

Fixed Points of Functional Meir-Keeler Contractions

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Abstract: A fixed point result involving functional Meir-Keeler maps is established on ordered metric spaces. Then, it is shown that some related statements, including the ones due to Choudhury and Kundu [Demonstr. Math., 46 (2013), 327-334] or Du and Rassias [Int. J. Nonlin. Anal. Appl., 11 (2020), 55-66] are ultimately reducible to such techniques.

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1 Introduction

Let X be a nonempty set. Call the subset Y of X , *almost singleton* (in short: *asingleton*), provided $[y_1, y_2 \in Y \text{ implies } y_1 = y_2]$; and *singleton* if, in addition, Y is nonempty; note that in this case $Y = \{y\}$, for some $y \in X$. Take a metric $d : X \times X \rightarrow R_+ := [0, \infty[$ over X , as well as a selfmap $T \in \mathcal{F}(X)$. [Here, for each couple A, B of nonempty sets, $\mathcal{F}(A, B)$ stands for the class of all functions from A to B ; when $A = B$, we write $\mathcal{F}(A)$ in place of $\mathcal{F}(A, A)$]. Denote $\text{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T . Concerning the existence and uniqueness of such points, a basic result (referred to as: *Banach fixed point theorem*; in short: (B-fpt)) may be stated as follows. Call the selfmap T , $(d; \lambda)$ -contractive (where $\lambda \geq 0$), if

$$(d\text{-lb-con}) \quad d(Tx, Ty) \leq \lambda d(x, y), \text{ for all } x, y \in X.$$

Theorem 1.1. *Suppose that the selfmap T is $(d; \lambda)$ -contractive, for some $\lambda \in [0, 1[$. In addition, let X be d -complete. Then, $\text{Fix}(T)$ is a singleton $\{z\}$, and $\lim_n T^n x = z$, for each $x \in X$.*

This result, obtained in 1922 by Banach [3], found a multitude of applications in operator equations theory; so, it was the subject of many extensions. The most general ones have the (set) *implicit* form

$$(i\text{-set}) \quad (d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)) \in \mathcal{M},$$

for all $x, y \in X$;

where $\mathcal{M} \subseteq R_+^6$ is a (nonempty) subset. A basic particular case of the general contractive property above is

$$(i\text{-set-}2) \quad (d(Tx, Ty), d(x, y)) \in \mathcal{K}, \text{ for all } x, y \in X;$$

where $\mathcal{K} \subseteq R_+^2$ is a (nonempty) subset. The classical example in this particular direction is the one due to Meir and Keeler [23]. Further refinements of the method were proposed by Ćirić [8] and Matkowski [22]; see also Jachymski [17]. In particular, when $\mathcal{M}$ is the zero-section of a certain function $F : R_+^6 \rightarrow R$; i.e.,

$$\mathcal{M} = \{(t_1, \dots, t_6) \in R_+^6; F(t_1, \dots, t_6) \leq 0\},$$

the implicit contractive condition above has the functional form:

$$(i\text{-fct}) \quad F(d(Tx, Ty), d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)) \leq 0, \\ \text{for all } x, y \in X.$$

In this setting, certain early statements have been obtained by Leader [20] and Turinici [37]. Finally, when the function F appearing here admits the explicit form

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - G(t_2, t_3, t_4, t_5, t_6), \quad (t_1, t_2, t_3, t_4, t_5, t_6) \in R_+^6,$$

(where $G : R_+^5 \rightarrow R_+$ is a function), one gets the explicit functional version of this (functional) contraction

$$(e\text{-fct}) \quad d(Tx, Ty) \leq G(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(Tx, y)), \\ \text{for all } x, y \in X.$$

For this case, some outstanding results were given in Boyd and Wong [6], Reich [29], Matkowski [21], and Piticari [28, Ch II]. Further aspects have been discussed in the survey paper by Rhoades [30]; see also Collaco and E Silva [10].

Concerning the recent developments in this area, we must mention the class of implicit functional contractions having as model the ones introduced in 1969 by Krasnoselskii and Stetsenko [19], re-discovered in 2001 by Rhoades [31], and refined in 2008 by Dutta and Choudhury [13]; see also Wardowski [43]. The fixed point or coincidence point results obtained with the aid of such contractions were appreciated as interesting enough to be used in the treatment of various operator equations involving univalued and multivalued maps; see, in this direction, the 2020 survey paper by Karapinar et al [18]. On the other hand, certain efforts have been made towards a structural extension of them, under the *convergence* setting in Petruşel and Rus [27]; see, in this direction, Batra and Vashistha [4].

Having these precise, it is our objective in the following to show that most of these fixed point statements are obtainable by means of a unitary Meir-Keeler procedure. For simplicity reasons, the standard metrical case will be considered. Further aspects, involving the pseudometric setting developed in the 2016 paper by Turinici [40], will be delineated elsewhere.

2 Dependent Choice Principle

Throughout this exposition, the axiomatic system in use is Zermelo-Fraenkel's (abbreviated: (ZF)), as described by Cohen [9, Ch 2]. The notations and basic facts to be considered in this system are more or less standard. Some important ones are discussed below.

(A) Let X be a nonempty set. By a *relation* over X , we mean any (nonempty) part $\mathcal{R} \subseteq X \times X$; then, $(X, \mathcal{R})$ will be referred to as a *relational structure*. For simplicity, we sometimes write $(x, y) \in \mathcal{R}$ as $x\mathcal{R}y$. Note that $\mathcal{R}$ may be regarded as a mapping between X and $\exp[X]$ (=the class of all subsets in X). In fact, denote

$$X(x, \mathcal{R}) = \{y \in X; x\mathcal{R}y\} \text{ (the section of } \mathcal{R} \text{ through } x), x \in X;$$

then, the desired mapping representation is $[\mathcal{R}(x) = X(x, \mathcal{R}); x \in X]$.

A basic example of relational structure is to be constructed as below. Let $N = \{0, 1, \dots\}$ be the set of *natural* numbers, endowed with the usual addition and (partial) order ($\leq$); note that

$(N, \leq)$ is well ordered: any (nonempty) subset of N has a first element.

For each $r \in N$, the section $N(r, >)$ is referred to as the *initial interval* (in N) induced by r . Any set P with $P \sim N$ (in the sense: there exists a bijection from P to N) will be referred to as *effectively denumerable*. In addition, given some natural number $n \geq 1$, any set Q with $Q \sim N(n, >)$ will be said to be *n-finite*; when n is generic here, we say that Q is *finite*. Finally, the (nonempty) set Y is called (at most) *denumerable* iff it is either effectively denumerable or finite.

Let X be a nonempty set. By a *sequence* in X , we mean any mapping $x : N \rightarrow X$, where $N = \{0, 1, \dots\}$ is the set of *natural* numbers. For simplicity reasons, it will be useful to denote it as $(x(n); n \geq 0)$, or $(x_n; n \geq 0)$; moreover, when no confusion can arise, we further simplify this notation as $(x(n))$ or (x_n) , respectively. Also, any sequence $(y_n := x_{i(n)}; n \geq 0)$ with

$$(i(n); n \geq 0) \text{ is strictly ascending (hence, } i(n) \rightarrow \infty \text{ as } n \rightarrow \infty)$$

will be referred to as a *subsequence* of $(x_n; n \geq 0)$. Note that, under such a convention, the relation "subsequence of" is transitive; i.e.:

$$\begin{aligned} (z_n) &= \text{subsequence of } (y_n) \text{ and } (y_n) = \text{subsequence of } (x_n) \\ \text{imply } (z_n) &= \text{subsequence of } (x_n). \end{aligned}$$

(B) Remember that, an outstanding part of (ZF) is the *Axiom of Choice* (abbreviated: (AC)); which, in a convenient manner, may be written as

(AC) For each couple (J, X) of nonempty sets and each function $F : J \rightarrow \exp(X)$, there exists a (selective) function $f : J \rightarrow X$, with $f(\nu) \in F(\nu)$, for each $\nu \in J$.

(Here, $\exp(X)$ stands for the class of all nonempty elements in $\exp[X]$). Sometimes, when the index set J is denumerable, the existence of such a selective function may be determined by using a weaker form of (AC), called: *Dependent Choice* principle (in short: (DC)). Call the relation $\mathcal{R}$ over X , *proper* when

$(X(x, \mathcal{R}) =) \mathcal{R}(x)$ is nonempty, for each $x \in X$.

Then, $\mathcal{R}$ is to be viewed as a mapping between X and $\exp(X)$; and the couple $(X, \mathcal{R})$ will be referred to as a *proper relational structure*. Further, given $a \in X$, let us say that the sequence $(x_n; n \geq 0)$ in X is $(a; \mathcal{R})$ -*iterative*, provided

$x_0 = a$, and $x_n \mathcal{R} x_{n+1}$ (i.e.: $x_{n+1} \in \mathcal{R}(x_n)$), for all n .

Proposition 2.1. *Let the relational structure $(X, \mathcal{R})$ be proper. Then, for each $a \in X$ there is at least one $(a, \mathcal{R})$ -iterative sequence in X .*

This principle – proposed, independently, by Bernays [5] and Tarski [36] – is deductible from (AC), but not conversely; cf. Wolk [45]. Moreover, by the developments in Moskhovakis [25, Ch 8] and Schechter [35, Ch 6], the *reduced system* (ZF-AC+DC) is comprehensive enough so as to cover the *usual* mathematics; see also Moore [24, Appendix 2].

A basic consequence of (DC) is the so-called *Denumerable Axiom of Choice* [in short: (AC(N))].

Proposition 2.2. *Let $F : N \rightarrow \exp(X)$ be a function. Then, for each $a \in F(0)$ there exists a function $f : N \rightarrow X$ with $f(0) = a$ and $f(n) \in F(n)$, $\forall n \in N$.*

Proof. Denote $Q = N \times X$; and let us introduce the (proper) relation $\mathcal{R}$ over it:

$$\mathcal{R}(n, x) = \{n + 1\} \times F(n + 1), \quad n \in N, x \in X.$$

Then, an application of (DC) to the proper relational structure $(Q, \mathcal{R})$ yields the desired conclusion; we do not give details. $\square$

As a consequence of the above facts,

(DC) $\implies$ (AC(N)) in the *strongly reduced system* (ZF-AC);

or, equivalently:

(AC(N)) is deductible in the *reduced system* (ZF-AC+DC).

The reciprocal of this is not true; see Moskhovakis [25, Ch 8, Sect 8.25] for details.

3 Statement of the problem

Let (X, d) be a metric space; and $(\leq)$ be a *quasi-order* [i.e.: reflexive, transitive relation] over X ; then, $(X, d, \leq)$ will be referred to as a *quasi-ordered metric space*. Call the subset $Y \in \exp[X]$, $(\leq)$ -*asingleton* if $[y_1, y_2 \in Y, y_1 \leq y_2]$ imply $y_1 = y_2$; and $(\leq)$ -*singleton* if, in addition, $Y \neq \emptyset$. Further, let $T \in \mathcal{F}(X)$ be a selfmap with

(s-pro) T is *semi-progressive* ($X(T, \leq) := \{x \in X; x \leq Tx\}$ is nonempty)

(incr) T is *increasing* ($x \leq y$ implies $Tx \leq Ty$).

We are interested in establishing sufficient conditions – involving iterative sequences – for the determination of elements in $\text{Fix}(T)$. The basic directions for getting these are described in our list below, comparable with the one proposed by Turinici [38]:

opic-0 We say that T is *fix- $(\leq)$ -asingleton*, when $\text{Fix}(T)$ is an $(\leq)$ -asingleton; and *fix- $(\leq)$ -singleton*, when $\text{Fix}(T)$ is a $(\leq)$ -singleton

opic-1 We say that T is a *semi Picard operator* (modulo $(d, \leq)$) if, for each $x \in X(T, \leq)$, $(T^n x; n \geq 0)$ is *d-asymptotic*: $\lim_n d(T^n x, T^{n+1} x) = 0$

opic-2 We say that T is a *Picard operator* (modulo $(d, \leq)$) when, for each $x \in X(T, \leq)$, $(T^n x; n \geq 0)$ is *d-Cauchy*: $d(T^n x, T^m x) \rightarrow 0$ as $n, m \rightarrow \infty, n \leq m$

opic-3 We say that T is a *strong Picard operator* (modulo $(d, \leq)$) if, for each $x \in X(T, \leq)$, $(T^n x; n \geq 0)$ is *d-convergent* and $T^\omega x := \lim_n (T^n x) \in \text{Fix}(T)$

opic-4 We say that T is a *Bellman Picard operator* (modulo $(d, \leq)$) when, for each $x \in X(T, \leq)$, $(T^n x; n \geq 0)$ is *d-convergent*, $T^\omega x := \lim_n (T^n x)$ belongs to $\text{Fix}(T)$, and $T^n x \leq T^\omega x, \forall n$.

In particular, when $(\leq) = X \times X$ (the *trivial quasi-order* on X) these conventions reduce to the ones in Rus [33, Ch 2, Sect 2.2]; because, in this case, $X(T, \leq) = X$.

Returning to the general setting, the sufficient (regularity) conditions attached to these properties are being founded on *ascending orbital full* (in short: (a-o-f)) concepts. Call the sequence $(z_n; n \geq 0)$ in X ,

(d1) *ascending*, provided $z_i \leq z_j$ whenever $i \leq j$;

(d2) *orbital*, if $(z_n = T^n x; n \geq 0)$, for some $x \in X$;

(d3) *full*, when $n \mapsto z_n$ is *injective* ($i \neq j$ implies $x_i \neq x_j$);

the intersection of these notions yields the precise one.

reg-1 Call X , *(a-o-f, d)-complete* provided (for each (a-o-f)-sequence) *d*-Cauchy implies *d*-convergent

reg-2 We say that T is *(a-o-f, d)-continuous*, if $((z_n) = (a-o-f)\text{-sequence and } z_n \xrightarrow{d} z)$ imply $Tz_n \xrightarrow{d} Tz$

reg-3) Call the quasi-order $(\leq)$, $(a\text{-}o\text{-}f, d)\text{-selfclosed}$ when $((z_n) = (a\text{-}o\text{-}f)\text{-sequence and } z_n \xrightarrow{d} z) \text{ imply } (z_n \leq z, \forall n)$.

As a basic completion of these, we have to introduce the contractive type conditions to be used. Denote $(P_0(x, y) = d(Tx, Ty); x, y \in X)$; and let $P : X \times X \rightarrow R_+$ be a map. We say that T is *Meir-Keeler* $(d, \leq; P)\text{-contractive}$, if

(mk-1) $x \leq y, P(x, y) > 0$ imply $P_0(x, y) < P(x, y)$;
referred to as: T is *strictly nonexpansive* (modulo $(d, \leq; P)$)

(mk-2) $\forall \varepsilon > 0, \exists \delta > 0: [x \leq y, \varepsilon < P(x, y) < \varepsilon + \delta] \implies P_0(x, y) \leq \varepsilon$;
referred to as: T has the *Meir-Keeler property* (modulo $(d, \leq; P)$).

Note that, by the former of these, the Meir-Keeler property may be written as

(mk-2a) $\forall \varepsilon > 0, \exists \delta > 0: [x \leq y, 0 < P(x, y) < \varepsilon + \delta] \implies P_0(x, y) \leq \varepsilon$;
referred to as: T has the *complete Meir-Keeler property* (modulo $(d, \leq; P)$).

The following asymptotic version of Meir-Keeler contractive property is sometimes useful in applications, as we will see. Call the selfmap T , *asymptotic Meir-Keeler* $(d, \leq; P)\text{-contractive}$ if

(a-mk-1) $x \leq y, P(x, y) > 0$ imply $P_0(x, y) < P(x, y)$;
referred to as: T is *strictly nonexpansive* (modulo $(d, \leq; P)$)

(a-mk-2) there are no sequences $(u_n), (v_n)$ in X and no elements $\varepsilon > 0$ with $(u_n \leq v_n, \forall n), P_0(u_n, v_n) \rightarrow \varepsilon+, P(u_n, v_n) \rightarrow \varepsilon+$; referred to as: T has the *asymptotic Meir-Keeler property* (modulo $(d, \leq; P)$).

Here, given the sequence $(r_n; n \geq 0)$ in R and the point $r \in R$, we denoted

$r_n \rightarrow r+, \text{ if } r_n \rightarrow r \text{ and } (r_n > r, \text{ for all } n).$

The relationships with the Meir-Keeler contractive condition are described by

Theorem 3.1. *We have, in $(ZF\text{-}AC+DC)$,*

T is *Meir-Keeler* $(d, \leq; P)\text{-contractive}$ iff
 T is *asymptotic Meir-Keeler* $(d, \leq; P)\text{-contractive}$.

Proof. i): Suppose that T is Meir-Keeler $(d, \leq; P)\text{-contractive}$; but [in contradiction with the conclusion] T is not endowed with the asymptotic Meir-Keeler property (modulo $(d, \leq; P)$); i.e.:

there are sequences $(u_n), (v_n)$ in X and elements $\varepsilon > 0$ with
 $(u_n \leq v_n, \forall n), P_0(u_n, v_n) \rightarrow \varepsilon+, P(u_n, v_n) \rightarrow \varepsilon+.$

Given this $\varepsilon > 0$, let $\delta > 0$ be the number associated to it, by the Meir-Keeler $(d, \leq; P)$ -contractive property. From the convergence relations above, there exists some rank $n(\delta)$ with

$$(\text{rela}) \quad u_n \leq v_n, \varepsilon < P_0(u_n, v_n) < \varepsilon + \delta, \varepsilon < P(u_n, v_n) < \varepsilon + \delta, \forall n \geq n(\delta).$$

But, according to the first and third part of (rela), we must have

$$P_0(u_n, v_n) \leq \varepsilon, \text{ for all } n \geq n(\delta).$$

This, however, contradicts the second part of (rela). Hence, our working assumption cannot be accepted; and the assertion follows.

ii): Suppose that T is asymptotic Meir-Keeler $(d, \leq; P)$ -contractive; but [in contradiction with the conclusion] T is not endowed with the Meir-Keeler property (modulo $(d, \leq; P)$); i.e. (for some $\varepsilon > 0$)

$$A(\delta) := \{(u, v) \in X \times X; u \leq v, \varepsilon < P(u, v) < \varepsilon + \delta, P_0(u, v) > \varepsilon\}$$

is nonempty, for each $\delta > 0$.

Taking a zero converging sequence $(\delta_n; n \geq 0)$ in R_+^0 , we get by the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a couple of sequences $(x_n; n \geq 0)$ and $(y_n; n \geq 0)$ in X , so as

$$(\forall n): (x_n, y_n) \in A(\delta_n); \text{ i.e. (by the strict nonexpansive condition)} \\ x_n \leq y_n, \varepsilon < P_0(x_n, y_n) < P(x_n, y_n) < \varepsilon + \delta_n.$$

As a direct consequence of this relation,

$$(x_n \leq y_n, \forall n), \text{ and } (P_0(x_n, y_n) \rightarrow \varepsilon+, P(x_n, y_n) \rightarrow \varepsilon+), \text{ as } n \rightarrow \infty.$$

This contradicts the asymptotic Meir-Keeler property (modulo $(d, \leq; P)$) of T . Hence, the Meir-Keeler property (modulo $(d, \leq; P)$) of T follows. $\square$

In the following, two basic examples of such contractions are constructed.

(Ex-I) Given the mapping $P : X \times X \rightarrow R_+$ and the function $\varphi \in \mathcal{F}(R_+^0, R)$, let us say that T is $(d, \leq; P; \varphi)$ -contractive, if

$$(\text{phi-con}) \quad P_0(x, y) \leq \varphi(P(x, y)), \forall x, y \in X, x \leq y, P(x, y) > 0.$$

The functions φ to be considered may be described as below. Let $\mathcal{F}(re)(R_+^0, R)$ stand for the subclass of all $\varphi \in \mathcal{F}(R_+^0, R)$, with

$$\varphi \text{ is regressive: } \varphi(t) < t, \text{ for each } t > 0.$$

Call $\varphi \in \mathcal{F}(re)(R_+^0, R)$, *Meir-Keeler admissible* if

(mk-adm) $\forall \gamma > 0, \exists \beta > 0, (\forall t): \gamma < t < \gamma + \beta \implies \varphi(t) \leq \gamma;$

and *Matkowski admissible*, provided

(M-adm) for each sequence $(t_n; n \geq 0)$ in R_+^0 with $(t_{n+1} \leq \varphi(t_n), \forall n)$, we have $\lim_n t_n = 0$.

The relationships between these notions are illustrated in

Theorem 3.2. *For each $\varphi \in \mathcal{F}(re)(R_+^0, R)$, we have*

φ is Meir-Keeler admissible iff φ is Matkowski admissible.

For a complete proof of this we refer to the paper by Turinici [41]. Some partial aspects of the problem can be found in Jachymski [16].

In particular, letting $\mathcal{F}(re, in)(R_+^0, R)$ stand for the class of all increasing functions in $\mathcal{F}(re)(R_+^0, R)$, we have (cf. Matkowski [21])

$\varphi \in \mathcal{F}(re, in)(R_+^0, R)$ is *Matkowski admissible* iff for each $t > 0$: $\lim_n \varphi^n(t) = 0$, as long as $(\varphi^n(t); n \geq 0)$ exists.

Here, as usual, φ^n stands for the n -th iterate of φ , for each $n \in \mathbb{N}$.

To get concrete examples of such functions, the constructions below are in effect. Let $\varphi \in \mathcal{F}(re)(R_+^0, R)$ and $s \in R_+^0$ be given. Denote, for each $\varepsilon > 0$,

$$\Lambda^+ \varphi(s) = \inf_{\varepsilon > 0} \Phi^*(s+)(\varepsilon); \text{ where } \Phi^*(s+)(\varepsilon) = \sup \varphi([s, s + \varepsilon]);$$

this will be referred to as: *right superior limit* of φ at s . From the regressive property of φ , we have

$$-\infty \leq \Lambda^+ \varphi(s) \leq s, \quad \forall s \in R_+^0;$$

but the alternative of the extremal terms being attained cannot be avoided. The following consequence of this definition will be useful.

Theorem 3.3. *Let $\varphi \in \mathcal{F}(re)(R_+^0, R)$ and $s \in R_+^0$ be arbitrary fixed. Then,*

(33-a) $\limsup_n (\varphi(t_n)) \leq \Lambda^+ \varphi(s),$
for each sequence (t_n) in R_+^0 with $t_n \rightarrow s+$

(33-b) *there exists a sequence (r_n) in R_+^0 with $r_n \rightarrow s+$ and $\varphi(r_n) \rightarrow \Lambda^+ \varphi(s)$.*

Proof. Denote, for simplicity,

$$\alpha = \Lambda^+ \varphi(s); \text{ hence, } \alpha = \inf_{\varepsilon > 0} \Phi^*(s+)(\varepsilon), \text{ and } -\infty \leq \alpha \leq s.$$

i) Given $\varepsilon > 0$, there exists a rank $p(\varepsilon) \geq 0$ such that $s < t_n < s + \varepsilon$, for all $n \geq p(\varepsilon)$; hence

$$\limsup_n(\varphi(t_n)) \leq \sup\{\varphi(t_n); n \geq p(\varepsilon)\} \leq \Phi^*(s+)(\varepsilon).$$

Passing to infimum over $\varepsilon > 0$, yields (see above)

$$\limsup_n(\varphi(t_n)) \leq \inf_{\varepsilon > 0} \Phi^*(s+)(\varepsilon) = \alpha;$$

and the claim follows.

ii): Define $(\beta_n := \Phi^*(s+)(2^{-n-1}s); n \geq 0)$; clearly,

(p1) $(\forall n)$: $-\infty < \beta_n \leq s + 2^{-n-1}s$, if we note that $-\infty < \varphi(t) < t < s + 2^{-n-1}s$, for all $t \in]s, s + 2^{-n-1}s[$;

(p2) (β_n) is descending, $(\beta_n \geq \alpha, \forall n)$, $\inf_n \beta_n = \alpha$; hence $\lim_n \beta_n = \alpha$.

By these properties, there may be constructed a sequence $(\gamma_n; n \geq 0)$ in R , with

$$\gamma_n < \beta_n, \forall n; \lim_n \gamma_n = \lim_n \beta_n = \alpha.$$

(For example, $(\gamma_n = \beta_n - 3^{-n}; n \geq 0)$ has such a property; so, (DC) is not used here). Define the map $n \mapsto E(n)$ from N to $\exp(R_+^0)$, as

$$E(n) = \{t \in]s, s + 2^{-n-1}s[; \beta_n \geq \varphi(t) > \gamma_n\}, n \in N;$$

clearly, this construction is meaningful, by the supremum definition. From the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a sequence $(r_n; n \geq 0)$ in R_+^0 may be obtained so as

$$(\forall n): r_n \in E(n); \text{ that is (by the above definition):} \\ s < r_n < s + 2^{-n-1}s \text{ and } \beta_n \geq \varphi(r_n) > \gamma_n.$$

This yields $(r_n \rightarrow s+ \text{ and } \varphi(r_n) \rightarrow \alpha)$; wherefrom, we are done. $\square$

In the following, some particular examples of Meir-Keeler (or, equivalently: Matkowski) admissible functions will be given.

I-1) Call $\varphi \in \mathcal{F}(re)(R_+^0, R)$, *Boyd-Wong admissible* if

$$(\text{bw-adm}) \quad \Lambda^+ \varphi(s) < s, \text{ for all } s > 0.$$

(This convention is related to the developments in Boyd and Wong [6]). In particular, $\varphi \in \mathcal{F}(re)(R_+^0, R)$ is Boyd-Wong admissible provided it is upper semicontinuous at the right on R_+^0 :

$$\Lambda^+ \varphi(s) \leq \varphi(s), \forall s \in R_+^0.$$

This, e.g., is fulfilled when φ is continuous at the right on R_+^0 ; for, in such a case,

$$\Lambda^+\varphi(s) = \varphi(s), \text{ for each } s \in R_+^0.$$

Proposition 3.4. *Each Boyd-Wong admissible function in $\mathcal{F}(re)(R_+^0, R)$ is Meir-Keeler admissible (or, equivalently: Matkowski admissible).*

Proof. Suppose that $\varphi \in \mathcal{F}(re)(R_+^0, R)$ is Boyd-Wong admissible, and fix $\gamma > 0$; hence $\Lambda^+\varphi(\gamma) < \gamma$. By definition, there exists $\beta = \beta(\gamma) > 0$ with $[\gamma < t < \gamma + \beta$ implies $\varphi(t) < \gamma]$; proving that φ is Meir-Keeler admissible. $\square$

I-2) Given $\varphi \in \mathcal{F}(re)(R_+^0, R)$, we call it *Geraghty admissible* [15], provided

$$(t_n; n \geq 0) = \text{sequence in } R_+^0 \text{ and } \varphi(t_n)/t_n \rightarrow 1 \text{ imply } t_n \rightarrow 0.$$

Proposition 3.5. *Each Geraghty admissible function is Boyd-Wong admissible; hence, Meir-Keeler admissible.*

Proof. Let $\varphi \in \mathcal{F}(re)(R_+^0, R)$ be Geraghty admissible; and suppose by contradiction that φ is not Boyd-Wong admissible. From a previous observation, there exists some $s \in R_+^0$ with $\Lambda^+\varphi(s) = s$. Combining with a preceding auxiliary fact, there exists a sequence $(r_n; n \geq 0)$ in R_+^0 with

$$r_n \rightarrow s+ \text{ and } \varphi(r_n) \rightarrow s; \text{ whence } \varphi(r_n)/r_n \rightarrow 1;$$

i.e.: φ is not Geraghty admissible. The obtained contradiction proves our claim. $\square$

Having these precise, we may establish the necessary connections between the introduced functional contractions and the Meir-Keeler ones.

Theorem 3.6. *Assume that T is $(d, \leq; P; \varphi)$ -contractive, where $\varphi \in \mathcal{F}(re)(R_+^0, R)$. Then, T is Meir-Keeler $(d, \leq; P)$ -contractive when φ is Meir-Keeler admissible (or, equivalently: Matkowski admissible). In particular, this is retainable whenever φ is Boyd-Wong admissible; hence, all the more, when it is Geraghty admissible.*

Proof. i) Let $x, y \in X$ be such that $x \leq y$ and $P(x, y) > 0$. By the contractive condition [and φ =regressive], one has $P_0(x, y) \leq \varphi(P(x, y)) < P(x, y)$; so that, T is strictly nonexpansive (modulo $(d, \leq; P)$).

ii) Let $\varepsilon > 0$ be arbitrary fixed; and $\delta > 0$ be the number assured by the Meir-Keeler admissible property of φ . Further, let $x, y \in X$ be such that $x \leq y$ and $\varepsilon < P(x, y) < \varepsilon + \delta$. By the contractive condition and admissible property,

$$P_0(x, y) \leq \varphi(P(x, y)) \leq \varepsilon;$$

so that, T has the Meir-Keeler property (modulo $(d, \leq; P)$).

ii): Evident, by the above auxiliary facts. $\square$

(Ex-II) Let (ψ, φ) be a couple of functions over $\mathcal{F}(R_+^0, R)$ endowed with

(norm) (ψ, φ) is *normal*:

ψ is increasing and φ is *strictly positive* $[\varphi(t) > 0, \forall t > 0]$.

(This concept is related with the developments in Rhoades [31]; see also Dutta and Choudhury [13]). The following extra condition will be considered here:

(r-s-pos) $(\varphi$ is *right sequentially positive*)

for each sequence (t_n) in R_+^0 and each $\varepsilon > 0$ with $t_n \rightarrow \varepsilon+$, the relation $\lim_n \varphi(t_n) = 0$ is impossible.

Given the mapping $P : X \times X \rightarrow R_+$ and the couple (ψ, φ) of functions over $\mathcal{F}(R_+^0, R)$, let us say that T is $(d, \leq; P; (\psi, \varphi))$ -*contractive*, provided

$$\begin{aligned} \psi(P_0(x, y)) &\leq \psi(P(x, y)) - \varphi(P(x, y)), \\ \forall x, y \in X, x \leq y, P_0(x, y) &> 0, P(x, y) > 0. \end{aligned}$$

The following auxiliary fact establishes the necessary connection between this contractive concept and the Meir-Keeler one.

Proposition 3.7. *Suppose that T is $(d, \leq; P; (\psi, \varphi))$ -contractive, for a normal couple (ψ, φ) over $\mathcal{F}(R_+^0, R)$ with φ =right sequentially positive. Then, in the reduced system $(ZF-AC+DC)$,*

*T is asymptotic Meir-Keeler $(d, \leq; P)$ -contractive;
or, equivalently: Meir-Keeler $(d, \leq; P)$ -contractive.*

Proof. i) Let $x, y \in X$ be such that $x \leq y, P(x, y) > 0$. If $P_0(x, y) = 0$, we are done; so, without loss, assume that $P_0(x, y) > 0$. As $P(x, y) > 0$, we have (along with φ =strictly positive), $\varphi(P(x, y)) > 0$; wherefrom

$$\psi(P_0(x, y)) < \psi(P(x, y)) \text{ (by the contractive condition);}$$

and this, via $[\psi$ =increasing], yields $P_0(x, y) < P(x, y)$. Putting these together, one derives that T is strictly nonexpansive (modulo $(d, \leq; P)$).

ii) We have to establish that T is endowed with the asymptotic Meir-Keeler property (modulo $(d, \leq; P)$). Suppose not:

there are sequences $(u_n), (v_n)$ in X and elements $\varepsilon > 0$
with $(u_n \leq v_n, \forall n), P_0(u_n, v_n) \rightarrow \varepsilon+, P(u_n, v_n) \rightarrow \varepsilon+$.

By the contractive property,

$$\begin{aligned}
&(\forall n): \psi(P_0(u_n, v_n)) \leq \psi(P(u_n, v_n)) - \varphi(P(u_n, v_n)); \\
&\text{or, equivalently (along with } \varphi = \text{strictly positive}): \\
&(0 <) \varphi(P(u_n, v_n)) \leq \psi(P(u_n, v_n)) - \psi(P_0(u_n, v_n)).
\end{aligned}$$

Passing to $\limsup$ as $n \rightarrow \infty$, gives

$$\begin{aligned}
0 &\leq \limsup \varphi(P(u_n, v_n)) \leq \psi(\varepsilon + 0) - \psi(\varepsilon + 0) = 0; \\
&\text{whence, } \lim_n \varphi(P(u_n, v_n)) = 0;
\end{aligned}$$

in contradiction with the choice of φ . Hence, the underlying asymptotic property holds; and we are done. $\square$

Note that, some other examples of such contractions are available. But, for the developments below, this will suffice.

4 Main result

Let $(X, d, \leq)$ be a quasi-ordered metric space; and T be a selfmap of X ; supposed to be semi-progressive and increasing. As precise, we have to determine points of $\text{Fix}(T)$, via iterative processes techniques. The basic directions as well as general regularity conditions under which this problem is to be solved were already listed. In addition, the contractive Meir-Keeler setting of our problem, expressed in terms of a certain mapping $P \in \mathcal{F}(X \times X, R_+)$, is being settled. It remains now to discuss the specific regularity conditions upon P to be used. Denote for each $x, y \in X$,

$$\begin{aligned}
A_1(x, y) &= d(x, y), \quad A_2(x, y) = \max\{d(x, Tx), d(y, Ty)\}, \\
L_0(x, y) &= \min\{d(x, y), d(Tx, Ty)\}, \\
L_1(x, y) &= \min\{d(x, y), d(Tx, Ty), d(x, Tx), d(y, Ty)\}, \\
M(x, y) &= \text{diam}\{x, Tx, y, Ty\}.
\end{aligned}$$

(I) The first condition upon P writes

$$P \text{ is } L\text{-positive (where } L \in \{L_0, L_1\}): L(x, y) > 0 \text{ implies } P(x, y) > 0.$$

It has the role of working with our iterative sequences (under $L = L_1$) as well as to assure the $\text{fix}(\leq)$ -asingleton property (under $L = L_0$). Note that

$$P \text{ is } L_0\text{-positive implies } P \text{ is } L_1\text{-positive;}$$

but the reciprocal is not in general true.

(II) The second condition upon the mapping P is expressed as

$$P \text{ is orbitally bounded: } P(x, Tx) \leq A_2(x, Tx), \text{ for all } x \in X;$$

it allows us deducing the d -asymptotic and full properties for the iterative sequences to be considered (see below).

(III) The third condition upon P writes

P is diametral: $P(x, y) \leq M(x, y)$, for all $x, y \in X$.

It has the role of deducing (in conjunction with Meir-Keeler contractive assumption) the d -Cauchy property for the iterative sequences in question.

(IV) The fourth condition upon the mapping P is a couple of orbital asymptotic ones, formulated as

(o-sg-asy) P is *orbitally singular asymptotic*:

for each (a-o-f) sequence (x_n) in X and each $z \in X$ with $x_n \xrightarrow{d} z$, $(x_n \leq z, L_1(x_n, z) > 0, \forall n)$, we have $\liminf_n P(x_n, z) < d(z, Tz)$

(s-o-sg-asy) P is *strongly orbitally singular asymptotic*:

for each (a-o-f) sequence (x_n) in X and each $z \in X$ with $x_n \xrightarrow{d} z$, $(x_n \leq z, L_1(x_n, z) > 0, \forall n)$, we have $(\exists) \lim_n P(x_n, z) < d(z, Tz)$.

Clearly, the former of these is weaker than the latter one. Note that the imposed conditions allow us deducing (in addition to the Meir-Keeler contractive assumption) the fixed point property for the limit of underlying iterative sequence.

We are now in position to state our basic fixed point result in this exposition.

Theorem 4.1. *Suppose that T is Meir-Keeler $(d, \leq; P)$ -contractive, (or, equivalently: asymptotic Meir-Keeler $(d, \leq; P)$ -contractive), where the mapping $P : X \times X \rightarrow R_+$ is orbitally bounded and diametral. In addition, let X be (a-o-f, d)-complete. Then,*

(41-a) T is a Picard operator (modulo $(d, \leq)$) when, in addition, P is L_1 -positive

(41-b) T is a strong Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (41-a), T is (a-o-f, d)-continuous

(41-c) T is a Bellman Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (41-a), $(\leq)$ is (a-o-f, d)-selfclosed and P is orbitally singular asymptotic

(41-d) T is fix- $(\leq)$ -asingleton when (in addition) P is L_0 -positive.

Proof. Let us firstly establish the fix- $(\leq)$ -asingleton property. Take some couple $z_1, z_2 \in \text{Fix}(T)$ with $z_1 \leq z_2$; and suppose by contradiction that $z_1 \neq z_2$; hence, $L_0(z_1, z_2) = d(z_1, z_2) > 0$. As P is L_0 -positive, we must have $P(z_1, z_2) > 0$; so, that $d(z_1, z_2) = d(Tz_1, Tz_2) < P(z_1, z_2)$ [since T is strictly nonexpansive (modulo $(d, \leq; P)$)]. On the other hand, as P is diametral, $P(z_1, z_2) \leq M(z_1, z_2) = d(z_1, z_2)$. Since the obtained relations are contradictory, our working assumption cannot be accepted; wherefrom, the assertion follows.

It remains to establish that T is a strong/Bellman Picard operator (modulo $(d, \leq)$). Fix some $x_0 \in X(T, \leq)$, and put $(x_n = T^n x_0; n \geq 0)$; it is an ascending and orbital sequence. If $x_n = x_{n+1}$ for some $n \geq 0$, we are done; so, without loss, one may assume that the *non-telescopic* property holds

$$(n\text{-tele}) \quad x_n \neq x_{n+1} \text{ (that is, } \rho_n := d(x_n, x_{n+1}) > 0), \forall n.$$

The argument will be divided into several parts.

Part 1. By the imposed condition,

$$(\forall n): L_1(x_n, x_{n+1}) = \min\{\rho_n, \rho_{n+1}\} > 0; \text{ hence } P(x_n, x_{n+1}) > 0,$$

if we remember that P is L_1 -positive. The Meir-Keeler $(d, \leq; P)$ -contractive condition applies to (x_n, x_{n+1}) , for each n ; and yields (by the strict nonexpansive condition upon T and the orbital boundedness of P)

$$\begin{aligned} (\text{iter-1}) \quad (\forall n): \rho_{n+1} &= P_0(x_n, x_{n+1}) < P(x_n, x_{n+1}) \leq \\ A_2(x_n, x_{n+1}) &= \max\{\rho_n, \rho_{n+1}\}; \end{aligned}$$

From the strict inequality between the extremal members of this relation, we get

$$(\text{iter-2}) \quad (\forall n): \rho_{n+1} < \rho_n; \text{ wherefrom, } \rho_{n+1} < P(x_n, x_{n+1}) \leq \rho_n.$$

Two consequences of this fact are retainable.

Conseq 1. By the first half of (iter-2), (ρ_n) is strictly descending. We claim that, in this case,

$$(x_n) \text{ is full: } i \neq j \text{ implies } x_i \neq x_j \text{ (whence, } d(x_i, x_j) > 0).$$

In fact, suppose by contradiction that there exist $i, j \in N$ with $i < j$, $x_i = x_j$. Then, by definition, $x_{i+1} = x_{j+1}$; so that $\rho_i = \rho_j$; in contradiction with $\rho_i > \rho_j$; and the assertion follows.

Conseq 2. By the same strict descending property of $(\rho_n; n \geq 0)$, we have that $\rho := \lim_n \rho_n$ exists in R_+ ; and $[\rho_n > \rho, \forall n]$. Assume that $\rho > 0$; and let $\sigma > 0$ be the number given by the Meir-Keeler property (modulo $(d, \leq; P)$) of T . From the convergence relation, there exists a rank $n(\sigma)$ such that

$$\begin{aligned} (\forall n \geq n(\sigma)): \rho &< \rho_n < \rho + \sigma; \text{ hence, by (iter-2),} \\ \rho &< \rho_{n+1} < P(x_n, x_{n+1}) \leq \rho_n < \rho + \sigma. \end{aligned}$$

This, by the quoted condition, yields (for the same n),

$$(\rho <) \quad \rho_{n+1} = P_0(x_n, x_{n+1}) \leq \rho;$$

a contradiction. Hence, $\rho = 0$; so that, $(x_n; n \geq 0)$ is a d -asymptotic sequence.

Part 2. Summing up, (x_n) is orbital, full, and d -asymptotic. We claim that, under the precise conditions, (x_n) is d -Cauchy. Let $\varepsilon > 0$ be arbitrary fixed; and $\delta > 0$ be the number associated by the Meir-Keeler property; without loss, one may assume that $\delta < \varepsilon$. From the obtained d -asymptotic property, there exists a rank $n(\delta) \geq 0$ such that

$$(d\text{-asy}) \quad d(x_n, x_{n+1}) < \delta/4 \text{ (hence, } d(x_n, x_{n+2}) < \delta/2), \forall n \geq n(\delta).$$

We claim, via ordinary induction, that for each $i \geq 1$, the property below holds

$$(d\text{-C};i) \quad d(x_n, x_{n+i}) < \varepsilon + \delta/2, \forall n \geq n(\delta);$$

wherefrom, the d -Cauchy property of $(x_n; n \geq 0)$ is clear. The case $i \in \{1, 2\}$ is evident, by (d-asy). Assume that, for a certain $k \geq 2$, (d-C;i) holds for all $i \in \{1, \dots, k\}$; we must establish that (d-C;k+1) holds too. So, let $n \geq n(\delta)$ be arbitrary fixed. From the inductive hypothesis,

$$\begin{aligned} d(x_n, x_{n+k}) &< \varepsilon + \delta/2; \text{ as well as} \\ d(x_{n+1}, x_{n+k}) &< \varepsilon + \delta/2, \quad d(x_{n+1}, x_{n+k+1}) < \varepsilon + \delta/2. \end{aligned}$$

On the other hand, by the asymptotic property,

$$d(x_n, x_{n+1}) < \delta/4, \quad d(x_{n+k}, x_{n+k+1}) < \delta/4.$$

Finally, the triangular inequality gives (in a direct way)

$$d(x_n, x_{n+k+1}) \leq d(x_n, x_{n+k}) + d(x_{n+k}, x_{n+k+1}) < \varepsilon + \delta/2 + \delta/4 < \varepsilon + \delta.$$

Putting these together, gives

$$M(x_n, x_{n+k}) < \varepsilon + \delta; \text{ whence, } P(x_n, x_{n+k}) < \varepsilon + \delta;$$

if we remember that P is diametral. On the other hand, as (x_n) is full,

$$\begin{aligned} L_1(x_n, x_{n+k}) &= \min\{\rho_n, \rho_{n+k}, d(x_n, x_{n+k}), d(x_{n+1}, x_{n+k+1})\} > 0; \\ \text{whence } P(x_n, x_{n+k}) &> 0 \end{aligned}$$

if we take the L_1 -positive property of P into account. Combining these with the complete Meir-Keeler property (modulo $(d, \leq; P)$), one derives

$$d(x_{n+1}, x_{n+k+1}) = P_0(x_n, x_{n+k}) \leq \varepsilon.$$

This, along with the triangular inequality, gives

$$d(x_n, x_{n+k+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+k+1}) < \varepsilon + \delta/2;$$

and the assertion is retainable.

Part 3. Since X is (a-o-f,d)-complete,

there exists (a unique) $z \in X$ with $x_n \xrightarrow{d} z$.

Two basic alternatives are to be discussed.

Alter 1. Suppose that T is (a-o-f,d)-continuous. Then, $u_n := Tx_n \xrightarrow{d} Tz$ as $n \rightarrow \infty$. On the other hand, $(u_n = x_{n+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$; so, $u_n \xrightarrow{d} z$ as $n \rightarrow \infty$. Combining with d =separated, yields $z = Tz$; which tells us that T is strongly Picard (modulo $(d, \leq)$).

Alter 2. Suppose that $(\leq)$ is (a-o-f,d)-selfclosed and P is orbitally singular asymptotic. By the convergence property above,

$$x_n \leq z, \text{ for all } n.$$

Further, as $(Tx_n = x_{n+1}; n \geq 0)$ is a full sequence,

$$H := \{n \in N; Tx_n = Tz\} \text{ is an asingleton;}$$

so that, the following separation property holds:

$$\begin{aligned} (\text{sepa}) \quad & \exists k = k(z) \geq 0, \text{ such that } n \geq k \text{ implies} \\ & P_0(x_n, z) = d(Tx_n, Tz) > 0; \text{ hence, } d(x_n, z) > 0. \end{aligned}$$

Denote for simplicity $(u_n = x_{n+k}; n \geq 0)$; note that (via (u_n) =full)

$$(\forall n): u_n \leq z, d(u_n, z) > 0, P_0(u_n, z) > 0, d(u_n, Tu_n) > 0.$$

Suppose by contradiction that $d(z, Tz) > 0$. By the relations above,

$$L_1(u_n, z) > 0; \text{ hence, } P(u_n, z) > 0 \text{ (as } P \text{ is } L_1\text{-positive).}$$

The contractive condition is therefore applicable to (u_n, z) , for each n ; and yields (by the strict nonexpansive (modulo $(d, \leq; P)$) property of T)

$$(\text{s-ineq}) \quad P_0(u_n, z) < P(u_n, z), \forall n.$$

Moreover, from the d -Cauchy and convergence relations, one gets (taking a metrical property of $d(.,.)$ into account)

$$\begin{aligned} (\text{conv}) \quad & d(u_n, z), d(Tu_n, z), d(u_n, Tu_n) \rightarrow 0; \\ & d(u_n, Tz), d(Tu_n, Tz) \rightarrow d(z, Tz). \end{aligned}$$

By the strict inequality relation, we get, passing to (inferior) limit as $n \rightarrow \infty$ (and remembering that P is orbitally singular asymptotic)

$$d(z, Tz) = \liminf_n P_0(u_n, z) \leq \liminf_n P(u_n, z) < d(z, Tz);$$

a contradiction. This tells us that the working hypothesis $d(z, Tz) > 0$ cannot be true; so that, $z = Tz$; which tells us that T is strongly Picard (modulo $(d, \leq)$). The proof is complete. $\square$

Note that, coincidence type versions of these facts are available, by means of related techniques in Roldán et al [32]. On the other hand, all these developments may be extended to quasi-metric structures, by following the methods in Turinici [39]. Finally, multivalued extensions of these facts are possible under the lines in Nadler [26]. We will discuss these elsewhere.

5 Du-Rassias results

Let $(X, d, \leq)$ be a quasi-ordered metric space; and T be a selfmap of X ; supposed to be semi-progressive and increasing. The problem to be solved was already precise: determine points of $\text{Fix}(T)$, via iterative processes starting from elements in $X(T, \leq)$. The basic directions and sufficient conditions under which this is to be solved were already listed. In addition, the contractive Meir-Keeler setting of our problem as well as a lot of specific regularity conditions upon our data are being settled. As a by-product of these, we stated the main result of this exposition, Theorem 4.1. It is our aim in the sequel to get a particular case of this result, with practical finality.

To begin with, remember that a lot of maps was introduced, as (for $x, y \in X$)

$$\begin{aligned} A_1(x, y) &= d(x, y), \quad A_2(x, y) = \max\{d(x, Tx), d(y, Ty)\}, \\ L_0(x, y) &= \min\{d(x, y), d(Tx, Ty)\}, \\ L_1(x, y) &= \min\{d(x, y), d(Tx, Ty), d(x, Tx), d(y, Ty)\}, \\ M(x, y) &= \text{diam}\{x, Tx, y, Ty\}. \end{aligned}$$

Then, let us complete this system with an extra lot of maps as (for $x, y \in X$)

$$\begin{aligned} B_1(x, y) &= (1/2)[d(x, Tx) + d(y, Ty)], \\ B_2(x, y) &= (1/2)[d(x, Ty) + d(y, Tx)], \\ B_3(x, y) &= (1/2)[d(x, Tx) + d(y, Tx)], \\ B_4(x, y) &= (1/2)[d(y, Tx) + d(y, Ty)], \\ C_1(x, y) &= (1/3)[d(x, Tx) + d(y, Ty) + d(y, Tx)], \\ C_2(x, y) &= (1/3)[d(x, Tx) + d(x, Ty) + d(y, Tx)], \\ C_3(x, y) &= (1/3)[d(y, Ty) + d(x, Ty) + d(y, Tx)], \\ E_1(x, y) &= (1/4)[d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)], \\ E_2(x, y) &= (1/5)[d(x, y) + d(x, Tx) + d(y, Ty) + d(x, Ty) + d(y, Tx)]. \end{aligned}$$

Finally, define the families of maps

$$\begin{aligned}\mathcal{Q} &= \{A_1, B_1, B_2, B_3, B_4, C_1, C_2, C_3, E_1, E_2\}, \\ P &= \max(\mathcal{Q}) = \max\{A_1, B_1, B_2, B_3, B_4, C_1, C_2, C_3, E_1, E_2\}.\end{aligned}$$

The following fixed point statement is available.

Theorem 5.1. *Suppose that T is Meir-Keeler $(d, \leq; P)$ -contractive, (or, equivalently: asymptotic Meir-Keeler $(d, \leq; P)$ -contractive), where the mapping $P : X \times X \rightarrow R_+$ is as before. In addition, let X be $(a\text{-}o\text{-}f, d)$ -complete. Then,*

(51-a) *T is a Picard operator (modulo $(d, \leq)$)*

(51-b) *T is a strong Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (51-a), T is $(a\text{-}o\text{-}f, d)$ -continuous*

(51-c) *T is a Bellman Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (51-a), $(\leq)$ is $(a\text{-}o\text{-}f, d)$ -selfclosed*

(51-d) *T has the $\text{fix}(\leq)$ -asingleton property.*

Proof. We show that all conditions in our main result are fulfilled by our data.

Part 1. For each $x \in X$, we have

$$\begin{aligned}A_1(x, Tx) &= d(x, Tx) \leq A_2(x, Tx), \\ B_1(x, Tx) &= (1/2)[d(x, Tx) + d(Tx, T^2x)] \leq A_2(x, Tx), \\ B_2(x, Tx) &= (1/2)d(x, T^2x) \leq B_1(x, Tx) \leq A_2(x, Tx), \\ B_3(x, Tx) &= (1/2)d(x, Tx) \leq A_2(x, Tx), \\ B_4(x, Tx) &= (1/2)d(Tx, T^2x) \leq A_2(x, Tx), \\ C_1(x, Tx) &= (1/3)[d(x, Tx) + d(Tx, T^2x)] \leq A_2(x, Tx), \\ C_2(x, Tx) &= (1/3)[d(x, Tx) + d(x, T^2x)] \leq \\ & (1/3)[2d(x, Tx) + d(Tx, T^2x)] \leq A_2(x, Tx), \\ C_3(x, Tx) &= (1/3)[d(Tx, T^2x) + d(x, T^2x)] \leq \\ & (1/3)[2d(Tx, T^2x) + d(x, Tx)] \leq A_2(x, Tx), \\ E_1(x, Tx) &= (1/4)[d(x, Tx) + d(Tx, T^2x) + d(x, T^2x)] \leq \\ & (1/2)[d(x, Tx) + d(Tx, T^2x)] = B_1(x, Tx) \leq A_2(x, Tx), \\ E_2(x, Tx) &= (1/5)[2d(x, Tx) + d(Tx, T^2x) + d(x, T^2x)] \leq \\ & (1/5)[3d(x, Tx) + 2d(Tx, T^2x)] \leq A_2(x, Tx).\end{aligned}$$

Putting these together, yields

each $Q \in \mathcal{Q}$ is orbitally bounded; hence, so is $P = \max(\mathcal{Q})$.

Part 2. For each $x, y \in X$, we have

$$\begin{aligned}A_i(x, y) &\leq M(x, y), \quad i \in \{1, 2\}; \quad B_j(x, y) \leq M(x, y), \quad j \in \{1, 2, 3, 4\}, \\ C_k(x, y) &\leq M(x, y), \quad k \in \{1, 2, 3\}; \quad E_h(x, y) \leq M(x, y), \quad h \in \{1, 2\}.\end{aligned}$$

This, by definition, yields

each $Q \in \mathcal{Q}$ is diametral; hence, so is $P = \max(\mathcal{Q})$.

Part 3. Let the (a-o-f) sequence (x_n) in X and the point $z \in X$ be such that

$$x_n \xrightarrow{d} z \text{ (hence, } Tx_n \xrightarrow{d} z) \text{ and } d(z, Tz) > 0.$$

We have, by definition (and a metrical property of d)

$$\begin{aligned} \lim_n A_1(x_n, z) &= 0, \\ \lim_n B_1(x_n, z) &= (1/2)d(z, Tz), \quad \lim_n B_2(x_n, z) = (1/2)d(z, Tz), \\ \lim_n B_3(x_n, z) &= 0, \quad \lim_n B_4(x_n, z) = (1/2)d(z, Tz) \\ \lim_n C_1(x_n, z) &= (1/3)d(z, Tz), \quad \lim_n C_2(x_n, z) = (1/3)d(z, Tz) \\ \lim_n C_3(x_n, z) &= (2/3)d(z, Tz), \\ \lim_n E_1(x_n, z) &= (1/2)d(z, Tz), \quad \lim_n E_2(x_n, z) = (2/5)d(z, Tz). \end{aligned}$$

This yields

each $Q \in \mathcal{Q}$ is strongly orbitally singular asymptotic;
hence, so is $P = \max(\mathcal{Q})$.

Part 4. Finally, again by definition,

P is A_1 -positive: $A_1(x, y) > 0$ implies $P(x, y) > 0$;
so that: P is both L_0 -positive and L_1 -positive.

Part 5. As a consequence of this, the main result is indeed applicable here; wherefrom, all is clear. $\square$

As a useful particular case of this, the following statement, extending the one in Du and Rassias [12], is holding.

Theorem 5.2. *Let the selfmap $T : X \rightarrow X$ be such that*

*(mk-orig) T is original Meir-Keeler $(d, \leq; P)$ -contractive:
 $\forall \varepsilon > 0, \exists \delta > 0: (x \leq y, \varepsilon \leq P(x, y) < \varepsilon + \delta)$ implies $P_0(x, y) < \varepsilon$,*

where the mapping $P : X \times X \rightarrow R_+$ is as before. In addition, let X be (a-o-f,d)-complete. Then,

(52-a) *T is a Picard operator (modulo $(d, \leq)$)*

(52-b) *T is a strong Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (52-a), T is (a-o-f,d)-continuous*

(52-c) *T is a Bellman Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (52-a), $(\leq)$ is (a-o-f,d)-selfclosed*

(52-d) *T has the fix-asingleton property.*

Proof. We claim that the following inclusion is holding

(mk-orig-mk) T is original Meir-Keeler $(d, \leq; P)$ -contractive implies
 T is Meir-Keeler $(d, \leq; P)$ -contractive.

And then, by the preceding statement, all is clear. There are two steps to be passed.

Step 1. Let $x, y \in X$ be such that $x \leq y$, $P(x, y) > 0$. Put $\varepsilon = P(x, y)$; and let $\delta > 0$ be the associated by (mk-orig) number. Then,

$$\varepsilon = P(x, y) < \varepsilon + \delta; \text{ whence, } P_0(x, y) < \varepsilon = P(x, y);$$

which tells us that T is strictly nonexpansive (modulo $(d, \leq; P)$).

Step 2. Let $\varepsilon > 0$ be given; and $\delta > 0$ be the associated by (mk-orig) number. Then, by the underlying condition,

$$\begin{aligned} (x \leq y, \varepsilon < P(x, y) < \varepsilon + \delta) &\implies (x \leq y, \varepsilon \leq P(x, y) < \varepsilon + \delta) \\ &\implies P_0(x, y) < \varepsilon \implies P_0(x, y) \leq \varepsilon; \end{aligned}$$

whence, T has the Meir-Keeler property (modulo $(d, \leq; P)$). Putting these together, we are done. $\square$

Note, finally, that a variant of Theorem 5.1 with respect to the mappings used in Samet et al [34] and Du et al [11] is also possible, by the same technique. Further aspects will be discussed elsewhere.

6 Particular aspects

Let $(X, d, \leq)$ be a quasi-ordered metric space; and $T : X \rightarrow X$ be a selfmap of X ; supposed to be semi-progressive and increasing. Roughly speaking, the particular statement above has been obtained by an appropriate choice of the mapping P appearing in the Meir-Keeler contractive condition. In the following, some other particular cases of our main result are stated, by working upon the contractive condition itself. Some connections between these and a lot of related developments in the area will be also discussed.

(A) Remember that $\varphi \in \mathcal{F}(re)(R_+^0, R)$ is called *Meir-Keeler admissible*, if

$$(\text{mk-adm}) \quad \forall \varepsilon > 0, \exists \delta > 0, (\forall t): \varepsilon < t < \varepsilon + \delta \implies \varphi(t) \leq \varepsilon.$$

Given the mapping $P : X \times X \rightarrow R_+$ and the function $\varphi \in \mathcal{F}(R_+^0, R)$, let us say that T is $(d, \leq; P; \varphi)$ -contractive, provided

$$P_0(x, y) \leq \varphi(P(x, y)), \forall x, y \in X, x \leq y, P(x, y) > 0.$$

As precise, any such contraction is necessarily Meir-Keeler $(d, \leq; P)$ -contractive, whenever $\varphi \in \mathcal{F}(re)(R_+^0, R)$ is Meir-Keeler admissible. As a direct consequence of this, we have (by means of our main result)

Theorem 6.1. *Suppose that the selfmap T is $(d, \leq; P; \varphi)$ -contractive, where the function $\varphi \in \mathcal{F}(re)(R_+^0, R)$ and the mapping $P : X \times X \rightarrow R_+$ are such that*

(phi-P) φ is Meir-Keeler admissible (or, equivalently: Matkowski admissible), and P is orbitally bounded, diametral.

In addition, let X be $(a-o-f, d)$ -complete. Then,

(61-a) *T is a Picard operator (modulo $(d, \leq)$) when, in addition, P is L_1 -positive*

(61-b) *T is a strong Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (61-a), T is $(a-o-f, d)$ -continuous*

(61-c) *T is a Bellman Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (61-a), $(\leq)$ is $(a-o-f, d)$ -selfclosed and P is orbitally singular asymptotic*

(61-d) *T has the $\text{fix}(\leq)$ -asingleton property when (in addition) P is L_0 -positive.*

Some particular cases of this result are described as follows. Let the system of maps over $\mathcal{F}(X \times X, R_+)$

$$\mathcal{Q} = \{A_1, B_1, B_2, B_3, B_4, C_1, C_2, C_3, E_1, E_2\}$$

be the already introduced one.

I) The regularity condition (phi-P) holds under

(phi-P-1) φ is Boyd-Wong admissible and $P = A_1$.

In this case, the corresponding version of Theorem 6.1 includes directly the related statement in Agarwal et al [1], proved by a direct method. But, as shown in that paper, this result includes (under $(\leq) = X \times X$) the well known contribution due to Boyd and Wong [6] or Matkowski [21]; hence, so does Theorem 6.1.

II) The same regularity condition (phi-P) holds (see above) under

(phi-P-2) φ is Geraghty admissible and $P = A_1$.

Then, the corresponding version of Theorem 6.1 includes the related statement in Amini-Harandi and Emami [2]. But (cf. a previous remark) (phi-P-2) is a particular case of (phi-P-1) This tells us that the result due to Amini-Harandi and Emami [2] is nothing but a particular case of the one in Agarwal et al [1].

III) Finally, the same regularity condition (phi-P) holds under

(phi-P-3) φ is Geraghty admissible and $P \in \mathcal{Q}$.

In particular, when $P = B_1$, the corresponding version of Theorem 6.1 includes the related statement in Choudhury and Kundu [7] proved by a distinct argument.

(B) Let (ψ, φ) be a pair of functions over $\mathcal{F}(R_+^0, R)$, with

(norm) (ψ, φ) is normal:

ψ is increasing and φ is strictly positive ($\varphi(t) > 0, \forall t \in R_+^0$)

The following extra condition will be considered here:

(r-s-pos) φ is right sequentially positive:

for each sequence $(t_n; n \geq 0)$ in R_+^0 and each element $\varepsilon > 0$

with $t_n \rightarrow \varepsilon+$, the relation $\lim_n \varphi(t_n) = 0$ is impossible.

Given the mapping $P : X \times X \rightarrow R_+$ and the couple (ψ, φ) of functions over $\mathcal{F}(R_+^0, R)$, let us say that T is $(d, \leq; P; (\psi, \varphi))$ -contractive, provided

$$\begin{aligned} \psi(P_0(x, y)) &\leq \psi(P(x, y)) - \varphi(P(x, y)), \\ \forall x, y \in X, x &\leq y, P_0(x, y) > 0, P(x, y) > 0. \end{aligned}$$

By a previous result, any such contraction is Meir-Keeler $(d, \leq; P)$ -contractive, whenever (ψ, φ) is normal and φ is right sequentially positive. As a direct consequence of this, we have (by means of our main result)

Theorem 6.2. *Suppose that T is $(d, \leq; P; (\psi, \varphi))$ -contractive, where the mapping $P : X \times X \rightarrow R_+$ and the couple (ψ, φ) of functions over $\mathcal{F}(R_+^0, R)$ are such that*

(psi-phi-P) (ψ, φ) is normal, φ is right sequentially positive, and P is orbitally bounded, diametral.

In addition, let X be $(a-o-f, d)$ -complete. Then,

(62-a) *T is a Picard operator (modulo $(d, \leq)$) when, in addition, P is L_1 -positive*

(62-b) *T is a strong Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (62-a), T is $(a-o-f, d)$ -continuous*

(62-c) *T is a Bellman Picard operator (modulo $(d, \leq)$) when, in addition to the setting of (62-a), $(\leq)$ is $(a-o-f, d)$ -selfclosed and P is orbitally singular asymptotic*

(62-d) *T has the fix- $(\leq)$ -asingleton property when (in addition) P is L_0 -positive.*

The obtained result extends the one in Dutta and Choudhury [13]. In fact, it also includes a related statement in Găvruta et al [14]; we do not give details.

In the following, a basic particular case of this last result is discussed.

Let $F : R_+^0 \rightarrow R$ and $\varphi : R_+^0 \rightarrow R$ be a couple of functions with

(nc-1) (F, φ) is normal: F is increasing and φ is strictly positive

(nc-2) φ is right sequentially positive (see above).

Note that this couple is just the one appearing in Theorem 6.2. As a consequence, the quoted result is applicable to contractions like

$$F(P_0(x, y)) \leq F(d(x, y)) - \varphi(d(x, y)), \\ x, y \in X, x \leq y, P_0(x, y) > 0, d(x, y) > 0;$$

referred to as: *Wardowski type contractions*. This tells us that Theorem 6.2 includes the basic fixed point result in Wardowski [44]; that extends an older statement by the same author [43]. In fact, some other statements in the area, described in the survey paper by Karapinar et al [18] may be obtained via these techniques; we do not give details. Note, finally, that our main result includes, partially, the ones described in Vujaković et al [42]. For a complete inclusion of all these, the implicit methods in Turinici [38] may be used; this will be discussed elsewhere.

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