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Pseudo-rings: a non-commutative extension of rings

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Abstract: We introduce the notion of pseudo-ring as a non-commutative extension of the ring. We provide finite examples and some results.

Keywords: ring, pseudo-ring, 'm-transposition' principle

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Dedicated to the memory of my wonderful Professors Nicolae Dinculeanu and Solomon Marcus on the occasion of their 100th birthday

1 Introduction

There are many books on ring theory, but we need here only some well known definitions. Note that a ring is a set endowed with two binary operations, a sum and a product, usually denoted by + and $\cdot$, and verifying, among other proprieties, that + is commutative and $\cdot$ is not commutative. Thus, the ring is only 'partially' a non-commutative algebra (is a particular case of non-commutative algebra), in which only the product $\cdot$ is not commutative.

The notion of *pseudo-ring* is old in the literature. From Wikipedia, the free encyclopedia, "in mathematics, and more specifically in abstract algebra, a *pseudo-ring* is *one of the following variants of a ring*:
- A rng (Rng (algebra) or non-unital ring or pseudoring), i.e. a structure satisfying all the axioms of a ring except for the existence of a multiplicative identity. [1]
- A set R with two binary operations + and $\cdot$ such that $(R,+)$ is an abelian group with identity 0, and $a\cdot(b+c)+a\cdot 0=a\cdot b+a\cdot c$ and $(b+c)\cdot a+0\cdot a=b\cdot a+c\cdot a$ for all $a,b,c\in R$. [6]
- An abelian group $(A,+)$ equipped with a subgroup B and a multiplication B × A → A making B a ring and A a B-module. [7]

None of these definitions are equivalent, so it is best to avoid the term "pseudo-ring" or to clarify which meaning is intended."

But note that, since the above algebras are variants of a ring, then, from gramatically point of view, their name should be 'pseudo ring' (i.e. a particular case of ring , named ‘pseudo’) and not *pseudo-ring.*

Therefore, we shall freely use the name *pseudo-ring* in this paper with a different meaning, namely for a *non-commutative generalization of a ring* (i.e. the addition + is no more commutative), just as the *non-commutative generalization of an MV algebra* was named *pseudo-MV algebra* [2], [3], the *non-commutative generalization of a BCK algebra* was named *pseudo-BCK algebra* etc. (see [4]). Thus, the *pseudo-ring* will be a ‘full’ non-commutative algebra, in which both + and $\cdot$ are not commutative.

Also note that, in the monograph [4], many non-commutative algebras of logic are studied, including the pseudo-BCK algebras, and, in parallel, many analogous non-commutative algebras are introduced and studied, including the m-pseduo-BCK algebras (‘m’ coming from ‘magma’). There exist connections between these two parallel non-commutative frameworks, in the involutive case. We have discovered in [4] the ‘secret’ of these non-commutative algebras, the ‘principle’ that governs them, that we have called ‘transposition’ principle (‘m-transposition’ principle, for magmas) - name borrowed from matrix theory.

This paper, like our previous works, would not be written in so little time and with so many examples without the help of the wonderful instrument which is PROVER9-MACE4, developed by William W. McCune (1953-2011) [5]. By PROVER9, we try to find a proof and, by MACE4, we try to find an example or a list of examples. Note that PROVER9-MACE4 can simultaneously try to find a proof, by PROVER9, and try to find a counterexample, by MACE4 - if we start both PROVER9 and MACE4.

The paper is organized in three sections. In Section 2, we recall the definition of a ring with finite examples and the ‘m-transposition’ principle. In Section 3, we introduce the notion of *pseudo-ring*, as a non-commutative generalization of a ring, and we present many finite examples and some results.

2 Preliminaries

2.1 The ring. Definition and examples

Recall the definition of a ring, in order to fix the notations:

Definition 2.1. *(Initial definition)*

A ring is an algebra $\mathcal{A} = (A, +, \cdot, -, 0)$ *of type* $(2, 2, 1, 0)$ *such that:*
(1) $(A, +, -, 0)$ *is a commutative group,*
(2) $(A, \cdot)$ *is a semigroup,*
(3) the operation $\cdot$ *is distributive versus the oparation* $+$*,*
i.e. $\mathcal{A}$ *verifies the following axioms: for all* $x, y, z \in A$*,*

(SU)	$0 + x = x$	*(0 is the zero of sum),*
(Scomm)	$x + y = y + x$	*(commutativity of sum),*
(Sass)	$x + (y + z) = (x + y) + z$	*(associativity of sum),*
(Re)	$x + (-x) = 0,$	
(Pass)	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$	*(associativity of product),*
(Pdis)	$z \cdot (x + y) = (z \cdot x) + (z \cdot y),$	
	$(x + y) \cdot z = (x \cdot z) + (y \cdot z)$	*(distributivity).*

A ring is *unitary* if there exists an element $1 \in A$ such that:
(pPU) $1 \cdot x = x$, $x \cdot 1 = x$ (1 is the *unit* of product),
i.e. if $(A, \cdot, 1)$ is a monoid. A *unitary ring* is denoted $\mathcal{A} = (A, +, \cdot, -, 0, 1)$.

A ring is $\cdot$*-commutative*, or *commutative* for short, if for all x, y, (Pcomm) $(x \cdot y = y \cdot x)$ holds. Note that, in mathematics, a *noncommutative ring* is a ring whose multiplication is not commutative; that is, there exist a and b in the ring such that $a \cdot b$ and $b \cdot a$ are different. Equivalently, a noncommutative ring is a ring that is not a commutative ring.

Here are some examples of finite rings found by Mace4.

Example 2.2. Finite rings

- **Example 1: non-unitary ring**

The ring $(A_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}, +, \cdot, -, 0)$ *with the following tables of operations:*

$+$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	3	0	2	5	7	4	6
2	2	0	3	1	6	4	7	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	7	4	6	1	3	0	2
6	6	4	7	5	2	0	3	1
7	7	6	5	4	3	2	1	0

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	3	3	0	0	3	3	0
5	0	3	3	0	0	3	3	0
6	0	3	3	0	0	3	3	0
7	0	3	3	0	0	3	3	0

x	0	1	2	3	4	5	6	7
$-x$	0	2	1	3	4	6	5	7

Note $1 \cdot 4 \neq 4 \cdot 1$*, hence* $\cdot$ *is not commutative and* 1 *is not a unit for* $\cdot$*.*

- **Example 2: unitary ring**

The ring $(A_{16} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, +, \cdot, -, 0, 1)$ *with the* *following tables of operations:*

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	3	0	2	5	7	4	6	9	11	8	10	13	15	12	14
2	2	0	3	1	6	4	7	5	10	8	11	9	14	12	15	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	7	4	6	1	3	0	2	13	15	12	14	9	11	8	10
6	6	4	7	5	2	0	3	1	14	12	15	13	10	8	11	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8 ,
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	11	8	10	13	15	12	14	1	3	0	2	5	7	4	6
10	10	8	11	9	14	12	15	13	2	0	3	1	6	4	7	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	15	12	14	9	11	8	10	5	7	4	6	1	3	0	2
14	14	12	15	13	10	8	11	9	6	4	7	5	2	0	3	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

·	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	2	1	3	4	6	5	7	8	10	9	11	12	14	13	15
3	0	3	3	0	0	3	3	0	0	3	3	0	0	3	3	0
4	0	4	4	0	0	4	4	0	0	4	4	0	0	4	4	0
5	0	5	6	3	4	1	2	7	8	13	14	11	12	9	10	15
6	0	6	5	3	4	2	1	7	8	14	13	11	12	10	9	15
7	0	7	7	0	0	7	7	0	0	7	7	0	0	7	7	0 ,
8	0	8	8	0	3	11	11	3	0	8	8	0	3	11	11	3
9	0	9	10	3	7	14	13	4	8	1	2	11	15	6	5	12
10	0	10	9	3	7	13	14	4	8	2	1	11	15	5	6	12
11	0	11	11	0	3	8	8	3	0	11	11	0	3	8	8	3
12	0	12	12	0	3	15	15	3	0	12	12	0	3	15	15	3
13	0	13	14	3	7	10	9	4	8	5	6	11	15	2	1	12
14	0	14	13	3	7	9	10	4	8	6	5	11	15	1	2	12
15	0	15	15	0	3	12	12	3	0	15	15	0	3	12	12	3

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$-x$	0	2	1	3	4	6	5	7	8	10	9	11	12	14	13	15

Note $4 \cdot 8 \neq 8 \cdot 4$, *hence* $\cdot$ *is not commutative and* 1 *is a unit for* $\cdot$.

- **Example 3: commutative non-unitary ring**

The ring $(A_6 = \{0,1,2,3,4,5\}, +, \cdot, -, 0)$ *with the following tables of operations:*

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	0	4	5	3
2	2	0	1	5	3	4
3	3	4	5	0	1	2
4	4	5	3	1	2	0
5	5	3	4	2	0	1

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	2	1	0	2	1
2	0	1	2	0	1	2
3	0	0	0	0	0	0
4	0	2	1	0	2	1
5	0	1	2	0	1	2

x	0	1	2	3	4	5
$-x$	0	2	1	3	5	4

Note that $1 \cdot 2 = 1 \neq 2$, *hence (pPU) does not hold.*

- **Example 4: commutative unitary ring**

The ring $(A_7 = \{0,1,2,3,4,5,6\}, +, \cdot, -, 0, 1)$ *with the following tables of operations:*

$+$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	3	0	5	2	6	4
2	2	0	4	1	6	3	5
3	3	5	1	6	0	4	2
4	4	2	6	0	5	1	3
5	5	6	3	4	1	2	0
6	6	4	5	2	3	0	1

$\cdot$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	1	4	3	6	5
3	0	3	4	6	5	2	1
4	0	4	3	5	6	1	2
5	0	5	6	2	1	3	4
6	0	6	5	1	2	4	3

x	0	1	2	3	4	5	6
$-x$	0	2	1	4	3	6	5

Note that the examples generated (found) by Mace4 (following the program written by us) are sets of the form $\{0,1\}$, or $\{0,1,2\}$, or $\{0,1,2,3\}$ etc. (of size 2, 3, 4 etc.) together with tables of operations on these sets and other informations.

2.2 The 'm-transposition' principle

Consider an algebra $\mathcal{A} = (A, \cdot, ^-, ^\sim, 1)$ of type $(2, 1, 1, 0)$.

We say that $\mathcal{A}$ is:

- *commutative*, if for all $x, y \in A$, (Pcomm) $(x \cdot y = y \cdot x)$ holds,
- *strong*, if $x^- = x^\sim$, for all $x \in A$,
- *strong commutative*, if it is commutative and strong.

Recall the 'm-transposition' principle for non-commutative algebras, principle that was introduced in the monograph [4] (see page 297).

Definition 2.3. *(See [4])*

(1) The basic operations $\cdot$, $\cdot^T$ *(where* $x \cdot^T y \overset{def.}{=} y \cdot x$*),* $^-$, $^\sim$ *and* 1 *are called elementary operations. Any other operation o, defined by using the elementary operations, shall be called a complex operation*

(1') We shall say that the pairs $(\cdot, \cdot^T)$ *and* $(^-, ^\sim)$ *are pairs of 'm-transposed' elementary operations, or that the 'm-transpose' of the elementary operation* $\cdot$ *is* $\cdot^T$ *and that the 'm-transpose' of the elementary operation* $^-$ *is* $^\sim$*, and vice-versa. We shall say that* 1 *is a 'm-symmetric' elementary operation.*

(1") Let o and ω *be two complex operations. We shall say that the 'm-transpose' of o is* ω *and vice-versa if, by interchanging the corresponding 'm-transposed' elementary operations in o, we obtain* ω*, and vice-versa. A complex operation o whose 'm-transpose' equals o, will be called 'm-symmetric'.*

(1"') We shall say that a pair of complex operations ($o1$, $o2$*) is a pair of 'm-transposed' complex operations, if the 'm-transpose' of* $o1$ *is* $o2$*, and vice-versa.*

(2) Let (P) and (Q) be two properties. We shall say that the 'm-transpose' of the property (P) is (Q) (and we write $(P)^T$*=(Q)), and vice-versa (i.e.* $(Q)^T$*=(P)) if, by interchanging the 'm-transposed' elementary operations* $\cdot$ *and* $\cdot^T$ *and also* $^-$ *and* $^\sim$ *in (P), we obtain (Q), and vice-versa. A property (P) whose 'm-transpose' equals (P) will be called 'm-symmetric'.*

(2') We shall say that a pair of properties (P) = ((P1), (P2)) is a pair of 'm-transposed' properties, if the 'm-transpose' of the property (P1) is (P2), and vice-versa.

(2") Any property is either a 'm-symmetric' one or belongs to (is an element of) a pair of 'm-transposed' properties.

(3) Any binary relation is defined either by a 'm-symmetric' property or by a property belonging to a pair of 'm-transposed' properties.

(3') Let r_1 *be a binary relation defined by the property (P1) and* r_2 *be a binary*

relation defined by the property (P2).
(i) If ((P1), (P2)) is a pair of 'm-transposed' properties, then $(r_1,\ r_2)$ *is called to be a pair of 'm-transposed' binary relations and* $r \overset{def.}{=} r_1 \bigwedge r_2$ *is a 'm-symmetric' binary relation.*
(ii) If ((P1), (P2)) is a pair of 'm-transposed' properties and (P1) (P2) (hence, $r_1 \ \ r_2$*), then we write:* $r \ ^{def.}\ r_1 \ \ r_2$.

(4) The 'm-transposed' proof of the property (Q), or the 'm-transpose' of the proof of (Q), is that proof of $(Q)^T$ *obtained from the proof of (Q) by interchanging the corresponding 'm-transposed' operations (elementary or complex), 'm-transposed' properties and 'm-transposed' relations.*

Following these definitions, note that the given definition of the ring must be rewritten in order to comply with them, namely we must replace (SU), (Re), (Pdis) by (pSU), (pRe), (pPdis), respectively, since we have:
(SU) = (pSU) + (Scomm),
(Re) = (pRe) + (Scomm),
(Pdis) = (pPdis) + (Scomm),
where:
(pSU) $0 + x = x$, $x + 0 = x$,
(pRe) $x + (-x) = 0$, $(-x) + x = 0$,
(pPdis) $z \cdot (x + y) = (z \cdot x) + (z \cdot y)$, $(y + x) \cdot z = (y \cdot z) + (x \cdot z)$.

Thus, we obtain the following equivalent definition of rings, which complies with the 'm-transposition' principle:

Definition 2.4. *(Second, equivalent definition)*
A ring is an algebra $\mathcal{A} = (A, +, \cdot, -, 0)$ *of type* $(2, 2, 1, 0)$ *such that:*
(1) $(A, +, -, 0)$ *is a commutative group,*
(2) $(A, \cdot)$ *is a semigroup,*
(3) the operation $\cdot$ *is distributive versus the oparation* $+$*,*
i.e. $\mathcal{A}$ *verifies the following axioms: for all* $x, y, z \in A$*,*

(pSU)	$0 + x = x$, $x + 0 = x$,	
(Scomm)	$x + y = y + x$	*(commutativity of sum),*
(Sass)	$x + (y + z) = (x + y) + z$	*(associativity of sum),*
(pRe)	$x + (-x) = 0$, $(-x) + x = 0$,	
(Pass)	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$	*(associativity of product),*
(pPdis)	$z \cdot (x + y) = (z \cdot x) + (z \cdot y)$, $(y + x) \cdot z = (y \cdot z) + (x \cdot z)$	*(pseudo-distributivity).*

Note that now, all the properties of a ring (which is a strong algebra) are 'm-symmetric' properties or pairs of 'm-transposed' properties, namely:
- (Scomm), (Sass), (Pass) are 'm-symmetric' properties,
- (pSU), (pRe), (pPdis) are pairs of 'm-transposed' properties.

Note also [4] that, when proving a pair of 'm-transposed' properties ((P1), (P2)), it is sufficient to prove (P1) (or (P2)), because the proof of (P2) (or of (P1), respectively) will be the *'m-transposed' proof* of (P1) (or of (P2)). **But, in this paper, we shall prove both (P1) and (P2).**

Remark 2.5. [4] In the strong commutative case, the 'm-transposition' principle is no longer visible.

Remark 2.6. Not all non-commutative algebras obey to the 'm-transposition' principle. Take the following example:
A *near-ring* (see Günter Pilz, Near-Rings, North-Holland, 1983) is a set R equipped with a group structure $(R, +, -, 0)$, a monoid structure $(R, \cdot, 1)$ and such that for any $x, y, z \in R$, we have only the right-distributivity:
(pPdis2) $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$.

Note that the property (pPdis2) is neither a 'm-symmetric' property nor a pair of 'm-transposed' properties.

Near-rings are important generalizations of rings that appear naturally when studying functions on groups, like all functions from a group to itself.

3 The pseudo-ring

We introduce the following definition.

Definition 3.1.

A pseudo-ring is an algebra $\mathcal{A} = (A, +, \cdot, -, 0)$ *of type* $(2, 2, 1, 0)$ *such that:*
(1) $(A, +, -, 0)$ *is a group,*
(2) $(A, \cdot)$ *is a semigroup,*
(3) the operation $\cdot$ *is pseudo-distributive versus the oparation* $+$*,*
i.e. $\mathcal{A}$ *verifies the following axioms: for all* $x, y, z \in A$*,*

(pSU) $0 + x = x,\ x + 0 = x$ *(0 is the zero of sum),*
(Sass) $x + (y + z) = (x + y) + z$ *(associativity of sum),*
(pRe) $x + (-x) = 0,\ (-x) + x = 0,$
(Pass) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ *(associativity of product),*
(pPdis) $z \cdot (x + y) = (z \cdot x) + (z \cdot y),$
$(y + x) \cdot z = (y \cdot z) + (x \cdot z)$ *(pseudo-distributivity).*

Note that all the properties of a pseudo-ring are ‘m-symmetric’ properties or pairs of ‘m-transposed’ properties, namely:
- (Sass), (Pass) are ‘m-symmetric’ properties,
- (pSU), (pRe), (pPdis) are pairs of ‘m-transposed’ properties.

If the pseudo-ring is +-commutative, i.e. if the operation + is commutative (for all x, y, (Scomm) holds), then the pseudo-ring is a *ring* (see Definition 2.4).

We shall say that a pseudo-ring is *proper*, if it is not a ring.

A pseudo-ring is *unitary*, if there exists an element $1 \in A$ such that:
(pPU) $1 \cdot x = x$, $x \cdot 1 = x$ (1 is the *unit* of product),
i.e. if $(A, \cdot, 1)$ is a monoid. A *unitary pseudo-ring* is denoted $\mathcal{A} = (A, +, \cdot, -, 0, 1)$.

Note that (pPU) is a pair of ‘m-transposed’ properties.

We present some properties.

Proposition 3.2. *Let* $\mathcal{A} = (A, +, \cdot, -, 0)$ *be a pseudo-ring. We have:*
(pr1) $-0 = 0$,
(pr2) $x + ((-x) + y) = y$, $(y + (-x)) + x = y$,
(pr3) $(-x) + (x + y) = y$, $(y + x) + (-x) = y$,
(pr4) $-(-x) = x$,
(pr5) $0 \cdot x = 0$, $x \cdot 0 = 0$.

Proof. (pr1): In (pRe) $(x + (-x) = 0)$, take $x := 0$, to obtain: $0 + (-0) = 0$, hence $-0 = 0$, by (pSU).

(pr2): $x + ((-x) + y) \overset{(Sass)}{=} (x + (-x)) + y \overset{(pRe)}{=} 0 + y \overset{(pSU)}{=} y$ and
$(y + (-x)) + x \overset{(Sass)}{=} y + ((-x) + x) \overset{(pRe)}{=} y + 0 \overset{(pSU)}{=} y$.

(pr3): $-x + (x + y) \overset{(Sass)}{=} ((-x) + x) + y \overset{(pRe)}{=} 0 + y \overset{(pSU)}{=} y$ and
$(y + x) + (-x) \overset{(Sass)}{=} y + (x + (-x)) \overset{(pRe)}{=} y + 0 \overset{(pSU)}{=} y$.

(pr4): In (pr2) $(x + ((-x) + y) = y)$, take $y := -(-x)$, to obtain: $x + ((-x) + (-(-x))) = -(-x)$, which becomes $x + 0 = -(-x)$, by (pRe), hence $x = -(-x)$, by (pSU).

(pr5): Firstly, we prove: $0 \cdot x = 0$. Indeed,
in (pPdis) $((x + y) \cdot z = (x \cdot z) + (y \cdot z))$, take $y := 0$, to obtain:
$(x + 0) \cdot z = (x \cdot z) + (0 \cdot z)$,
which, by (pSU), becomes:
(a) $x \cdot z = (x \cdot z) + (0 \cdot z)$;
then, in (pr3) $((-x) + (x + y) = y)$, take $X := x \cdot y$, $Y := 0 \cdot y$, to obtain:
$(-(x \cdot y)) + ((x \cdot y) + (0 \cdot y)) = 0 \cdot y$,
which, by (a), becomes:
$(-(x \cdot y)) + (x \cdot y) = 0 \cdot y$,
which, by (pRe), becomes: $0 = 0 \cdot y$.

Now, we prove (but it is not compulsory, by the 'm-transposition' principle): $x \cdot 0 = x$. Indeed,
in (pPdis) ($z \cdot (y + x) = (z \cdot y) + (z \cdot x)$), take $y := 0$, to obtain:
$z \cdot (0 + x) = (z \cdot 0) + (z \cdot x)$,
which, by (pSU), becomes:
(a') $z \cdot x = (z \cdot 0) + (z \cdot x)$;
then, in (pr3) ($(y + x) + (-x) = y$), take $X := y \cdot x$, $Y := y \cdot 0$, to obtain:
$((y \cdot 0) + (y \cdot x)) + (-(y \cdot x)) = y \cdot 0$,
which, by (a'), becomes:
$(y \cdot x) + (-(y \cdot x)) = y \cdot 0$,
which, by (pRe), becomes: $0 = y \cdot 0$. □

Proposition 3.3. *Let $\mathcal{A} = (A, +, \cdot, -, 0, 1)$ be a unitary pseudo-ring. Then, (Scomm) holds, i.e. $\mathcal{A}$ is a unitary ring.*

Proof. (Based on a proof by PROVER9, Length of proof was 26, in 0.01 seconds).

Firstly, we prove:

$$(x \cdot (-y)) + (x \cdot y) = 0. \tag{3.1}$$

Indeed, in (pPdis) ($z \cdot (x + y) = (z \cdot x) + (z \cdot y)$), take $X := -y$, to obtain:
$z \cdot ((-y) + y) = (z \cdot (-y)) + (z \cdot y)$,
which, by (pRe) on left side, becomes:
$z \cdot 0 = (z \cdot (-y)) + (z \cdot y)$,
which, by (pr5), becomes:
$0 = (z \cdot (-y)) + (z \cdot y)$, i.e. (3.1) holds.

Now, we prove:

$$(y \cdot (-1)) + ((z \cdot (-1)) + [y + z]) = 0. \tag{3.2}$$

Indeed, in (3.1), take $y := 1$, to obtain:
$(x \cdot (-1)) + (x \cdot 1) = 0$,
which, by (pPU), becomes:
(a) $(x \cdot (-1)) + x = 0$;
then, in (Sass) ($(x + y) + z = x + (y + z)$), take $X := (y + z) \cdot (-1)$, to obtain:
$([(y + z) \cdot (-1)] + y) + z = [(y + z) \cdot (-1)] + (y + z)$,
which, by (a), becomes:
$([(y + z) \cdot (-1)] + y) + z = 0$,
which, by (pPdis), becomes:
$([(y \cdot (-1)) + (z \cdot (-1))] + y) + z = 0$,
which, by (Sass), becomes:
$((y \cdot (-1)) + [(z \cdot (-1)) + y]) + z = 0$,

which, by (Sass) again, becomes:
$(y \cdot (-1)) + ([(z \cdot (-1)) + y] + z) = 0$,
which, by (Sass) again, becomes:
$(y \cdot (-1)) + ((z \cdot (-1)) + [y + z]) = 0$, i.e. (3.2) holds.

Now, we prove:

$$x \cdot (-y) = -(x \cdot y). \tag{3.3}$$

Indeed, in (pr3) ($(-x) + (x + y) = y$), take $X := x \cdot (-y)$, $Y := x \cdot y$, to obtain:
$(-(x \cdot (-y))) + ((x \cdot (-y)) + (x \cdot y)) = x \cdot y$,
which, by (3.1) , becomes:
$(-(x \cdot (-y))) + 0 = x \cdot y$,
which, by (pSU), becomes:
$-(x \cdot (-y)) = x \cdot y$,
which implies:
$-(-(x \cdot (-y))) = -(x \cdot y)$,
which, by (pr4), becomes:
$x \cdot (-y) = -(x \cdot y)$, i.e. (3.3) holds.

Now, we prove:

$$(-y) + ([-z] + [y + z]) = 0. \tag{3.4}$$

Indeed, from (3.2), by (3.3), we obtain:
$(-(y \cdot 1)) + ([-(z \cdot 1)] + [y + z]) = 0$,
which, by (pPU) twice, becomes:
$(-y) + ([-z] + [y + z]) = 0$, i.e. (3.4) holds.

Now, we prove:

$$(-y) + (x + y) = x. \tag{3.5}$$

Indeed, in (pr2) ($x + ((-x) + y) = y$), take $Y := (-y) + (x + y)$, to obtain:
$x + ((-x) + [(-y) + (x + y)]) = (-y) + (x + y)$,
which, by (3.4), becomes:
$x + 0 = (-y) + (x + y)$,
which, by (pSU), becomes:
$x = (-y) + (x + y)$, i.e. (3.5) holds.

Finally, in (pr2) ($x + ((-x) + y) = y$), take $Y := y + x$, to obtain:
$x + ((-x) + (y + x)) = y + x$,
which, by (3.5), becomes:
$x + y = y + x$, i.e. (Scomm) holds. □

We present now some examples of finite proper pseudo-rings.

Example 3.4. Finite proper pseudo-rings

• **Example 1: the first proper pseudo-ring (of size 8) from a list of 10000 models (examples) found by Mace4**
The pseudo-ring ($A_8 = \{0,1,2,3,4,5,6,7\}, +, \cdot, -, 0$) *with the following tables of operations:*

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	4	0	6	1	7	3	5
3	3	5	1	7	0	6	2	4
4	4	2	6	0	7	1	5	3
5	5	3	7	1	6	0	4	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

,

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	7	0	7	7	0	7	0
3	0	7	0	7	7	0	7	0
4	0	7	0	7	7	0	7	0
5	0	7	0	7	7	0	7	0
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0

,

x	0	1	2	3	4	5	6	7
$-x$	0	1	2	4	3	5	6	7

.

• **Example 2.** *(nr. 768, the last of size 8 from 10000 models)*
The pseudo-ring ($A_8 = \{0,1,2,3,4,5,6,7\}, +, \cdot, -, 0$) *with the following tables of operations:*

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	3	0	2	6	7	5	4
2	2	0	3	1	7	6	4	5
3	3	2	1	0	5	4	7	6
4	4	7	6	5	3	0	1	2
5	5	6	7	4	0	3	2	1
6	6	4	5	7	2	1	3	0
7	7	5	4	6	1	2	0	3

,

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	3	3	0	3	3	0	0
2	0	3	3	0	3	3	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	3	3	3	3
5	0	0	0	0	3	3	3	3
6	0	3	3	0	0	0	3	3
7	0	3	3	0	0	0	3	3

,

x	0	1	2	3	4	5	6	7
$-x$	0	2	1	3	5	4	7	6

.

• **Example 3.** *(nr.769, the first of size 12 from 10000 models)*
The pseudo-ring ($A_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}, +, \cdot, -, 0$) *with the following tables of operations:*

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	3	2	5	4	7	6	9	8	11	10
2	2	4	0	5	1	3	8	10	6	11	7	9
3	3	5	1	4	0	2	9	11	7	10	6	8
4	4	2	5	0	3	1	10	8	11	6	9	7
5	5	3	4	1	2	0	11	9	10	7	8	6
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	6	9	8	11	10	1	0	3	2	5	4
8	8	10	6	11	7	9	2	4	0	5	1	3
9	9	11	7	10	6	8	3	5	1	4	0	2
10	10	8	11	6	9	7	4	2	5	0	3	1
11	11	9	10	7	8	6	5	3	4	1	2	0

,

·	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0
6	0	1	1	0	0	1	6	7	7	6	6	7
7	0	1	1	0	0	1	6	7	7	6	6	7
8	0	1	1	0	0	1	6	7	7	6	6	7
9	0	1	1	0	0	1	6	7	7	6	6	7
10	0	1	1	0	0	1	6	7	7	6	6	7
11	0	1	1	0	0	1	6	7	7	6	6	7

,

x	0	1	2	3	4	5	6	7	8	9	10	11
$-x$	0	1	2	4	3	5	6	7	8	10	9	11

.

- **Example 4.** *(nr. 6600, the last of size 12 from 10000 models)*

The pseudo-ring $(A_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}, +, \cdot, -, 0)$ *with the following tables of operations:*

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	3	0	5	2	4	7	9	6	11	8	10
2	2	0	4	1	5	3	8	6	10	7	11	9
3	3	5	1	4	0	2	9	11	7	10	6	8
4	4	2	5	0	3	1	10	8	11	6	9	7
5	5	4	3	2	1	0	11	10	9	8	7	6
6	6	8	7	10	9	11	0	2	1	4	3	5
7	7	6	9	8	11	10	1	0	3	2	5	4
8	8	10	6	11	7	9	2	4	0	5	1	3
9	9	7	11	6	10	8	3	1	5	0	4	2
10	10	11	8	9	6	7	4	5	2	3	0	1
11	11	9	10	7	8	6	5	3	4	1	2	0

,

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	5	5	0	0	5	10	7	7	10	10	7
2	0	5	5	0	0	5	10	7	7	10	10	7
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	5	5	0	0	5	10	7	7	10	10	7
6	0	5	5	0	0	5	10	7	7	10	10	7
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	5	5	0	0	5	10	7	7	10	10	7
10	0	5	5	0	0	5	10	7	7	10	10	7
11	0	0	0	0	0	0	0	0	0	0	0	0

,

x	0	1	2	3	4	5	6	7	8	9	10	11
$-x$	0	2	1	4	3	5	6	7	8	9	10	11

.

• **Example 5.** *(nr. 6601, the first of size 16, in 257 seconds, from 10000 models)*
The pseudo-ring $(A_{16} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}, +, \cdot, -, 0)$ *with the following tables of operations:*

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	4	0	6	1	7	3	5	10	12	8	14	9	15	11	13
3	3	5	1	7	0	6	2	4	11	13	9	15	8	14	10	12
4	4	2	6	0	7	1	5	3	12	10	14	8	15	9	13	11
5	5	3	7	1	6	0	4	2	13	11	15	9	14	8	12	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	12	8	14	9	15	11	13	2	4	0	6	1	7	3	5
11	11	13	9	15	8	14	10	12	3	5	1	7	0	6	2	4
12	12	10	14	8	15	9	13	11	4	2	6	0	7	1	5	3
13	13	11	15	9	14	8	12	10	5	3	7	1	6	0	4	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

,

·	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	8	9	9	8	1	0	8	9	0	1	1	0	9	8
3	0	1	8	9	9	8	1	0	8	9	0	1	1	0	9	8
4	0	1	8	9	9	8	1	0	8	9	0	1	1	0	9	8
5	0	1	8	9	9	8	1	0	8	9	0	1	1	0	9	8
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	1	8	9	9	8	1	0	8	9	0	1	1	0	9	8
9	0	1	8	9	9	8	1	0	8	9	0	1	1	0	9	8
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	1	8	9	9	8	1	0	8	9	0	1	1	0	9	8
15	0	1	8	9	9	8	1	0	8	9	0	1	1	0	9	8

,

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$-x$	0	1	2	4	3	5	6	7	8	9	10	12	11	13	14	15

.

- **Example 6.** *(nr. 10000, of size 16, in 270 seconds, from 10000 models)*

The pseudo-ring $(A_{16} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, +, \cdot, -, 0)$ *with the following tables of operations:*

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	10	11	8	9	13	12	15	14
2	2	4	0	6	1	7	3	5	12	15	14	13	8	11	10	9
3	3	5	1	7	0	6	2	4	13	14	15	12	10	9	8	11
4	4	2	6	0	7	1	5	3	14	13	12	15	11	8	9	10
5	5	3	7	1	6	0	4	2	15	12	13	14	9	10	11	8
6	6	7	4	5	2	3	0	1	11	10	9	8	14	15	12	13
7	7	6	5	4	3	2	1	0	9	8	11	10	15	14	13	12
8	8	10	15	14	13	12	11	9	7	0	6	1	2	3	4	5
9	9	11	12	13	14	15	10	8	0	7	1	6	5	4	3	2
10	10	8	14	15	12	13	9	11	6	1	7	0	3	2	5	4
11	11	9	13	12	15	14	8	10	1	6	0	7	4	5	2	3
12	12	14	9	10	11	8	13	15	5	2	3	4	0	6	1	7
13	13	15	11	8	9	10	12	14	4	3	2	5	1	7	0	6
14	14	12	10	9	8	11	15	13	3	4	5	2	6	0	7	1
15	15	13	8	11	10	9	14	12	2	5	4	3	7	1	6	0

,

·	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	7	7	0	0	7	7	0	0	0	7	7	7	0	0	7
3	0	7	7	0	0	7	7	0	0	0	7	7	7	0	0	7
4	0	7	7	0	0	7	7	0	0	0	7	7	7	0	0	7
5	0	7	7	0	0	7	7	0	0	0	7	7	7	0	0	7
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	7	0	7	7	0	7	0	7	7	0	0	7	0	0	7
9	0	7	0	7	7	0	7	0	7	7	0	0	7	0	0	7
10	0	7	0	7	7	0	7	0	7	7	0	0	7	0	0	7
11	0	7	0	7	7	0	7	0	7	7	0	0	7	0	0	7
12	0	0	7	7	7	7	0	0	7	7	7	7	0	0	0	0
13	0	0	7	7	7	7	0	0	7	7	7	7	0	0	0	0
14	0	0	7	7	7	7	0	0	7	7	7	7	0	0	0	0
15	0	0	7	7	7	7	0	0	7	7	7	7	0	0	0	0

,

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$-x$	0	1	2	4	3	5	6	7	9	8	11	10	12	14	13	15

.

*

If the operation $\cdot$ is commutative, i.e. for all x, y, (Pcomm) $(x \cdot y = y \cdot x)$ holds, then we say that the pseudo-ring is $\cdot$ *-commutative.*

We present now some examples of finite $\cdot$-commutative proper pseudo-rings.

Example 3.5. Finite $\cdot$-commutative proper pseudo-rings

- **Example 1.**

The pseudo-ring $(A_6 = \{0, 1, 2, 3, 4, 5\}, +, \cdot, -, 0)$ *with the following tables of operations:*

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	0	4	5	3
2	2	0	1	5	3	4
3	3	5	4	0	2	1
4	4	3	5	1	0	2
5	5	4	3	2	1	0

,

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	5	5	5
4	0	0	0	5	5	5
5	0	0	0	5	5	5

,

x	0	1	2	3	4	5
$-x$	0	2	1	3	4	5

.

- **Example 2.**

The pseudo-ring $(A_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}, +, \cdot, -, 0)$ *with the following tables of operations:*

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	3	0	2	5	7	4	6
2	2	0	3	1	6	4	7	5
3	3	2	1	0	7	6	5	4
4	4	6	5	7	0	2	1	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	5	6	4	3	1	2	0

,

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	3	3	0	0	3	3	0
2	0	3	3	0	0	3	3	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	4	4	4	4
5	0	3	3	0	4	7	7	4
6	0	3	3	0	4	7	7	4
7	0	0	0	0	4	4	4	4

,

x	0	1	2	3	4	5	6	7
$-x$	0	2	1	3	4	5	6	7

.

• **Example 3.**
The pseudo-ring $(A_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}, +, \cdot, -, 0)$ *with the following tables of operations:*

$+$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	4	5	2	3	7	6	10	11	8	9
2	2	5	3	0	1	4	8	11	9	6	7	10
3	3	4	0	2	5	1	9	10	6	8	11	7
4	4	3	5	1	0	2	10	9	11	7	6	8
5	5	2	1	4	3	0	11	8	7	10	9	6
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	6	10	11	8	9	1	0	4	5	2	3
8	8	11	9	6	7	10	2	5	3	0	1	4
9	9	10	6	8	11	7	3	4	0	2	5	1
10	10	9	11	7	6	8	4	3	5	1	0	2
11	11	8	7	10	9	6	5	2	1	4	3	0

,

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	5	5	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	5	5	5	5	5	5
5	0	0	0	0	0	0	5	5	5	5	5	5
6	0	5	0	0	5	5	11	6	11	11	6	6
7	0	5	0	0	5	5	6	11	6	6	11	11
8	0	5	0	0	5	5	11	6	11	11	6	6
9	0	5	0	0	5	5	11	6	11	11	6	6
10	0	5	0	0	5	5	6	11	6	6	11	11
11	0	5	0	0	5	5	6	11	6	6	11	11

,

x	0	1	2	3	4	5	6	7	8	9	10	11
$-x$	0	1	3	2	4	5	6	7	9	8	10	11

.

We present now some special examples of finite $\cdot$-commutative proper pseudo-rings, found by MACE4 with the following program:

0 + x=x.
x + 0=x.
x + (y + z) = (x + y) + z.
x + (-x)=0.
-x + x=0.
x * (y * z)=(x * y) * z.
z * (x + y) = (z * x) + (z * y).
(y + x) * z=(y * z) + (x * z).
x * y = y * x.
a + b != b + a.

Example 3.6. Special finite $\cdot$-commutative proper pseudo-rings

- **Example 1.** *(the first example (of size 6) from a list of 1500 models (examples) found by* Mace4*)*

The pseudo-ring $\mathcal{A}_6 = (A_6 = \{0,1,2,3,4,5\}, +, \cdot, -, 0)$ *with the following tables of operations:*

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	5	1	4	0	2
4	4	2	5	0	3	1
5	5	3	4	1	2	0

,

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0

,

x	0	1	2	3	4	5
$-x$	0	1	2	4	3	5

.

- **Example 2.** *(nr. 13, the first of size 8, from 1500 models)*

The pseudo-ring $\mathcal{A}_8 = (A_8 = \{0,1,2,3,4,5,6,7\}, +, \cdot, -, 0)$ *with the following tables of operations:*

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	4	0	6	1	7	3	5
3	3	5	1	7	0	6	2	4
4	4	2	6	0	7	1	5	3
5	5	3	7	1	6	0	4	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

,

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0

,

x	0	1	2	3	4	5	6	7
$-x$	0	1	2	4	3	5	6	7

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• **Example 3.** *(nr. 117, the first of size 10, from 1500 models)*
The pseudo-ring $\mathcal{A}_{10} = (A_{10} = \{0,1,2,3,4,5,6,7,8,9\}, +, \cdot, -, 0)$ *with the following tables of operations:*

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	0	3	2	5	4	7	6	9	8
2	2	4	0	6	1	8	3	9	5	7
3	3	5	1	7	0	9	2	8	4	6
4	4	2	6	0	8	1	9	3	7	5
5	5	3	7	1	9	0	8	2	6	4
6	6	8	4	9	2	7	0	5	1	3
7	7	9	5	8	3	6	1	4	0	2
8	8	6	9	4	7	2	5	0	3	1
9	9	7	8	5	6	3	4	1	2	0

,

·	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0

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x	0	1	2	3	4	5	6	7	8	9
$-x$	0	1	2	4	3	5	6	8	7	9

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• **Example 4.** *(nr. 135, the first of size 12, from 1500 models)*
The pseudo-ring $\mathcal{A}_{12} = (A_{12} = \{0,1,2,3,4,5,6,7,8,9,10,11\}, +, \cdot, -, 0)$ *with the following tables of operations:*

+	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	3	2	5	4	7	6	9	8	11	10
2	2	4	0	5	1	3	8	10	6	11	7	9
3	3	5	1	4	0	2	9	11	7	10	6	8
4	4	2	5	0	3	1	10	8	11	6	9	7
5	5	3	4	1	2	0	11	9	10	7	8	6
6	6	7	8	9	10	11	0	1	2	3	4	5
7	7	6	9	8	11	10	1	0	3	2	5	4
8	8	10	6	11	7	9	2	4	0	5	1	3
9	9	11	7	10	6	8	3	5	1	4	0	2
10	10	8	11	6	9	7	4	2	5	0	3	1
11	11	9	10	7	8	6	5	3	4	1	2	0

,

·	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0

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x	0	1	2	3	4	5	6	7	8	9	10	11
$-x$	0	1	2	4	3	5	6	7	8	10	9	11

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• **Example 5.** *(nr. 1137, the first of size 14, from 1500 models)*
The pseudo-ring $\mathcal{A}_{14} = (A_{14} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}, +, \cdot, -, 0)$ *with the following tables of operations:*

$+$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12
2	2	4	0	6	1	8	3	10	5	12	7	13	9	11
3	3	5	1	7	0	9	2	11	4	13	6	12	8	10
4	4	2	6	0	8	1	10	3	12	5	13	7	11	9
5	5	3	7	1	9	0	11	2	13	4	12	6	10	8
6	6	8	4	10	2	12	0	13	1	11	3	9	5	7 ,
7	7	9	5	11	3	13	1	12	0	10	2	8	4	6
8	8	6	10	4	12	2	13	0	11	1	9	3	7	5
9	9	7	11	5	13	3	12	1	10	0	8	2	6	4
10	10	12	8	13	6	11	4	9	2	7	0	5	1	3
11	11	13	9	12	7	10	5	8	3	6	1	4	0	2
12	12	10	13	8	11	6	9	4	7	2	5	0	3	1
13	13	11	12	9	10	7	8	5	6	3	4	1	2	0

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0 ,
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$-x$	0	1	2	4	3	5	6	8	7	9	10	12	11	13 .

• **Example 6.** *(nr. 1161, the first of size 16, from 1500 models)*
The pseudo-ring $\mathcal{A}_{16} = (A_{16} = \{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\}, +, \cdot, -, 0)$ *with the following tables of operations:*

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	4	0	6	1	7	3	5	10	12	8	14	9	15	11	13
3	3	5	1	7	0	6	2	4	11	13	9	15	8	14	10	12
4	4	2	6	0	7	1	5	3	12	10	14	8	15	9	13	11
5	5	3	7	1	6	0	4	2	13	11	15	9	14	8	12	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8 ,
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	12	8	14	9	15	11	13	2	4	0	6	1	7	3	5
11	11	13	9	15	8	14	10	12	3	5	1	7	0	6	2	4
12	12	10	14	8	15	9	13	11	4	2	6	0	7	1	5	3
13	13	11	15	9	14	8	12	10	5	3	7	1	6	0	4	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

·	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0 ,
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$-x$	0	1	2	4	3	5	6	7	8	9	10	12	11	13	14	15 .

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