main tool in [6] is the rescaling argument and [16] relies on the bootstrap argument based on the interpolation and the maximal regularity theory. However it seems to be difficult to apply these devices for our problem (P) because of the presence of the nonlinear boundary condition.

The main purpose of this paper is to derive the uniform boundedness in $H^1(\Omega)$ and $L^\infty(\Omega)$ for every global solution of (P) by following the same strategy as that in [14]. However, we can not directly apply arguments in [14], since the functional associated with the Laplacian with nonlinear boundary conditions is not homogeneous, which is one of basic tools used in [14]. Nevertheless by introducing a new substitutive argument to avoid the use of the homogeneity of functionals, we are able to derive uniform bounds for global solutions in $H^1(\Omega)$. Moreover with the aid of Moser's iteration scheme, the uniform bound in $L^\infty(\Omega)$ is also obtained.

This paper is composed as follows. In the next section, we deal with the local well-posedness of (P) in $H^1(\Omega)$ and $L^\infty(\Omega)$. In order to work in $H^1(\Omega)$, we reduce

(P) to abstract evolution equations in a real Hilbert space governed by subdifferential operators and apply a nonmonotone perturbation theory developed in [15]. For the analysis in $L^\infty(\Omega)$, we rely on the L^∞-energy method given in [14].

In §3, our main theorem is stated and its proof is given by following the strategy in [14] and by relying on Moser's iteration scheme, whose main tool is proved in Appendix.

2 Preliminaries and Local Well-posedness

In this section, we are going to show the local well-posedness for (P) by applying the theory of the evolution equation governed by subdifferential operators. Throughout this paper, H designates a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $\Phi(H)$ be the set of all convex and lower-semicontinuous functionals $\phi : H \to (-\infty, +\infty]$ such that its effective domain $D(\phi) := \{u \in H; \phi(u) < \infty\}$ is nonempty. For each $u \in D(\phi)$, we call the set

$$\partial\phi(u) = \{f \in H; \phi(v) - \phi(u) \geq (f, v-u) \quad \forall v \in D(\phi)\}$$

subdifferential of ϕ at u. Then $\partial\phi : H \to 2^H$ becomes a (possibly multivalued) maximal monotone operator with domain $D(\partial\phi) := \{u \in D(\phi); \partial\phi(u) \neq \emptyset\}$, which is called by subdifferential operator. We remark that $D(\partial\phi) \subset D(\phi) \subset \overline{D(\phi)} = \overline{D(\partial\phi)}$ holds (for the proofs see [2, 3]).

Define the functional φ on $L^2(\Omega)$ by

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx + \dfrac{1}{q}\int_{\partial\Omega} |u|^q d\sigma & u \in D(\varphi) := \{u \in H^1(\Omega); u \in L^q(\partial\Omega)\}, \\ +\infty & u \in L^2(\Omega) \setminus D(\varphi), \end{cases}$$

Then we can see that $\varphi \in \Phi(L^2(\Omega))$ and the subdifferential operator associated with φ is given as follows (see [4]):

$$\begin{cases} D(\partial\varphi) = \{u \in H^2(\Omega)\,;\, \partial_\nu u + |u|^{q-2}u = 0 \quad \text{a.e. on } \partial\Omega\}, \\ \partial\varphi(u) = -\Delta u. \end{cases} \tag{2.1}$$

Furthermore the following elliptic estimate for $\partial\varphi$ holds, i.e., there exist some constants c_1, $c_2 > 0$ such that

$$\|u\|_{H^2(\Omega)} \leq c_1 \| -\Delta u + u\|_{L^2(\Omega)} + c_2 \quad \forall u \in D(\partial\varphi) \tag{2.2}$$

and $\overline{D(\partial\varphi)} = L^2(\Omega)$ (see [4, 2]).

Hereafter we denote the L^p norm by $\|\cdot\|_p$ $(1 \leq p \leq \infty)$, a general constant by $C > 0$ which may vary from place to place and Sobolev critical exponent by 2^*, i.e., $2^* = \frac{2N}{N-2}$ for $N \geq 3$, $2^* = \infty$ for $N = 1, 2$.

2.1 Local well-posedness in $D(\varphi)$

We first show the existence of time local solutions of (P) for the initial values which belong to the effective domain $D(\varphi)$ of φ (note that $D(\varphi) \subset H^1(\Omega)$). We here emphasize that even though $\partial\varphi(u) = -\Delta u$ looks like a linear operator, this is not the case since $D(\partial\varphi)$ does not have the linear structure. Therefore we can not rely on Duhamel's principle. Instead we here rely on the following abstract theory of nonlinear evolution equations associated with subdifferential operator.

Proposition 2.1. ([15]) *Let $\phi \in \Phi(H)$ and the following assumptions* (A1) - (A3) *be satisfied.*

(A1) For any $L > 0$, the set $\{u \in H\,;\, \phi(u) + \|u\|^2 \le L\}$ is compact in H.

(A2) $B : H \to H$ satisfies the following ϕ-demiclosedness condition:
If $u_n \to u$ strongly in $C([0,T];H)$, $\partial\phi(u_n) \rightharpoonup \partial\phi(u)$ weakly in $L^2(0,T;H)$ and $B(u_n) \rightharpoonup b$ weakly in $L^2(0,T;H)$, then $b = B(u)$ holds a.e. in $t \in [0,T]$.

(A3) There exist a monotone increasing function $\ell(\cdot) : [0,\infty) \to [0,\infty)$ and $k \in [0,1)$ such that

$$\|B(u)\|^2 \le k\|\partial\phi(u)\|^2 + \ell(\phi(u) + \|u\|) \qquad \forall u \in D(\partial\phi).$$

Then for any $u_0 \in D(\phi)$, there exists a positive number $T_0 = T_0(\|u_0\|, \phi(u_0)) \in [0,T]$ such that the following abstract Cauchy problem in H;

$$\frac{d}{dt}u(t) + \partial\phi(u(t)) + B(u(t)) = 0, \quad t > 0, \qquad u(0) = u_0,$$

possesses a strong solution $u \in C([0,T_0];H)$ such that

$$\frac{d}{dt}u,\ \partial\phi(u),\ B(u) \in L^2(0,T_0;H). \tag{2.3}$$

Then we can apply Proposition 2.1 for the existence part of the following result.

Proposition 2.2. *Let $p \in (2, 2^*)$ and $u_0 \in D(\varphi)$. Then there exists $T_0 = T_0(\varphi(u_0)) >$ 0 such that* (P) *possesses a unique solution u satisfying the following regularity*

$$u \in C([0,T_0];L^2(\Omega)); \quad \partial_t u,\ \Delta u,\ |u|^{p-2}u \in L^2(0,T_0;L^2(\Omega)). \tag{2.4}$$

Proof. (Existence) Put $B(u) = -|u|^{p-2}u$, then (P) is reduced to the following abstract evolution equation in $H := L^2(\Omega)$:

$$\frac{d}{dt}u(t) + \partial\varphi(u(t)) + B(u(t)) = 0, \qquad u(0) = u_0. \tag{2.5}$$

In order to show the existence of a solution of (2.5), we are going to apply Proposition 2.1. To do this, we have only to check three assumptions (A.1), (A.2) and (A.3). It is clear that (A1) follows from the boundedness of the domain Ω and the Rellich-Kondrachov compactness theorem. Since $-B(u)$ is maximal monotone and the maximal monotone operator satisfies the demiclosedness property (in the standard sense), assumption (A2) is also satisfied. To verify (A3), we note that there exists $\lambda = \lambda(p, N) \in (0, 2]$ such that

$$\|u\|_{2(p-1)}^{2(p-1)} \le C\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \qquad \forall u \in H^2(\Omega), \tag{2.6}$$

which will be proved in the next section (see Lemma 3.6).

Then by virtue of (2.6), the elliptic estimate (2.2) and Young's inequality, we obtain

$$\begin{aligned}
\|B(u)\|_2^2 &= \|u\|_{2(p-1)}^{2(p-1)} \\
&\le C\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \\
&\le C\left(\|-\Delta u + u\|_2^{2-\lambda} + 1\right)\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \\
&\le C\left(\|-\Delta u\|_2^{2-\lambda} + \|u\|_2^{2-\lambda} + 1\right)\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \\
&\le \varepsilon\|-\Delta u\|_2^2 + \frac{\lambda}{2}\left(\frac{2-\lambda}{2\varepsilon}\right)^{\frac{2-\lambda}{\lambda}} C^{\frac{2}{\lambda}}\|u\|_{H^1(\Omega)}^{\frac{2(2p-4+\lambda)}{\lambda}} + C\left(\|u\|_2^{2-\lambda} + 1\right)\|u\|_{H^1(\Omega)}^{2p-4+\lambda},
\end{aligned}$$

for every $\varepsilon > 0$. Hence since $\|u\|_{H^1(\Omega)}^2 \le 2\varphi(u) + \|u\|_2^2$, in view of (2.1), we can assure (A3). Thus by Proposition 2.1 we observe that (P) admits a local solution $u \in C([0, T_0]; L^2(\Omega))$ satisfying (2.3).

(Uniqueness) Let u and v be two solutions of (P) on $[0, T_0]$ with the initial values $u_0 \in D(\varphi)$ and $v_0 \in D(\varphi)$, respectively. Then $w := u - v$ satisfies

$$\begin{cases}
\partial_t w - \Delta w = |u|^{p-2}u - |v|^{p-2}v & t > 0,\ x \in \Omega, \\
\partial_\nu w + |u|^{q-2}u - |v|^{q-2}v = 0 & t > 0,\ x \in \partial\Omega, \\
w(0, x) = u_0(x) - v_0(x) & x \in \Omega.
\end{cases} \tag{P_w}$$

Multiplying (P_w) by w and using integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \|\nabla w(t)\|_2^2 + \int_{\partial\Omega}\left(|u|^{q-2}u - |v|^{q-2}v\right) w \, d\sigma \\
&\qquad = \int_\Omega \left(|u|^{p-2}u - |v|^{p-2}v\right) w \, dx.
\end{aligned}$$

Since $u \mapsto |u|^{q-2}u$ is monotone increasing, $\int_{\partial\Omega} \left(|u|^{q-2}u - |v|^{q-2}v\right) w \, d\sigma \geq 0$.

Moreover we note

$$\left||x|^{p-2}x - |y|^{p-2}y\right| = \left|\int_x^y (p-1)|s|^{p-2}ds\right| \leq (p-1)\left(|x|^{p-2} + |y|^{p-2}\right)|x-y|$$

for all x, $y \in \mathbb{R}^1$. Hence by Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega \left(|u|^{p-2}u - |v|^{p-2}v\right) w \, dx &\leq (p-1)\int_\Omega \left(|u|^{p-2} + |v|^{p-2}\right) w^2 dx \\ &\leq (p-1)\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)\|w(t)\|_p^2.\end{aligned}$$

We here recall the following Gagliardo-Nirenberg interpolation inequality (see [11])

$$\|u\|_p \leq C\left(\|\nabla u\|_2^\eta \|u\|_2^{1-\eta} + \|u\|_2\right) \qquad \forall u \in H^1(\Omega),$$

where $\eta \in (0,1)$ is determined by $\frac{1}{p} = \eta\left(\frac{1}{2} - \frac{1}{N}\right) + (1-\eta)\frac{1}{2}$. By this inequality and Young's inequality, we obtain

$$\begin{aligned}&\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)\|w(t)\|_p^2 \\ &\leq C\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)\left(\|\nabla w(t)\|_2^{2\eta}\|w(t)\|_2^{2(1-\eta)} + \|w(t)\|_2^2\right) \\ &\leq \frac{1}{2(p-1)}\|\nabla w(t)\|_2^2 + C\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)^{\frac{1}{1-\eta}}\|w(t)\|_2^2 \\ &\quad + C\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)\|w(t)\|_2^2.\end{aligned}$$

Since u and v satisfy the regularity (2.3) of Proposition 2.1, $\varphi(u)$ and $\varphi(v)$ are absolute continuous on $[0, T_0]$ (see [3]). Noting that $p \in (2, 2^*)$ implies $\|u\|_p \leq C(\varphi(u) + \|u\|_2^2)^{1/2}$, we deduce that $\|u\|_p$ and $\|v\|_p$ are bounded above by some constant $M > 0$ uniformly on $[0, T_0]$. Thus we get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \frac{1}{2}\|\nabla w(t)\|_2^2 \leq C\left((2M^{p-2})^{\frac{1}{1-\eta}} + 2M^{p-2}\right)\|w(t)\|_2^2.$$

Then by Gronwall's inequality, we obtain

$$\|u(t) - v(t)\|_2^2 \leq \|u_0 - v_0\|_2^2 \, e^{2C\left((2M^{p-2})^{\frac{1}{1-\eta}} + 2M^{p-2}\right)t} \qquad \forall t \in [0, T_0],$$

whence follows the uniqueness. $\square$

2.2 Local well-posedness in $L^\infty(\Omega)$

In this subsection, we are going to show the local well-posedness in $L^\infty(\Omega)$ without any restriction on the growth order p. The main tool here is "L^∞-energy method" developed in [13], for which we need prepare the following lemma (see Lemma 2.2 of [13]).

Lemma 2.3. *Let $y(t)$ be a bounded measurable non-negative function on $[0,T]$ and suppose that there exists $y_0 \geq 0$ and a monotone non-decreasing function $\ell(\cdot) : (0,+\infty) \to (0,+\infty)$ such that*

$$y(t) \leq y_0 + \int_0^t \ell(y(s))ds \qquad a.e.\ t \in (0,T).$$

Then for any $y_1 > y_0$, there exists a number $T_0 = T_0(y_0, y_1, \ell(\cdot)) \in (0,T]$ such that

$$y(t) \leq y_1 \qquad a.e.\ t \in [0,T_0]. \tag{2.7}$$

Proof. Put $z(t) = y_0 + \int_0^t \ell(y(s))ds$, then $z(t) \in C([0,T];\mathbb{R}^1)$ and $y(t) \leq z(t)$. So $z(t)$ satisfies

$$z(t) \leq y_0 + \int_0^t \ell(z(s))ds \qquad \text{for all } t \in [0,T]. \tag{2.8}$$

We here claim that

$$z(t) \leq y_1 \qquad \text{for all } t \in [0,T_0], \quad T_0 = \min\Big(\frac{y_1 - y_0}{2\,\ell(y_1)}, T\Big). \tag{2.9}$$

In fact, suppose that (2.9) does not hold, i.e., there exists $t_0 \in (0,T_0]$ such that $z(t_0) > y_1$, then since $z(t)$ is continuous on $[0,T]$ and $z(0) = y_0 < y_1$, there exists $t_1 \in (0,t_0)$ such that $z(t_1) = y_1$ and $z(t) < y_1 \quad \forall t \in [0,t_1)$. Then, by (2.8), we get

$$\begin{aligned} y_1 = z(t_1) &\leq y_0 + \int_0^{t_1} \ell(z(s))ds \\ &\leq y_0 + \ell(y_1)\,T_0 \leq y_0 + \frac{y_1 - y_0}{2} < y_1, \end{aligned}$$

which leads to a contradiction. Thus (2.9) is verified and hence (2.7) is derived from the fact that $y(t) \leq z(t)$ for all $t \in [0,T]$. □

Now the local well-posedness of (P) in $L^\infty(\Omega)$ can be stated as follows.

Proposition 2.4. *Let $u_0 \in L^\infty(\Omega)$, then there exists $T_0 = T_0(\|u_0\|_\infty) > 0$ such that* (P) *possesses a unique solution u satisfying the following regularity*

$$\begin{gathered} u \in C([0,T_0]; L^2(\Omega)) \cap L^\infty(0,T_0; L^\infty(\Omega)), \\ \sqrt{t}\, \partial_t u,\ \ \sqrt{t}\, \Delta u,\ \ \sqrt{t}\, |u|^{p-2}u \in L^2(0,T_0; L^2(\Omega)). \end{gathered} \tag{2.10}$$

Proof. (Uniqueness) Let u and v be two solutions of (P) with the same initial data $u_0 \in L^\infty(\Omega)$ satisfying the regularity (2.10). Then $w := u - v$ satisfies (P_w) with $w(0) = 0$. Multiplying (P_w) by w, we now get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 &\le \int_\Omega \left(|u|^{p-2}u - |v|^{p-2}v\right) w\, dx \\ &\le (p-1)\int_\Omega \left(|u|^{p-2} + |v|^{p-2}\right) w^2 dx \\ &\le (p-1)\left(\|u\|^{p-2}_{L^\infty(0,T;L^\infty(\Omega))} + \|v\|^{p-2}_{L^\infty(0,T;L^\infty(\Omega))}\right)\|w(t)\|_2^2 \\ &\le C\,\|w(t)\|_2^2, \end{aligned}$$

whence follows from Gronwall's inequality

$$\|w(t)\|_2^2 \le \|w(0)\|_2^2 e^{2CT} = 0 \qquad \forall t \in (0,T).$$

(Existence) We consider the following auxiliary problem:

$$\begin{cases} \partial_t u - \Delta u = |[u]_M|^{p-2}\, u & t > 0,\ x \in \Omega, \\ \partial_\nu u + |u|^{q-2}u = 0 & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases} \tag{2.11}$$

Here

$$M = \|u_0\|_\infty + 2 \tag{2.12}$$

and $[u]_M$ is a cut-off function of u defined by

$$[u]_M = \begin{cases} M & u \ge M, \\ u & |u| \le M, \\ -M & u \le -M. \end{cases}$$

Set $B_M(u) = -|[u]_M|^{p-2}u$, then the auxiliary problem (2.11) can be reduced to the following evolution equation in $L^2(\Omega)$:

$$\frac{d}{dt}u(t) + \partial\varphi(u(t)) + B_M(u(t)) = 0, \qquad u(0) = u_0. \tag{2.13}$$

Since $B_M : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous, we know that (2.13) has a unique global solution $u \in C([0,T]; L^2(\Omega))$ for $u_0 \in L^2(\Omega)$ satisfying the same regularity (except L^∞-estimate) of Proposition 2.4 with T_0 replaced by T by applying the abstract theory developed by H. Brézis (see Proposition 3.12 in [3]).

Furthermore we can show that $u_0 \in L^\infty(\Omega)$ assures $u(t) \in L^\infty(\Omega)$ for all $t \geq 0$. To see this, put $v(t) := e^{-M^{p-2}t}u(t)$, then $v(t)$ satisfies

$$\partial_t v(t) - \Delta v(t) = \Big(|[u]_M|^{p-2} - M^{p-2}\Big) v(t), \quad v(0) = u_0. \tag{2.14}$$

Multiplying (2.14) by $[v(t) - M]^+ := \max(v(t) - M, 0)$ and noting that $|[u]_M|^{p-2} - M^{p-2} \leq 0$, we get

$$\frac{1}{2}\frac{d}{dt}\left\|[v(t) - M]^+\right\|_2^2 + \int_\Omega |\nabla [v(t) - M]^+|^2\, dx \leq 0. \tag{2.15}$$

Here we used the fact that

$$\begin{aligned}
-\int_\Omega \Delta v [v - M]^+ dx &= \int_\Omega |\nabla [v - M]^+|^2 dx - \int_{\partial\Omega} \partial_\nu v [v - M]^+ d\sigma \\
&= \int_\Omega |\nabla [v - M]^+|^2 dx + \int_{\partial\Omega} |u|^{q-2} v [v - M]^+ d\sigma \\
&\geq \int_\Omega |\nabla [v - M]^+|^2 dx + \int_{\partial\Omega} |u|^{q-2} M [v - M]^+ d\sigma \\
&\geq \int_\Omega |\nabla [v - M]^+|^2 dx.
\end{aligned}$$

Hence $\|[v(t) - M]^+\|_2 \leq \|[u_0 - M]^+\|_2 = 0$ (which is assured by (2.12)), i.e., $v(t) \leq M$ so we get $u(t) \leq M e^{M^{p-2}t}$ for a.e. $t \in [0, \infty)$.

Multiply again (2.14) by $[v(t) + M]^- = \max\,(-v(t) - M, 0)$. Then in parallel with (2.15), we get

$$\frac{1}{2}\frac{d}{dt}\left\|[v(t) + M]^-\right\|_2^2 + \int_\Omega |\nabla [v(t) + M]^-|^2\, dx \leq 0, \tag{2.16}$$

whence follows $u(t) \geq -M e^{M^{p-2}t}$. Thus we get $|u(t)|_{L^\infty} \leq M e^{M^{p-2}t}$. In particular, we observe that $u(t) \in L^\infty$ for a.e. $t \in [0, \infty)$. Hence noticing that $|u|^{r-2}u \in L^2(\Omega)$ and $|\,|[u]_M|^{p-2}\,| \leq |u|^{p-2}$, we multiply (2.11) by $|u|^{r-2}u$ to obtain

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + (r-1)\int_\Omega |\nabla u|^2 |u|^{r-2} dx + \int_{\partial\Omega} |u|^{q+r-2} d\sigma \\
&= \int_\Omega |[u]_M|^{p-2} |u|^r dx \\
&\leq \int_\Omega |u|^{p+r-2} dx \leq \|u(t)\|_\infty^{p-2} \|u(t)\|_r^r.
\end{aligned}$$

Since the second term and third term of left hand side are nonnegative,

$$\|u(t)\|_r^{r-1}\frac{d}{dt}\|u(t)\|_r \le \|u(t)\|_\infty^{p-2}\|u(t)\|_r^r.$$

Divide both sides by $\|u(t)\|_r^{r-1}$ and integrate with respect to t on $[0,t]$, then we get

$$\|u(t)\|_r \le \|u_0\|_r + \int_0^t \|u(\tau)\|_\infty^{p-2}\|u(\tau)\|_r d\tau$$

Note that even though $\|u(t)\|_r^{r-1}$ attains zero, we can justify this argument by Proposition 1 in [9]. Letting r tend to ∞, we derive

$$\|u(t)\|_\infty \le \|u_0\|_\infty + \int_0^t \|u(\tau)\|_\infty^{p-1} d\tau.$$

Hence applying Lemma 2.3 with $y_0 = \|u_0\|_\infty,\ y_1 = y_0 + 1$ and $\ell(y) = y^{p-1}$, we see that there exists $T_0 > 0$ such that

$$\|u(t)\|_\infty \le \|u_0\|_\infty + 1 \qquad \text{a.e. } t \in [0, T_0].$$

Since $M > \|u_0\|_\infty + 1$ by (2.12), we can see that u gives a solution for (P) on $[0, T_0]$ by the definition of cut-off function $[u]_M$. □

Remark 2.5. If $\|u_0\|_\infty > 0$, then applying Lemma 2.3 with $y_0 = \|u_0\|_\infty,\ y_1 = 2\,y_0$ and $\ell(y) = y^{p-1}$ and choosing $T_0 = \frac{1}{2^p\|u_0\|_\infty^{p-2}}$, we can show

$$\|u(t)\|_\infty \le 2\,\|u_0\|_\infty \qquad \text{a.e. } t \in [0, T_0].$$

From this observation we can deduce that the maximal existence time $T_m(u)$ of u is larger than $\frac{1}{2^p\|u_0\|_\infty^{p-2}}$, which can be sufficiently large for sufficiently small $\|u_0\|_\infty > 0$.

3 Uniform Bounds for Global Solutions

In this section, we discuss the existence of uniform bounds for global solutions of (P). In order to investigate this, we make most use of a variational structure of our problem, which can be characterized by the following functionals. Set

$$\psi(u) = \frac{1}{p}\|u\|_p^p, \tag{3.1}$$

$$J(u) = \varphi(u) - \psi(u) \tag{3.2}$$

and

$$j(u) = -\|\nabla u\|_2^2 - \int_{\partial\Omega} |u|^q d\sigma + \|u\|_p^p. \tag{3.3}$$

Let u be a global solution of (2.5) satisfying (2.4). Then multiplying (2.5) by u and $du(t)/dt$, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 = j(u(t)) \qquad \forall t \in [0,\infty) \tag{3.4}$$

and

$$\frac{d}{dt}J(u(t)) + \left\|\frac{du}{dt}(t)\right\|_2^2 = 0 \qquad a.e.\ t \in (0,\infty). \tag{3.5}$$

Hence, in particular, it is obvious that J is monotone non-increasing in $(0,\infty)$ and

$$J(u(t)) \le J_0 := J(u_0) \qquad \text{for all } \ t \ge 0. \tag{3.6}$$

Now our main theorems can be stated as follows.

Theorem 3.1. *Assume that $q \in (1,p)$, $p \in (2,2^*)$ and $u_0 \in D(\varphi)$. Let u be a global strong solution of* (P) *satisfying* (2.4)*. Then we have*

$$\|u(t)\|_2 \le \left[\frac{q_2\, p\, J_0\, |\Omega|^{\frac{p-2}{2}}}{p-q_2}\right]^{1/p} \qquad \forall t \ge 0, \tag{3.7}$$

$$\sup_{t\ge 0} \varphi(u(t)) < \infty, \tag{3.8}$$

where $q_2 := \max\,(2,q)$.

Theorem 3.2. *Assume that $q \in (1,p)$, $p \in (2,2^*)$ and $u_0 \in L^\infty(\Omega)$. Let u be a global strong solution of* (P) *satisfying* (2.10)*. Then there exists $C_\infty = C_\infty(p,q,|\Omega|)$ such that*

$$\|u(t)\|_2 \le C_\infty\, \|u_0\|_\infty \qquad \forall t \ge 0, \tag{3.9}$$

$$\sup_{t\ge 0} \|u(t)\|_\infty < \infty. \tag{3.10}$$

To prove these theorems, we prepare some lemmas.

Lemma 3.3. *Let $q_2 = \max\,(2,q) < p$ and let u be a global solution of* (2.5) *satisfying* (2.4)*. Then we have*

$$0 \le J(u(t)) \le J_0 \qquad \forall t \ge 0, \tag{3.11}$$

$$\|u(t)\|_2 \le B_{L^2} := \left[\frac{q_2\, p\, J_0\, |\Omega|^{\frac{p-2}{2}}}{p-q_2}\right]^{1/p} \qquad \forall t \ge 0. \tag{3.12}$$

Furthermore there exists a constant C_0 depending only on p, q, J_0 and $|\Omega|$ such that

$$\sup_{t\geq 0}\int_t^{t+1}\left(\psi(u(s))^2+\varphi(u(s))^2\right)\,ds\leq C_0. \tag{3.13}$$

Proof. By (3.4), (3.3)and (3.6), we get

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_2^2 &= -2\left(\|\nabla u(t)\|_2^2+\int_{\partial\Omega}|u(t)|^q d\sigma-\|u(t)\|_p^p\right)\\
&\geq -2\left(\frac{q_2}{2}\|\nabla u(t)\|_2^2+\frac{q_2}{q}\int_{\partial\Omega}|u(t)|^q d\sigma-\frac{q_2}{p}\|u(t)\|_p^p\right)+\frac{2(p-q_2)}{p}\|u(t)\|_p^p\\
&\geq -2\,q_2\,J(u(t))+\frac{2(p-q_2)}{p}\|u(t)\|_p^p \qquad (3.14)\\
&\geq -2\,q_2\,J(u(t))+\frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \qquad (3.15)\\
&\geq -2\,q_2\,J_0+\frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \quad \forall t\in[0,\infty). \qquad (3.16)
\end{aligned}$$

Suppose that $J(u(t_1)) < 0$ for some $t_1 \in [0,\infty)$, then by (3.5) we get $J(u(t)) < 0$ for all $t \in [t_1,\infty)$, which together with (3.15) yields

$$\frac{d}{dt}\|u(t)\|_2^2\geq\frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \qquad \forall t\in[t_1,\infty). \tag{3.17}$$

Since $p > q_2 \geq 2$ and $J(u(t_1)) < 0$ implies $\|u(t_1)\|_2 > 0$, it follows from (3.17) that $\|u(t)\|_2$ blows up in finite time, which leads to a contradiction. Thus (3.11) is derived.

Suppose now that $\|u(t_2)\|_2 > B_{L^2}$ for some $t_2 \in [0,\infty)$, i.e., $\frac{d}{dt}\|u(t_2)\|_2^2 > 0$, then $\|u(t)\|_2$ is monotone increasing in the neighborhood of $t = t_2$. Hence, by (3.16), we can easily see that

$$\frac{d}{dt}\|u(t)\|_2^2\geq\delta:=-2J_0+\frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t_2)\|_2^p>0 \quad \forall t\in[t_2,\infty),$$

which implies that $\|u(t)\|_2$ is strictly monotone increasing and tends to ∞ as $t\to\infty$. Hence there exists $t_3 > t_2$ such that

$$\frac{d}{dt}\|u(t)\|_2^2\geq\frac{(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \qquad \forall t\in[t_3,\infty).$$

This leads to a contradiction as before. Thus (3.12) is verified.

Furthermore, since $d\|u(t)\|_2^2/dt = 2\,(u(t), du(t)/dt)_{L^2} \leq 2\,\|u(t)\|_2\,\|du(t)/dt\|_2$, (3.5), (3.6) and (3.12) assure that $\int_t^{t+1} |d\|u(s)\|_2^2/ds|^2 ds$ is uniformly bounded. Hence, in view of (3.2) and (3.6), we can derive (3.13) from (3.14). □

As a consequence of lemma 3.3 and monotonicity of $J(u(t))$, we can conclude that

$$\lim_{t\to\infty} J(u(t)) =: J_\infty \geq 0. \tag{3.18}$$

Remark 3.4. Estimate (3.12) implies that if $J_0 = 0$, then there is no global solution of (2.5) except the trivial solution $u(t) \equiv 0$.

Lemma 3.5. *Let $q_2 := \max\,(2, q) < p$ and let u be a global solution of* (2.5) *satisfying* (2.4). *Then we have*

$$\liminf_{t\to\infty} \varphi(u(t)) \leq \frac{pJ_0 + 1}{p - q_2}. \tag{3.19}$$

Proof. Suppose that

$$\liminf_{t\to\infty} \varphi(u(t)) > \frac{pJ_0 + 1}{p - q_2}.$$

Then we can see that there exists $t_0 > 0$ such that

$$\varphi(u(t)) \geq \frac{pJ_0 + 1}{p - q_2} \qquad \forall t \geq t_0. \tag{3.20}$$

By (3.4) and (3.20), it holds that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 &= j(u(t)) \\
&= -\|\nabla u(t)\|_2^2 - \int_{\partial\Omega} |u(t)|^q d\sigma + \|u(t)\|_p^p \\
&\geq -\frac{q_2}{2}\|\nabla u(t)\|_2^2 - \frac{q_2}{q}\int_{\partial\Omega} |u(t)|^q d\sigma + \|u(t)\|_p^p \\
&= -q_2\varphi(u(t)) + p\psi(u(t)) \\
&= -pJ(u(t)) + (p - q_2)\varphi(u(t)) \\
&\geq -pJ_0 + (p - q_2)\varphi(u(t)) \geq 1 \qquad \forall t \geq t_0.
\end{aligned} \tag{3.21}$$

Hence we get

$$\|u(t)\|_2^2 \geq \|u(t_0)\|_2^2 + 2(t - t_0) \qquad \forall t \geq t_0,$$

whence it follows that $\|u(t)\|_2 \to \infty$ as $t \to \infty$, which contradicts (3.12). □

Lemma 3.6. *Let $p \in (2, 2^*)$, then there exists a constant $\lambda = \lambda(N,p) \in (0,2]$ such that*

$$\|u\|_{2(p-1)}^{2(p-1)} \le C\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \qquad \forall u \in H^2(\Omega) \tag{3.22}$$

for some $C > 0$.

Proof. First of all, if $N = 1, 2$; or $N \ge 3$ and $p \le \frac{2(N-1)}{N-2}$, then we can take $\lambda = 2$ by Sobolev's embedding $H^1(\Omega) \subset L^{2(p-1)}(\Omega)$. For the case of $N \ge 3$ and $p > \frac{2(N-1)}{N-2}$, we note that the following Gagliardo-Nirenberg inequality holds:

$$\|v\|_{2(p-1)} \le C\|v\|_{H^2(\Omega)}^{\theta}\|v\|_{\frac{2N}{N-2}}^{1-\theta} \qquad \forall v \in H^2(\Omega), \tag{3.23}$$

where $\theta \in (0,1)$ satisfies

$$\tfrac{1}{2(p-1)} = \theta\left(\tfrac{1}{2} - \tfrac{2}{N}\right) + (1-\theta)\tfrac{N-2}{2N}.$$

Then we see that $\frac{2(N-1)}{N-2} < p < \frac{2N}{N-2}$ implies $0 < \theta = \frac{(N-2)p-2N+2}{2(p-1)} < 1$ and $0 < 2(p-1)\theta = (N-2)p - 2N + 2 < 2$. Since $H^1(\Omega)$ is continuously embedded in $L^{\frac{2N}{N-2}}(\Omega)$, it follows from (3.23) that (3.22) holds with $\lambda = 2N-(N-2)p \in (0,2)$. $\square$

Lemma 3.7. *Let $p \in (2, 2^*)$ and u be a global solution of* (P)*. Then there exists a monotone decreasing function $T_0(\cdot) : [0,\infty) \to (0,\infty)$ such that for every $t_0 > 0$*

$$\varphi(u(t)) \le \varphi(u(t_0)) + 1 \qquad \forall t \in [t_0, t_0 + T_0(\varphi(u(t_0)))].$$

Proof. Multiplying (P) by $-\Delta u = \partial\varphi(u(t))$, we get by (3.22),

$$\begin{aligned}
\frac{d}{dt}\varphi(u(t)) + \|\Delta u(t)\|_2^2 &\le \int_\Omega |\Delta u||u|^{p-1}dx \\
&\le \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|u(t)\|_{2(p-1)}^{2(p-1)} \\
&\le \frac{1}{2}\|\Delta u(t)\|_2^2 + C\|u(t)\|_{H^2(\Omega)}^{2-\lambda}\|u(t)\|_{H^1(\Omega)}^{2p-4+\lambda}.
\end{aligned}$$

By (2.2) and Young's inequality, for any $\eta > 0$, there exists C_η such that

$$\begin{aligned}
\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} &\le \eta\,\|u\|_{H^2(\Omega)}^2 + C_\eta\|u\|_{H^1(\Omega)}^{\frac{2(2p-4+\lambda)}{\lambda}} \\
&\le \eta\, C\left(\|\Delta u\|_2^2 + \|u\|_2^2 + 1\right) + C_\eta\|u\|_{H^1(\Omega)}^{\frac{2(2p-4+\lambda)}{\lambda}} \\
&\le \eta\, C\|\Delta u\|_2^2 + M_\eta(\varphi(u)),
\end{aligned}$$

where $M_\eta(\cdot)$ is a monotone increasing function on $\mathbb{R}^+$ of the form $M(s) = C_\eta(s+1)^{\frac{2p-4+\lambda}{\lambda}} + C\,\eta(s+1)$ and we used the fact that $\|u\|^2_{H^1(\Omega)} \le C(\varphi(u)+1)$, which is verified by the Poincaré-Friedrichs inequality, that is, $\|u\|_2^2 \le C(\|\nabla u\|_2^2 + \int_{\partial\Omega}|u|^q d\sigma + 1)$ holds for any $q \in (1,\infty)$. Thus, taking $\eta > 0$ sufficiently small, we obtain

$$\frac{d}{dt}\varphi(u(t)) \le M_\eta(\varphi(u(t))).$$

Hence by applying Lemma 2.3, we can conclude the claim of this lemma (cf. [13]). □

Lemma 3.8. *Let* $q_2 := \max\,(2,q) < p$ *and let* u *be a global solution of* (2.5) *satisfying* (2.4). *Then we have*

$$\limsup_{t\to\infty}\varphi(u(t)) \le \frac{pJ_0+1}{p-q_2} + 3. \tag{3.24}$$

Proof. Suppose that

$$\limsup_{t\to\infty}\varphi(u(t)) > \frac{pJ_0+1}{p-q_2} + 3.$$

Then, by (3.19) of Lemma 3.5, there exist a couple of sequences $\{t_n^i\}_{n=1}^\infty$, $\{t_n^s\}_{n=1}^\infty$ such that

$$t_n^i < t_n^s < t_{n+1}^i, \qquad t_n^i \to \infty \quad \text{as } n\to\infty, \tag{3.25}$$

$$\varphi(u(t_n^i)) = \frac{pJ_0+1}{p-q_2} + 1, \qquad \varphi(u(t_n^s)) = \frac{pJ_0+1}{p-q_2} + 3, \tag{3.26}$$

$$\varphi(u(t)) \ge \frac{pJ_0+1}{p-q_2} + 1 \qquad \forall t \in [t_n^i, t_n^s]. \tag{3.27}$$

Integrating (3.5) over $[0,t]$, we obtain

$$\int_0^t \left\|\frac{du}{d\tau}(\tau)\right\|_2^2 = J_0 - J(u(t)) \le J_0 - J_\infty.$$

Therefore $\frac{du}{dt} \in L^2(0,\infty;L^2(\Omega))$. Hence

$$\varepsilon(t) := \left\|\frac{du}{d\tau}\right\|_{L^2(t,\infty;L^2(\Omega))} \to 0 \qquad \text{as } t\to\infty. \tag{3.28}$$

In view of (3.20) and (3.27), by the same argument as for (3.21), we have

$$1 < \frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 \le \|u(t)\|_2 \left\|\frac{du}{dt}(t)\right\|_2 \qquad \forall t \in [t_n^i, t_n^s]. \tag{3.29}$$

Hence $\|u(t)\|_2^2$ is monotone increasing in $t \in [t_n^i, t_n^s]$, so we get

$$\|u(t)\|_2^2 \leq \|u(t_n^s)\|_2^2 \leq C\left(\varphi(u(t_n^s)) + 1\right) \qquad \forall t \in [t_n^i, t_n^s]. \tag{3.30}$$

Integrating (3.29) over $[t_n^i, t_n^s]$ and making use of (3.30), we get

$$\begin{aligned}
t_n^s - t_n^i &< \int_{t_n^i}^{t_n^s} \|u(\tau)\|_2 \Big\|\frac{du}{d\tau}(\tau)\Big\|_2 d\tau \\
&\leq C\left(\varphi(u(t_n^s)) + 1\right) \int_{t_n^i}^{t_n^s} \Big\|\frac{du}{d\tau}(\tau)\Big\|_2 d\tau \\
&\leq C\left(\varphi(u(t_n^s)) + 1\right) \left(\int_{t_n^i}^{t_n^s} \Big\|\frac{du}{d\tau}(\tau)\Big\|_2^2 d\tau\right)^{\frac{1}{2}} \left(\int_{t_n^i}^{t_n^s} d\tau\right)^{\frac{1}{2}} \\
&\leq C\Big(\frac{pJ_0 + 1}{p - q_2} + 4\Big)\sqrt{t_n^s - t_n^i}\ \varepsilon(t_n^i).
\end{aligned}$$

Therefore from (3.28), we can derive that $t_n^s - t_n^i \to 0$ as $n \to \infty$, which contradicts Lemma 3.7 and (3.26) with a sufficiently large n. □

Now we are ready to give a proof of Theorem 3.1.

Proof of Theorem 3.1. The assertion (3.7) is nothing but (3.12) given in Lemma 3.3. By (3.24) of Lemma 3.8, there exists $T_1 > 0$ such that $\sup_{t \geq T_1} \varphi(u(t)) \leq \frac{p\,J_0+1}{p-q_2} + 4$. Since $\varphi(u(t))$ is continuous on $[0, \infty)$, we have $\sup_{0 \leq t \leq T_1} \varphi(u(t)) < \infty$. Hence (3.8) is verified. □

In order to discuss the uniform bounds of solutions in $L^\infty(\Omega)$, we prepare the following device, which is a variant of results by Alikakos [1] and Nakao [10]. Its proof is given in Appendix and can be done along essentially the same lines in the proof of Lemma 3.1 in [10].

Lemma 3.9. *Let $w \in W^{1,2}_{loc}([0, \infty); L^2(\Omega)) \cap L^\infty_{loc}([0, \infty); L^\infty(\Omega) \cap H^1(\Omega))$ and assume that w satisfies*

$$\frac{d}{dt}\|w(t)\|_r^r + c_0 r^{-\theta_0} \||w(t)|^{\frac{r}{2}}\|^2_{H^1(\Omega)} \leq c_1 r^{\theta_1} \|w(t)\|_r^r \qquad a.e.\ t \in (0, \infty) \tag{3.31}$$

for all $r \in [2, \infty)$, where $c_0 > 0$ and c_1, θ_0, $\theta_1 \geq 0$. Then there exist some positive constants a, b, c such that

$$\sup_{t \geq 0} \|w(t)\|_\infty \leq a^{\frac{1}{2}} 2^{\theta_1 + (\theta_0 + \theta_1)b} M_0,$$

where $M_0 = \max\,(1,\ c\,\|w(0)\|_\infty,\ \sup_{t \geq 0} \|w(t)\|_2)$.

Proof of Theorem 3.2. If $\|u_0\|_\infty = 0$, then the unique solution of (P) is the trivial solution $u(t) \equiv 0$, so (3.10) is obvious. Let $\|u_0\|_\infty > 0$, then as is stated in Remark 2.5, we have

$$\|u(t)\|_\infty \le 2\,\|u_0\|_\infty \qquad a.e.\ t \in [0,T_0] \quad \text{with } T_0 = \frac{1}{2^p\|u_0\|_\infty^{p-2}}. \tag{3.32}$$

In order to apply results prepared for the proof of Theorem 3.1, we are going to derive a priori bounds for $\varphi(u(t))$. Multiplying (2.5) by u, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \varphi(u(t)) \le \|u(t)\|_p^p \le \|u(t)\|_\infty^p\,|\Omega|,$$

where we used the fact that $\varphi(0) = 0$ and the definition of subdifferential yield $\varphi(u) \le (\partial\varphi(u), u)_{L^2}$. Integrating this over $(0, T_0)$ and using (3.32), we obtain

$$\int_0^{T_0} \varphi(u(t))\,dt \le 2^p\,\|u_0\|_\infty^p|\Omega|\,\frac{1}{2^p\|u_0\|_\infty^{p-2}} + \frac{1}{2}\|u_0\|_\infty^2 = (|\Omega| + \frac{1}{2})\,\|u_0\|_\infty^2. \tag{3.33}$$

We now multiply (2.5) by $t\,du(t)/dt$ to get

$$t\,\|\frac{du}{dt}(t)\|_2^2 + t\,\frac{d}{dt}\varphi(u(t)) \le \frac{t}{2}\,\|\frac{du}{dt}(t)\|_2^2 + \frac{t}{2}\,\|u(t)\|_{2(p-1)}^{2(p-1)}.$$

Integrating this over $(0, T_0)$, we get

$$T_0\,\varphi(u(T_0)) \le \int_0^{T_0}\varphi(u(t))\,dt + \frac{T_0^2}{4}\sup_{0\le t\le T_0}\|u(t)\|_\infty^{2(p-1)}\,|\Omega|.$$

Hence in view of (3.32) and (3.33), we obtain

$$\begin{aligned}\varphi(u(T_0)) &\le 2^p\,\|u_0\|_\infty^{p-2}\,(|\Omega| + \frac{1}{2})\|u_0\|_\infty^2 + 2^{p-4}\,\|u_0\|_\infty^p\,|\Omega| \\ &\le 2^{p+1}\,(|\Omega| + \frac{1}{2})\,\|u_0\|_\infty^p.\end{aligned} \tag{3.34}$$

Consequently, from (3.34) and (3.12) of Lemma 3.3, we can derive

$$\sup_{T_0\le t<\infty}\|u(t)\|_2 \le \Big[\frac{q_2\,p\,|\Omega|^{\frac{p-2}{2}}\,2^{p+1}\,(|\Omega| + \frac{1}{2})}{p - q_2}\Big]^{1/p}\,\|u_0\|_\infty. \tag{3.35}$$

Hence since $\|u(t)\|_2 \le \|u(t)\|_\infty\,|\Omega|^{1/2} \le 2\,\|u_0\|_\infty\,|\Omega|^{1/2}$ for all $t \in [0,T_0]$, (3.9) is derived. In order to derive the uniform bound of solutions in $L^\infty(\Omega)$ on $[T_0,\infty)$, we rely on Lemma 3.9.

To do this, we rewrite (P) in the following way:

$$\partial_t u - \Delta u + u = |u|^{p-2}u + u. \tag{3.36}$$

Multiplying (3.36) by $|u|^{r-2}u$ ($r \geq 2$), we obtain

$$\frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r - \int_\Omega |u|^{r-2}u\Delta u dx + \|u(t)\|_r^r = \int_\Omega |u|^{p+r-2}dx + \|u(t)\|_r^r. \tag{3.37}$$

We first note that the left-hand side of (3.37), denoted by (LHS), can be estimated from below as follows:

$$\begin{aligned}
(\mathrm{LHS}) &= \frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + (r-1)\int_\Omega |\nabla u|^2|u|^{r-2}dx + \int_{\partial\Omega} |u|^{q+r-2}d\sigma + \|u(t)\|_r^r \\
&\geq \frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + \frac{4(r-1)}{r^2}\int_\Omega \left|\nabla |u|^{\frac{r}{2}}\right|^2 dx + \||u(t)|^{\frac{r}{2}}\|_2^2 \\
&\geq \frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + \frac{4(r-1)}{r^2}\||u(t)|^{\frac{r}{2}}\|_{H^1(\Omega)}^2,
\end{aligned}$$

Here in order to give an estimate for the right-hand side of (3.37), denoted by (RHS), we use Hölder's inequality of the following form:

$$\int_\Omega |u|^{p+r-2}dx \leq \|u\|_r^{r(1-\alpha)}\,\|u\|_p^{p-2}\,\|u\|_{\frac{sr}{2}}^{\alpha r} \quad \text{with} \quad \alpha = \tfrac{(p-2)s}{p(s-2)}. \tag{3.38}$$

This is valid for all $\alpha \in (0,1)$, which holds if and only if $p < s$. So we take $s = 2^*$ for $N = 3$ and $s = 2p$ for $N = 2$ to get

$$\|u\|_{\frac{sr}{2}}^{\alpha r} = \|\,|u|^{\frac{r}{2}}\|_s^{2\alpha} \leq C\|\,|u|^{\frac{r}{2}}\|_{H^1(\Omega)}^{2\alpha}. \tag{3.39}$$

Then, recalling that $\|u\|_p \leq C(\varphi(u)+1)^{1/2}$ which is uniformly bounded by (3.8), we obtain by (3.38) and (3.39)

$$\begin{aligned}
(\mathrm{RHS}) &\leq \|u(t)\|_r^{r(1-\alpha)}\|u(t)\|_p^{p-2}\|u(t)\|_{\frac{sr}{2}}^{\alpha r} + \|u(t)\|_r^r \\
&\leq C\|u(t)\|_r^{r(1-\alpha)}\Big(\sup_{t\geq T_0}\varphi(u(t))+1\Big)^{\frac{p-2}{2}}\|\,|u(t)|^{\frac{r}{2}}\|_s^{2\alpha} + \|u(t)\|_r^r \\
&\leq \tfrac{2(r-1)}{r^2}\|\,|u(t)|^{\frac{r}{2}}\|_{H^1(\Omega)}^2 + C\left(\tfrac{2(r-1)}{r^2}\right)^{-\frac{\alpha}{1-\alpha}}\|u(t)\|_r^r + \|u(t)\|_r^r,
\end{aligned}$$

Thus since $\frac{r^2}{2(r-1)} \leq r$ and $\frac{2(r-1)}{r} \geq 1$ for all $r \geq 2$, from (3.37) we deduce

$$\frac{d}{dt}\|u(t)\|_r^r + \left\|\,|u(t)|^{\frac{r}{2}}\right\|_{H^1(\Omega)}^2 \leq Cr^{\frac{1}{1-\alpha}}\|u(t)\|_r^r \qquad \forall t \in [T_0,\infty). \tag{3.40}$$

Then (3.40) implies that u satisfies (3.31) with $c_0 = 1$, $c_1 = C$, $\theta_0 = 0$ and $\theta_1 = \frac{1}{1-\alpha}$. Thus the desired bound of u in $L^\infty([T_0,\infty); L^\infty(\Omega))$ is derived from Lemma 3.9 and (3.9). $\square$

Remark 3.10. It is possible to show that the global bounds of $\varphi(u(t))$ and $\|u(t)\|_\infty$ depend only on initial data $\varphi(u_0)$ and $\|u_0\|_\infty$ (as well as on $p, q, |\Omega|$) respectively, if p satisfies the following more restrictive condition: $2 < p < 2_*$, where $2_* = \infty$ for $N = 1$ and $2_* = 2 + \frac{12}{3N-4}$ for $N \geq 2$ ($2_* < 2^*$ for $N \geq 2$, see [5] and [12]).

Appendix

We here give a proof of Lemma 3.9.

Proof of Lemma 3.9. For each $k \in \mathbb{N}$, setting

$$r_k = 2^{k+1}, \quad \alpha_k = c_1 r_k^{\theta_1}, \quad \nu_k = c_0 r_k^{-\theta_0}, \quad v = w^{2^k},$$

by (3.31), we get the following inequality

$$\frac{d}{dt}\|v(t)\|_2^2 \leq -\nu_k \|v(t)\|_{H^1(\Omega)}^2 + \alpha_k \|v(t)\|_2^2. \tag{A.1}$$

We here note that the following Gagliardo-Nirenberg interpolation inequality

$$\|v\|_2^2 \leq C\|v\|_{H^1(\Omega)}^{2\theta} \|v\|_1^{2(1-\theta)} \leq \epsilon_k \|v\|_{H^1(\Omega)}^2 + C_{\epsilon_k} \|v\|_1^2$$

holds with $\theta = \frac{N}{N+2}$. Here $C_{\epsilon_k} = C^{\frac{1}{1-\theta}} \epsilon_k^{-\frac{\theta}{1-\theta}}$ and we take $\epsilon_k > 0$ sufficiently small so that $\epsilon_k \alpha_k + \epsilon_k^2 \leq \nu_k$ and $C_{\epsilon_k} \geq 1$. Then we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |w|^{r_k} dx &\leq -\epsilon_k^2 \|v(t)\|_{H^1(\Omega)}^2 + \alpha_k C_{\epsilon_k} \|v(t)\|_1^2 \\
&\leq -\epsilon_k \|v(t)\|_2^2 + (\epsilon_k + \alpha_k)\, C_{\epsilon_k} \|v(t)\|_1^2 \\
&\leq -\epsilon_k \int_\Omega |w|^{r_k} dx + (\epsilon_k + \alpha_k)\, C_{\epsilon_k} \Big(\int_\Omega |w|^{r_{k-1}} dx\Big)^2 \\
&\leq -\epsilon_k \int_\Omega |w|^{r_k} dx + (\epsilon_k + \alpha_k)\, C_{\epsilon_k} \Big(\sup_{t \geq 0}\int_\Omega |w|^{r_{k-1}} dx\Big)^2,
\end{aligned}$$

whence follows

$$\sup_{t\geq 0}\int_\Omega |w(t)|^{r_k} dx \leq \max\Big\{\delta_k \Big(\sup_{t\geq 0}\int_\Omega |w(t)|^{r_{k-1}} dx\Big)^2, \int_\Omega |w(0)|^{r_k} dx\Big\}, \tag{A.2}$$

where $\delta_k = \frac{(\epsilon_k + \alpha_k) C_{\epsilon_k}}{\epsilon_k} \geq 1$. Indeed, it is easy to see that $y'(t) \leq -\epsilon y(t) + C$ implies $\sup_{t\geq 0} y(t) \leq \max\{\frac{C}{\epsilon}, y(0)\}$.

Then the iterative use of (A.2) gives

$$\begin{aligned} &\int_\Omega |w|^{r_k} dx \le \delta_k\, \delta_{k-1}^2 \cdots \delta_1^{2(k-1)}\, M_0^{r_k}, \\ &M_0 := \max\,(1, c\,\|w(0)\|_\infty,\ \sup_{t\ge 0} \|w(t)\|_2) \quad \text{with} \quad c = \max\,(1, |\Omega|). \end{aligned} \tag{A.3}$$

Set $\epsilon_k = \eta\, 2^{-(\theta_0+\theta_1)k}$ and choose $\eta > 0$ sufficiently small so that $\epsilon_k \alpha_k + \epsilon_k^2 \le \nu_k$ and $C_{\epsilon_k} \ge 1$ are satisfied, then rewriting $C_{\epsilon_k} = C\epsilon_k^{-\gamma}$ with $\gamma = \frac{\theta}{1-\theta} > 0$, we have

$$\begin{aligned} \delta_k = \frac{(\epsilon_k + \alpha_k) C_{\epsilon_k}}{\epsilon_k} &= C(\epsilon_k + \alpha_k)\epsilon_k^{-\gamma-1} \\ &\le C\,\nu_k\,\epsilon_k^{-\gamma-2} \\ &\le C c_0\, 2^{-\theta_0(k+1)} \eta^{-(\gamma+2)}\, 2^{(\theta_0+\theta_1)(\gamma+2)k} \\ &= C\, 2^{-\theta_0} c_0\, \eta^{-(\gamma+2)}\, 2^{\{\theta_1 + (\theta_0+\theta_1)(\gamma+1)\}k} \\ &=: a\, 2^{\{\theta_1+(\theta_0+\theta_1)b\}k}, \end{aligned}$$

where we put $a = C\, 2^{-\theta_0} c_0\, \eta^{-(\gamma+2)}$ and $b = \gamma + 1$. Then by virtue of (A.3) with inductive reasoning, we easily obtain

$$\|w(t)\|_{r_k} \le a^{p_k} 2^{q_k} M_0, \tag{A.4}$$

where

$$p_k = \frac{2^k - 1}{2^{k+1}}, \qquad q_k = \frac{(2^{k+1} - k - 2)\{\theta_1 + (\theta_0 + \theta_1)b\}}{2^{k+1}}.$$

Since

$$p_k \uparrow \frac{1}{2}, \qquad q_k \uparrow \theta_1 + (\theta_0 + \theta_1)b \qquad \text{as} \quad k \uparrow \infty,$$

from (A.4) we can derive (see [13])

$$\|w(t)\|_\infty \le a^{\frac{1}{2}} 2^{\{\theta_1 + (\theta_0+\theta_1)b\}} M_0 \qquad a.e.\ t \in [0, \infty).$$

□

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