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Some differentiability properties of solutions of a Caputo type fractional differential inclusion

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Abstract: We establish several fractional variational inclusions for solutions of a nonconvex fractional differential inclusion involving a Caputo type fractional derivative.

Keywords: Differential inclusion, fractional derivative, variational inclusion, tangent cone, set-valued derivative.

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1 Introduction

In the last decades the literature provides a huge development of the theory of differential equations and inclusions of fractional order ([5, 11, 17, 18] etc.). This is due, mainly, to the fact that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [6] allows to use Cauchy conditions which have physical meanings.

A Caputo type fractional derivative of a function with respect to another function ([17]) that extends and unifies several fractional derivatives existing in the literature like Caputo, Caputo-Hadamard, Caputo-Katugampola was intensively studied in recent years [1, 2, 3, 9, 10] etc. where existence results and qualitative properties of the solutions for fractional differential equations defined by this fractional derivative are obtained.

The present paper is concerned with fractional differential inclusions of the form

$$D_C^{\alpha,\psi}x(t) \in F(t,x(t)) \quad a.e.\ ([0,T]), \quad x(0)=x_0, \quad x'(0)=x_1, \tag{1.1}$$

where $\alpha \in (1,2]$, $D_C^{\alpha,\psi}$ is the fractional derivative mentioned above,, $F:[0,T]\times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $x_0, x_1 \in \mathbf{R}$.

Our goal is to extend the results concerning the differentiability of solutions of differential inclusions with respect to initial conditions to the solutions of problem (1.1). In Control Theory, mainly, if we want to obtain necessary optimality conditions, it is essential to have several "differentiability" properties of solutions with respect to initial conditions. One of the most powerful result in the theory of differential equations, the classical Bendixson-Picard-Lindelőf theorem states that the maximal flow of a differential equation is differentiable with respect to initial conditions and its derivatives verify the variational equation. This result has been generalized in various ways to differential inclusions by considering several variational inclusions and proving corresponding theorems that extend Bendixson-Picard-Lindelőf theorem. The results we extend known as the contingent, the intermediate (quasitangent) and the circatangent variational inclusion are obtained in the "classical case" of differential inclusions. For this results and for a complete discussion on this topic we refer to [4].

The proofs of our results follows by an approach similar to the classical case of differential inclusions ([4, 14]) and use a recent result ([9]) concerning the existence of solutions of problem (1.1). Similar results for fractional differential inclusions defined by Caputo fractional derivative may be found in [7] and for fractional differential inclusions defined by Caputo-Katugampola fractional derivative are obtained in [8]; therefore, the present paper extends and unifies all these results.

The paper is organized as follows: in Section 2 we present the notations and the preliminary results to be used in the sequel and in Section 3 we provide our main results.

2 Preliminaries

Let $T > 0$, $I := [0, T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Denote by $\mathcal{P}(\mathbf{R})$ the family of all nonempty subsets of $\mathbf{R}$ and by $\mathcal{B}(\mathbf{R})$ the family of all Borel subsets of $\mathbf{R}$.

As usual, we denote by $C(I, \mathbf{R})$ the Banach space of all continuous functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I, \mathbf{R})$ the Banach space of all integrable functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|\mathrm{d}t$. On $C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ we consider the following norm

$$|(x, f)|_{C\times L} = |x|_C + |f|_1 \quad \forall\ (x, f) \in C(I, \mathbf{R}) \times L^1(I, \mathbf{R}).$$

Consider $\beta > 0$, $f(.) \in L^1(I, \mathbf{R})$ and $\psi(.) \in C^n(I, \mathbf{R})$ such that $\psi'(t) > 0\ \forall\ t \in I$.

Definition 2.1. ([17]) a) The ψ - Riemann-Liouville fractional integral of f of order β is defined by

$$I^{\beta,\psi} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} f(s)ds,$$

where Γ is the (Euler's) Gamma function defined by $\Gamma(\beta) = \int_0^\infty t^{\beta-1}e^{-t}dt$.

b) The ψ - Riemann-Liouville fractional derivative of f of order β is defined by

$$D^{\beta,\psi} f(t) = \frac{1}{\Gamma(n-\beta)}(\frac{1}{\psi'(t)}\frac{d}{dt})^n \int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-\beta-1} f(s)ds,$$

where $n = [\beta]+1$.

c) The ψ - Caputo fractional derivative of f of order β is defined by

$$D_C^{\beta,\psi} f(t) = D^{\beta,\psi}[f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(0)}{k!}(\psi(t)-\psi(0))^k],$$

where $f_\psi^{[k]}(t) = (\frac{1}{\psi'(t)}\frac{d}{dt})^k x(t)$, $n = \beta$ if $\alpha \in \mathbf{N}$ and $n = [\beta]+1$, otherwise.

We note that if $\beta = m \in \mathbf{N}$ then $D_C^{\beta,\psi} f(t) = f_\psi^{[m]}(t)$ and if $n = [\beta]+1$ then $D_C^{\beta,\psi} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f_\psi^{[n]}(s)ds$. Also, if $\psi(t) \equiv t$ one obtains Caputo's fractional derivative, if $\psi(t) \equiv \ln(t)$ one obtains Caputo-Hadamard's fractional derivative and, finally, if $\psi(t) \equiv t^\sigma$ one obtains Caputo-Katugampola's fractional derivative.

Definition 2.2. By a solution of the problem (1.1) we mean a function $x \in C(I, \mathbf{R})$ for which there exists a function $h \in L^1(I, \mathbf{R})$ satisfying $h(t) \in F(t, x(t))$ *a.e.* (I), $D_C^{\alpha,\psi} x(t) = h(t)$ *a.e.* (I) and $x(0) = x_0$, $x'(0) = x_1$.

In this case we say that $(x(.), h(.))$ is a *trajectory-selection pair* of (1.1).

We shall use the following notations for the solution sets of (1.1).

$$\mathcal{S}(x_0, x_1) = \{(x(.), h(.)); \quad (x(.), h(.)) \text{ is a trajectory-selection pair of (1.1)}\}.$$

Hypothesis. (i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ *measurable.*

(ii) *There exists* $L(.) \in L^1(I, (0, \infty))$ *such that, for almost all* $t \in I, F(t,.)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H(F(t,x), F(t,y)) \le L(t)|x-y| \quad \forall x, y \in \mathbf{R},$$

where $d_H(.,.)$ is the Hausdorff distance

$$d(A,B) = \max\{d^*(A,B), d^*(B,A)\}, d^*(A,B) = \sup\{d(a,B); a \in A\}.$$

In what follows $\alpha \in [1,2)$ and $\psi(.) \in C^1(I, \mathbf{R})$ with $\psi'(t) > 0 \ \forall\, t \in I$. The next result ([8]) is an extension of Filippov's theorem concerning the existence of solutions to a Lipschitzian differential inclusion ([13]) to fractional differential inclusions of the form (1.1). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem ([13]) consists in proving the existence of a solution starting from a given almost solution. Moreover, the result provides an estimate between the starting almost solution and the solution of the differential inclusion.

Consider $y_0, y_1 \in \mathbf{R}$, $g(.) \in L^1(I, \mathbf{R})$ and $y(.)$ is a solution of the problem

$$D_C^{\alpha,\psi} y(t) = g(t) \quad y(0) = y_0, \quad y'(0) = y_1.$$

Denote

$$\eta = \frac{1}{1 - I^{\alpha,\psi} L(T)} (|y(0) - x_0| + T|y'(0) - x_1| + I^{\alpha,\psi} q(T)).$$

Theorem 2.3. *Assume that Hypothesis is satisfied, assume that $I^{\alpha,\psi}L(T) < 1$ and let $y(.) \in C(I, \mathbf{R})$ be such that there exists $q(.) \in L^1(I, \mathbf{R})$ with $I^{\alpha,\psi}q(T) < +\infty$ and $d(D_C^{\alpha,\psi} y(t), F(t, y(t))) \leq q(t)$ a.e. (I).*

Then there exists $(x(.), h(.))$ a trajectory-selection pair of (1.1) satisfying for all $t \in I$

$$|x(t) - y(t)| \leq \eta \quad \forall t \in I,$$

$$|h(t) - g(t)| \leq L(t)\eta + q(t) \quad a.e.\ (I).$$

3 The main results

Let Y be a normed space, $X \subset Y$ and $x \in \overline{X}$ (the closure of X). From the multitude of the tangent cones in the literature (e.g., [4]) we recall only the contingent, the quasitangent and Clarke's tangent cones, defined, respectively by

$$K_x X = \{v \in Y; \quad \exists s_m \to 0+,\ \exists v_m \to v :\ x + s_m v_m \in X\},$$

$$Q_x X = \{v \in Y; \quad \forall s_m \to 0+,\ \exists v_m \to v :\ x + s_m v_m \in X\},$$

$$C_x X = \{v \in Y; \forall (x_m, s_m) \to (x, 0+),\ x_m \in X,\ \exists y_m \in X :\ \frac{y_m - x_m}{s_m} \to v\}.$$

This cones are related as follows: $C_x X \subset Q_x X \subset K_x X$.

Corresponding to each type of tangent cone, say $\tau_x X$, one may introduce (e.g., [4]) a *set-valued directional derivative* of a multifunction $G(\cdot) : X \subset Y \to \mathcal{P}(Y)$ (in particular of a single-valued mapping) at a point $(x, y) \in \mathrm{Graph}(G)$ as follows

$$\tau_y G(x; v) = \{w \in Y; \quad (v, w) \in \tau_{(x,y)} \mathrm{Graph}(G)\}, \quad v \in \tau_x X.$$

Let $(y(.), g(.))$ be a trajectory-selection pair of problem (1.1). Our intention is to "linearize" (1.1) along $(y(.), g(.))$ by replacing it by several fractional variational inclusions.

Consider, first, the quasitangent variational inclusion

$$\begin{cases} D_C^{\alpha,\psi} w(t) \in Q_{g(t)}(F(t,.))(y(t); w(t)) \quad a.e.\ (I) \\ w(0) = u, \quad w'(0) = v, \end{cases} \tag{3.1}$$

where $u, v \in \mathbf{R}$.

Theorem 3.1. *Consider the solution map $\mathcal{S}(.,.)$ as a set valued map from $\mathbf{R} \times \mathbf{R}$ into $C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ and assume that Hypothesis is satisfied and $I^{\alpha,\psi} L(T) < 1$.*

Then, for any $u, v \in \mathbf{R}$ and any trajectory-selection pair (w, π) of the linearized inclusion (3.1) one has

$$(w, \pi) \in Q_{(y,g)} \mathcal{S}((y(0), y'(0); (u, v)).$$

Proof. Let $u, v \in \mathbf{R}$ and let $(w, \pi) \in C(I, \mathbf{R}) \times L^1(I, \mathbf{R})$ be a trajectory-selection pair of (3.1). By the definition of the quasitangent derivative and from the Lipschitzianity of $F(t, .)$, for almost all $t \in I$, we have

$$\begin{aligned} &\lim_{h \to 0+} d(D_C^{\alpha,\psi} w(t), \tfrac{F(t, y(t) + hw(t)) - D_C^{\alpha,\psi} y(t)}{h}) = \\ &\lim_{h \to 0+} d(\pi(t), \tfrac{F(t, y(t) + hw(t)) - g(t)}{h}) = 0. \end{aligned} \tag{3.2}$$

Moreover, since $g(t) \in F(t, y(t))$ a.e. (I), from Hypothesis, for all enough small $h > 0$ and for almost all $t \in I$, one has

$$\begin{aligned} &d(D_C^{\alpha,\psi}(y(t) + hw(t)), F(t, y(t) + hw(t))) = d(g(t) + h\pi(t), F(t, y(t) + \\ &hw(t))) \le h(|\pi(t)| + L(t)|w(t)|) \end{aligned}$$

By standard arguments (e.g., Lemmas 1.4 and 1.5 in [14]) the function $t \to \mathrm{d}(g(t) + h\pi(t), F(t, y(t) + hw(t)))$ is measurable. Therefore, using the Lebesgue dominated convergence theorem we infer

$$\int_0^T d(D_C^{\alpha,\psi}(y(t) + hw(t)), F(t, y(t) + hw(t)))dt = o(h), \tag{3.3}$$

where $\lim_{h \to 0+} \frac{o(h)}{h} = 0$.

We apply Theorem 2.3 and by (3.3) we deduce the existence of $M \ge 0$ and of trajectory-selection pairs $(y_h(.), g_h(.))$ of the fractional differential inclusion in (1.1) satisfying

$$|y_h - y - hw|_C + |g_h - g - h\pi|_1 \le Mo(h),$$

$$y_h(0)=y(0)+hu,\quad y_h'(0)=y'(0)+hv,$$

which implies

$$\lim_{h\to 0+}\frac{y_h-y}{h}=w\quad\text{in}\quad C(I,\mathbf{R}),$$

$$\lim_{h\to 0+}\frac{g_h-g}{h_n}=\pi\quad\text{in}\quad L^1(I,\mathbf{R}).$$

Therefore

$$\lim_{h\to 0+}\mathrm{d}_{C\times L}((w,\pi),\frac{\mathcal{S}((y(0)+hu,y'(0)+hv))-(y,g)}{h})=0$$

and the proof is complete. □

We consider next the variational inclusion defined by the Clarke directional derivative of the set-valued map $F(t,.)$, i.e., the so called circatangent variational inclusion

$$\begin{cases} D_C^{\alpha,\psi}w(t)\in C_{g(t)}(F(t,.))(y(t);w(t)) & a.e.\ (I)\\ w(0)=u,\quad w'(0)=v. \end{cases}\tag{3.4}$$

Theorem 3.2. *Consider the solution map $\mathcal{S}(.,.)$ as a set valued map from $\mathbf{R}\times\mathbf{R}$ into $C(I,\mathbf{R})\times L^1(I,\mathbf{R})$ and assume that Hypothesis is satisfied and $I^{\alpha,\psi}L(T)<1$.*

Then, for any $u,v\in\mathbf{R}$ and any trajectory-selection pair (w,π) of the linearized inclusion (3.4) one has

$$(w,\pi)\in C_{(y,g)}\mathcal{S}((y(0),y'(0);(u,v)).$$

Proof. Let $u,v\in\mathbf{R}$, let $(w,\pi)\in C(I,\mathbf{R})\times L^1(I,\mathbf{R})$ be a trajectory-selection pair of (3.4), let (y_n,g_n) be a sequence of trajectory-selection pairs of (1.1) that converges to $(y,g)\in C(I,\mathbf{R})\times L^1(I,\mathbf{R})$ and let $h_n\to 0+$. Then there exists a subsequence $g_j(.):=g_{n_j}(.)$ such that

$$\lim_{j\to\infty}g_j(t)=g(t)\quad a.e.\ (I)\tag{3.5}$$

Denote $\lambda_j:=h_{n_j}$. From (3.4) and from the definition of the Clarke directional derivative, for almost all $t\in I$ we have

$$\begin{aligned}&\lim_{h\to 0+}d(D_C^{\alpha,\psi}w(t),\tfrac{F(t,y_j(t)+\lambda_jw(t))-D_C^{\alpha,\psi}y_j(t)}{\lambda_j})=\\&\lim_{h\to 0+}d(\pi(t),\tfrac{F(t,y_j(t)+\lambda_jw(t))-g_j(t)}{\lambda_j})=0.\end{aligned}\tag{3.6}$$

Since $g_j(t)\in F(t,y_j(t))$ a.e. (I), for almost all $t\in I$, we get

$$\begin{aligned}&d(D_C^{\alpha,\psi}(y_j(t)+\lambda_jw(t)),F(t,y_j(t)+\lambda_jw(t)))=d(g_j(t)+\lambda_j\pi(t),F(t,y_j(t)\\&+\lambda_jw(t)))\le\lambda_j(|\pi(t)|+L(t)|w(t)|).\end{aligned}$$

The last inequality together with Lebesgue's dominated convergence theorem implies

$$\int_0^T d(D_C^{\alpha,\psi}(y_j(t)+\lambda_j w(t)), F(t, y_j(t)+\lambda_j w(t)))dt = o(\lambda_j), \tag{3.7}$$

where $\lim_{j\to\infty} \frac{o(\lambda_j)}{\lambda_j} = 0$.

We apply Theorem 2.3 and by (3.7) we deduce the existence of $M \geq 0$ and of trajectory-selections pairs $(\overline{y}_j(.), \overline{g}_j(.))$ of the fractional differential inclusion in (1.1) satisfying

$$|\overline{y}_j - y_j - \lambda_j w|_C + |\overline{g}_j - g_j - \lambda_j \pi|_1 \leq M o(\lambda_j),$$
$$\overline{y}_j(0) = y(0) + \lambda_j u, \quad \overline{y}_j'(0) = y'(0) + \lambda_j v.$$

It follows that

$$\lim_{j\to\infty} \frac{\overline{y}_j - y}{\lambda_j} = w \quad \text{in} \quad C(I, \mathbf{R}),$$

$$\lim_{j\to\infty} \frac{\overline{g}_j - g}{\lambda_j} = \pi \quad \text{in} \quad L^1(I, \mathbf{R}),$$

which completes the proof. □

Finally, we consider the contingent variational inclusion

$$\begin{cases} D_C^{\alpha,\psi} w(t) \in \overline{co} K_{g(t)}(F(t,.))(y(t); w(t)) \quad a.e.\ (I) \\ w(0) = u, \quad w'(0) = v. \end{cases} \tag{3.8}$$

Theorem 3.3. *Consider the solution map $\mathcal{S}(.,.)$ as a set valued map from $\mathbf{R}\times\mathbf{R}$ into $C(I,\mathbf{R}) \times L^\infty(I,\mathbf{R})$, with $L^\infty(I,\mathbf{R})$ supplied with the weak-* topology and assume that Hypothesis is satisfied.*

Then for any $u, v \in \mathbf{R}$ one has

$$K_{(y,g)}\mathcal{S}((y(0), y'(0); (u,v)) \subset \{(w,\pi); (w,\pi) \textit{ is a trajectory-selection pair of (3.8)}\}.$$

Proof. Let $u, v \in \mathbf{R}$ and let $(w,\pi) \in K_{(y,g)}\mathcal{S}((y(0), y'(0); (u,v))$. According to the definition of the contingent derivative there exist $h_n \to 0+$, $u_n \to u$, $v_n \to v$, $w_n(.) \to w(.)$ in $C(I,\mathbf{R})$, $\pi_n(.) \to \pi(.)$ in weak-* topology of $L^\infty(I,\mathbf{R})$ and $c > 0$ such that

$$\begin{aligned} &|\pi_n(t)| \leq c \quad a.e.\ (I), \\ &g(t) + h_n \pi_n(t) \in F(t, y(t) + h_n w_n(t)) \quad a.e.\ (I), \\ &w_n(0) = u_n, w_n'(0) = v_n. \end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned} &w_n(.) \quad \text{converges pointwise to} \quad w(.) \\ &\pi_n(.) \quad \text{converges weak in } L^1(I,\mathbf{R}) \text{ to} \quad \pi(.) \end{aligned} \tag{3.10}$$

We apply Mazur's theorem (e.g., [12]) and we find that there exists

$$v_m(t) = \sum_{p=m}^{\infty} a_m^p \pi_p(t)$$

$v_m(.) \to \pi(.)$ (strong) in $L^1(I, \mathbf{R})$, where $a_m^p \geq 0$, $\sum_{p=m}^{\infty} a_m^p = 1$ and for any m, $a_m^p \neq 0$ for a finite number of p.

Therefore, a subsequence (again denoted) $v_m(.)$ converges la $\pi(.)$ a.e.. From (3.9) for any p and for almost all $t \in I$

$$w_p'(t) \in \frac{1}{h_p}(F(t, y(t) + h_p w_p(t)) - g(t)) \cap cB$$

Let $t \in I$ be such that $v_m(t) \to \pi(t)$ and $g(t) \in F(t, y(t))$. Fix $n \geq 1$ and $\epsilon > 0$. From (3.9) there exists m such that $h_p \leq 1/n$ and $|w_p(t) - w(t)| \leq 1/n$ for any $p \geq m$.

If, we denote

$$\phi(z, h) := \frac{1}{h}(F(t, y(t) + hz) - g(t)) \cap cB$$

then

$$v_m(t) \in \mathrm{co}(\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h))$$

and if $m \to \infty$, we get

$$\pi(t) \in \overline{\mathrm{co}}(\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h)).$$

Since, $\phi(z, h) \subset cB$, we infer that

$$\pi(t) \in \overline{\mathrm{co}} \cap_{\epsilon > 0, n \geq 1} (\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h) + \epsilon B).$$

On the other hand,

$$\cap_{\epsilon > 0, n \geq 1} (\cup_{h \in (0, \frac{1}{n}], z \in B(w(t), \frac{1}{n})} \phi(z, h) + \epsilon B) \subset K_{g(t)} F(t, .)(y(t); w(t))$$

and the proof is done. □

Remark 3.4. If in Theorems 3.1-3.3 $\psi(t) \equiv t$ we obtain the results in [7] and if in Theorem 3.1-3.3 $\psi(t) \equiv t^{\sigma}$ we get the results in [8].

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