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A coincidence point result for Wardowski contraction

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Abstract: In the present paper we obtain a coincidence point result for Wardowski contraction which generalizes analogous results known in the literature.

Keywords: Coincidence point, complete metric space, contraction.

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1 Introduction

Since the seminal result of Banach [2] the fixed point theory of nonexpansive maps has been a rapidly growing area of research. See, for example, [3, 10, 11, 15, 16, 18, 19, 21, 22, 24, 25] and the references mentioned therein. A great progress has taken place in this area including studies of feasibility and common fixed point problems, which find various important applications [5, 6, 9, 24, 25]. In [23] D. Wardowski introduced an interesting class of mappings which contains Banach contractions and showed the existence of fixed points for these mappings. Wardowski type contractions were studied in [8, 17, 20]. In the present paper we obtain a coincidence point result for Wardowski contraction which generalizes analogous results known in the literature.

Assume that (X, ρ) is a complete metric space. In our paper we use the following notation. For each mapping $S : X \to X$, denote by S^0 the identity self-mapping of X and set

$$S^{i+1} = S \circ S^i$$

for each integer $i \geq 0$. We suppose that the sum over empty set is zero. For each $x \in X$ and each $r > 0$, set

$$B(x, r) = \{y \in X :\ \rho(x, y) \leq r\}.$$

For each $x \in X$ and each nonempty set $A \subset X$ set

$$\rho(x, A) = \inf(\rho(x, y) :\ y \in A\}.$$

2 Preliminaries

When considering a pair of mappings (T, g) from the non-empty set X to itself, we state the following terms:

1) T and g are commutative mappings if $T(g(x)) = g(T(x))$ for every x in X;

2) If $T(x) = g(x)$ for some $x \in X$, then the point x is called a coincidence point, while the point $y = T(x) = g(x)$ is called a point of coincidence.

3) A pair (T, g) is called weakly compatible if T and g commute at the coincidence point, i.e, if $T(g(x)) = g(T(x))$ whenever $T(x) = g(x)$;

4) The mappings T and g of the pair (T, g) in the metric space (X, d) are compatible if $d(T(g(x_n)), g(T(x_n)))$ tends to zero when n tends to $+\infty$ where x_n is a convergent sequence in the metric space (X, d).

All these notions are closely related to the notion of a common fixed point of the mapping T and g, especially in relation from the uniqueness of that common fixed point. The following result is from [1]:

For other interesting details see [12] and [13].

Proposition 2.1. *[1] Let T and g be weakly compatible self maps of a set X. If T and g have a unique point of coincidence $w = T(x) = g(x)$, then w is the unique common fixed point of T and g.*

We add the following two lemmas that will be, as in the paper [8], useful in the proofs of some results. Both these lemmas are used to prove the Cauchyness of the Picard sequence $x_n = T(x_{n-1}), n \in \mathbb{N}$ or the Jungck's sequence $Tx_n = g(x_{n+1}), n \in \mathbb{N}$, where $x_0 \in X$ is given point in a metric space X and $T, g : X \to X$ with $T(X) \subset g(X)$.

Lemma 2.2. *([8], [14]) Let $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ be a Picard sequence in a metric space (X, d) such that $d(x_{n+1}, x_n) < d(x_n, x_{n-1})$ is satisfied for all $n \in \mathbb{N}$. Then $x_n \neq x_m$ whenever $n \neq m$.*

Lemma 2.3. *([8], [14]) Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1})$ tends to 0 as $n \to +\infty$. If $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to ε from above, as $k \to +\infty$:*

$$\left\{d\left(x_{n(k)}, x_{m(k)}\right)\right\}, \left\{d\left(x_{n(k)}, x_{m(k)-1}\right)\right\}, \left\{d\left(x_{n(k)+1}, x_{m(k)}\right)\right\},$$

$$\left\{d\left(x_{n(k)+1}, x_{m(k)-1}\right)\right\}, \left\{d\left(x_{n(k)+1}, x_{m(k)+1}\right)\right\}, \ldots.$$

3 The result

Assume that (X,ρ) is a complete metric space endowed with the metric ρ, $\tau > 0$, $F : (0,\infty) \to R^1$ is an increasing function and that $g, T : X \to X$ are mappings such that for every pair of points $x, y \in X$ satisfying $g(x) \neq g(y)$ and $T(x) \neq T(y)$, we have

$$F(\rho(T(x),T(y))) + \tau \le F(\rho(g(x),g(y))). \tag{3.1}$$

This class of mappings was studied in [4] where it was shown the existence of a coincidence point. In [4] it was assumed that F is strictly increasing function and satisfies two other conditions. Here we only assume that F is merely increasing.

Recall that the mapping g, T are compatible if for each sequence $\{x_n\}_{n=1}^{\infty} \subset X$ satisfying

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} T(x_n) = t,$$

we have $T(t) = g(t)$.

Theorem 3.1. *Assume that $T(X) \subset g(X)$ and that at least one of the following properties hold:*

g, T are compatible; $g(X)$ is a closed set.

Then there exists $x_* \in X$ such that $g(x_*) = T(x_*)$ and if $y_1, y_2 \in X$ satisfy $g(y_i) = T(y_i)$, $i = 1, 2$, then $g(y_1) = g(y_2)$.

Proof. First we show the uniqueness. Assume that $y_1, y_2 \in X$ and

$$g(y_i) = T(y_i),\ i = 1, 2.$$

If

$$g(y_1) \neq g(y_2),$$

then by (3.1),

$$\tau + F(\rho(T(y_1),T(y_2))) \le F(\rho(g(y_1),g(y_2))),$$

a contradiction.

Now we prove the existence. Let $x_0 \in X$ and define $\{x_n\}_{n=1}^{\infty} \subset X$ such that for each integer $n \ge 0$,

$$g(x_{n+1}) = T(x_n). \tag{3.2}$$

Set for each integer $n \ge 0$,

$$y_n = T(x_n). \tag{3.3}$$

If $k \ge 1$ is an integer and

$$g(x_{k+1}) = g(x_k),$$

then

$$T(x_k) = g(x_k)$$

and the assertion of the theorem holds. Therefore we may assume without loss of generality that

$$g(x_{k+1}) \neq g(x_k) \text{ for each integer } k \geq 1.$$

This implies that

$$y_{k+1} \neq y_k \text{ for each integer } k \geq 0.$$

Let $n \geq 1$ be an integer. By (3.1) and (3.3),

$$\begin{aligned} \tau + F(\rho(y_n, y_{n+1})) &= \tau + F(\rho(T(x_n), T(x_{n+1})) \\ &\leq F(\rho(g(x_n), g(x_{n+1})) = F(\rho(y_n, y_{n+1})) \end{aligned} \tag{3.4}$$

and

$$\rho(y_n, y_{n+1}) < \rho(y_n, y_{n-1}). \tag{3.5}$$

By (3.5), for each integer $n \geq 1$,

$$F(\rho(y_n, y_{n+1})) \leq F(\rho(y_0, y_1)) - n\tau. \tag{3.6}$$

We show that

$$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = 0.$$

Assume the contrary. Then there exists $\epsilon > 0$ such that

$$\epsilon \leq \rho(y_n, y_{n+1}),\ n = 0, 1, \dots$$

and for each integer $n \geq 0$,

$$F(\epsilon) \leq F(\rho(y_n, y_{n+1})).$$

This contradicts (3.6). The contradiction we have reached proves that

$$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = 0. \tag{3.7}$$

Since the function F is increasing there exists a countable set $E \subset (0, \infty)$ such the function F is continuous at every point of $(0, \infty) \setminus E$. Let $\epsilon \in (0, 1)$. We show that $\rho(y_{n_1}, y_{n_2}) \leq \epsilon$ for each pair of sufficiently large numbers n_1, n_2. Clearly, we may assume without loss of generality that the function F is continuous at ϵ. Therefore there exists

$$\delta \in (0, \epsilon/4) \tag{3.8}$$

such that

$$|F(\xi) - F(\epsilon)| \leq \tau/4 \text{ for each } \xi \in [\epsilon - 4\delta, \epsilon + 4\delta]. \tag{3.9}$$

In view of (3.7), there exists an integer $n_0 \geq 1$ such that for each integer $n \geq n_0$,

$$\rho(y_n, y_{n+1}) \leq \delta. \tag{3.10}$$

Assume that $n_0 \leq i < j$ are integers. We show that $\rho(y_i, y_j) \leq \epsilon$. Assume the contrary. Then

$$\rho(y_i, y_j) > \epsilon. \tag{3.11}$$

By (3.10) and (3.11),

$$\rho(y_i, y_{i+1}) < \delta, \ j > i + 1. \tag{3.12}$$

We may assume without loss of generality that for each $s \in \{i + 1, \ldots, j - 1\}$,

$$\rho(y_i, y_s) \leq \epsilon$$

and in particular,

$$\rho(y_i, y_{j-1}) \leq \epsilon. \tag{3.13}$$

By (3.10) and (3.11),

$$y_{i+1} \neq y_j, \ y_i \neq y_{j-1}. \tag{3.14}$$

It follows from (3.2), (3.7), (3.9), (3.13) and (3.14) that

$$F(\rho(y_j, y_{i+1})) = F(\rho(T(x_j), T_{i+1}))) \leq F(\rho(g(x_j), g_{i+1}))) - \tau$$

$$\leq F(\rho(y_{j-1}), y_i)) - \tau \leq F(\epsilon) - \tau. \tag{3.15}$$

In view of (3.13) and (3.15),

$$\rho(y_j, y_{j+1}) < \rho(y_j, y_{j-1}) \leq \epsilon,$$

$$\rho(y_j, y_{j+1}) \leq \epsilon - 4\delta. \tag{3.16}$$

By (3.12) and (3.16),

$$\rho(y_i, y_j) \leq \rho(y_i, y_{i+1}) + \rho(y_{i+1}, y_j) \leq \delta + \epsilon - 4\delta < \epsilon.$$

This contradicts (3.11). The contradiction we have reached proves that $\rho(y_i, y_j) \leq \epsilon$ for each pair of integers $j > i \geq n_0$. Thus $\{y_i\}_{i=0}^{\infty}$ is a Cauchy sequence and there exists

$$y_* = \lim_{n\to\infty} y_n = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} g(x_{n+1}).$$

If g, T are compatible, then $g(x_*) = T(x_*)$ and the assertion of the theorem holds.

Assume that the set $g(X)$ is closed. Then there exists $x_* \in X$ such that

$$g(x_*) = y_* = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} g(x_n). \tag{3.17}$$

Set

$$E = \{n \in \{1, 2, \dots\} : \ y_* \neq g(x_n)\}.$$

By our assumptions the set E is infinite. Let $n \in E$. If $T(x_*) \neq T(x_n)$, then by (3.1),

$$F(\rho(T(x_n), T(x_*))) + \tau \leq F(\rho(g(x_n), g(x_*))),$$

$$\rho(T(x_n), T(x_*)) < \rho(g(x_n), g(x_*)).$$

Otherwise $T(x_*) = T(x_n)$. Therefore

$$\lim_{n\in E,\, n\to\infty} \rho(T(x_n), T(x_*)) = 0.$$

Together with (17) this implies that $g(x_*) = T(x_*)$. Theorem 3.1 is proved. □

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