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Convergence Criteria for Operator Equations

Mircea Sofonea

Abstract: In this paper we deal with two nonlinear equations in real Hilbert spaces. The first one is of the form $Au = f$ in which A is a strongly monotone Lipschitz continuous operator and the second one is of the form $Au + \mathcal{S}u = f$ in which $\mathcal{S}$ is a history-dependent operator. The unique solvability of these equations represents well known results. Here, our interest is in providing necessary and sufficient conditions which guarantee the convergence of an arbitrary sequence to the solution. Our main results are gathered in Theorems 3.1, 3.2, 5.1 and 5.2. They represents useful tools which allows us to deduce continuous dependence results of the solution with respect to the data. They also can be employed to prove that the solution of these equations represents the limit of the solution of some elliptic and history-dependent variational inequalities, respectively. We illustrate our abstract results with examples from Solid and Contact Mechanics and provide the corresponding mechanical interpretations.

Keywords: Strongly monotone operator, history-dependent operator, convergence criterion, convergence result, elastic constitutive law, viscoelastic constitutive law, frictional contact model.

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1 Introduction

Equations involving nonlinear operators abound in Functional Analysis, Solid Mechanics and Engineering Sciences. A first elementary example is provided by the constitutive law of an elastic material, that is

$$\boldsymbol{\sigma} = \mathcal{F}\boldsymbol{\varepsilon}. \tag{1.1}$$

Here and below $d \in \{1, 2, 3\}$, $\mathbb{S}^d$ denotes the set of second order tensors on $\mathbb{R}^d$, $\boldsymbol{\sigma} \in \mathbb{S}^d$ represents the stress tensor, $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ denotes the linearized strain tensor and $\mathcal{F} : \mathbb{S}^d \to \mathbb{S}^d$ is the elasticity constitutive operator, assumed to be nonlinear. A particular example of elastic constitutive law of the form (1.1) is provided by

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon} + \alpha\,(\boldsymbol{\varepsilon} - P_K\boldsymbol{\varepsilon}), \tag{1.2}$$

where $\mathcal{A}$ is a linear or nonlinear operator, α is a positive elasticity coefficient, K is a given nonempty closed convex subset of $\mathbb{S}^d$ and $P_K : \mathbb{S}^d \to K$ denotes the projection operator. Various examples of such convex sets can be found in [7, 22] and the references therein. Moreover, static displacement-tractions problems for elastic materials of the form (1.1) or (1.2) lead to operator equations of the form

$$A\boldsymbol{u} = \boldsymbol{f}. \tag{1.3}$$

Here $\boldsymbol{u} \in V$ represents the displacement field, $A : V \to V$ is a nonlinear operator and $\boldsymbol{f} \in V$ is a given element describing the applied body forces and surface tractions, V being the space of admissible displacements fields. References in the field, including existence and uniqueness results, can be found in [12, 17, 20, 21], for instance.

Besides the elastic constitutive laws, a popular constitutive law used in the literature is the so-called viscoelastic constitutive law with long memory,

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(t) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(s)\,ds \qquad \forall\, t \in [0,T]. \tag{1.4}$$

Here $\mathcal{A}$ is the elasticity operator, $\mathcal{B}$ represents the relaxation tensor and $[0,T]$ denotes the time interval of interest Examples and mechanical interpretations in the study of viscoelastic materials of the form (1.4) can be found in [3, 6, 7, 8, 18], for instance. Using this constitutive law in the study of equilibrium displacement-tractions problems give rise to time-dependent equations of the form

$$A\boldsymbol{u}(t) + \mathcal{S}\boldsymbol{u}(t) = \boldsymbol{f}(t) \qquad \forall\, t \in [0,T], \tag{1.5}$$

in which the operator $\mathcal{S}$ is determined by the relaxation tensor $\mathcal{B}$. For more details on this topic we send the reader to [20, 21].

Equations (1.1)–(1.3), on one hand, as well as equations (1.4)–(1.5), on the other hand, motivate us to consider the following nonlinear problems.

Problem $\mathcal{P}$. *Given $A : X \to X$ and $f \in X$, find $u \in X$ such that*

$$Au = f. \tag{1.6}$$

Problem $\mathcal{Q}$. *Given $A : X \to X$, $\mathcal{S} \in C([0,T];X) \to C([0,T];X)$ and $f : [0,T] \to X$, find $u : [0,T] \to X$ such that*

$$Au(t) + \mathcal{S}u(t) = f(t) \qquad \forall\, t \in [0,T]. \tag{1.7}$$

Here and everywhere below X will represent a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$, $T>0$ and $C([0,T];X)$ denotes the space of continuous functions defined on the time interval $[0,T]$ with values in X, endowed with the norm of the uniform convergence, that is

$$\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X \qquad \forall\, u \in C([0,T];X). \tag{1.8}$$

Results of existence and uniqueness in the study of Problems $\mathcal{P}$ and $\mathcal{Q}$ have been obtained in the literature by using various functional methods, including fixed point arguments. We shall recall such results in the next section of the manuscript.

Besides the unique solvability, convergence results represent an important topic in the study of nonlinear problems. References in the field include [1, 2, 3, 4, 5, 10, 15, 16, 19], where convergences results are proved for various classes of equations, inequalities, minimization and fixed point problems. Nevertheless, in most of these references, only sufficient conditions which guarantee the corresponding convergence results have been considered. The problem of establishing *convergence criteria*, i.e., necessary and sufficient conditions for convergence, is an important topic which, at the best of our knowledge, is widely open.

Our aim in this paper is to fill this gap and is three fold. The first one is to state and prove convergence criteria in the study of nonlinear equations (1.6) and (1.7). The second one is to show how these criteria can be used to prove the continuous dependence of the solution with respect to the data. And, finally, our third aim is to illustrate the use of these abstract results in Solid and Contact Mechanics.

The rest of the manuscript is structured as follows. In Section 2 we recall existence and uniqueness results in the study of Problems $\mathcal{P}$ and $\mathcal{Q}$, then we provide some preliminary material. In Section 3 we state and prove convergence criteria for the nonlinear equation (1.6). These criteria are formulated in terms of various equivalent inequalities. Section 4 is devoted to applications of these abstract results in the study of the elastic equations (1.1)–(1.3). In Section 5 we state and prove convergence criteria for the nonlinear equation (1.7) which, again, are expressed in terms of inequalities. We apply these results in Section 6, in the study of the viscoelastic equations (1.4) and (1.5). We end this paper with Section 7 in which we present some concluding remarks and problems for furthcoming research.

2 Preliminary results

In the study of Problems $\mathcal{P}$ and $\mathcal{Q}$ we consider the following assumptions.

$$\begin{cases} A \text{ is a Lipschitz continuous strongly monotone operator, i.e.:} \\ \text{(a) There exists } M_A > 0 \text{ such that} \\ \qquad \|Au - Av\|_X \le M_A \|u - v\|_X \quad \forall\, u,\, v \in X. \\ \text{(b) There exists } m_A > 0 \text{ such that} \\ \qquad (Au - Av, u - v)_X \ge m_A \|u - v\|_X^2 \quad \forall\, u,\, v \in X. \end{cases} \tag{2.1}$$

$$\begin{cases} \mathcal{S} \text{ is a history-dependent operator, i.e.,} \\ \text{there exists } L_{\mathcal{S}} > 0 \text{ such that} \\ \quad \|\mathcal{S}u(t) - \mathcal{S}v(t)\|_X \le L_{\mathcal{S}} \displaystyle\int_0^t \|u(s) - v(s)\|_X ds \\ \qquad \forall\, u,\ v \in C([0,T];X),\ t \in [0,T]. \end{cases} \tag{2.2}$$

$$f \in C([0,T];X). \tag{2.3}$$

$$f \in X. \tag{2.4}$$

We recall the following existence and uniqueness results.

Theorem 2.1. *Assume* (2.1) *and* (2.4). *Then, Problem* $\mathcal{P}$ *has a unique solution* $u \in X$.

Theorem 2.2. *Assume* (2.1), (2.2) *and* (2.3). *Then, Problem* $\mathcal{Q}$ *has a unique solution. Moreover, the solution has the regularity* $u \in C([0,T];X)$.

The proof of Theorem 2.1 can be found in [20, p.22], based on the Banach fixed point argument. The proof of Theorem 2.2 can be found in [20, p.65], based on a fixed point property for history-dependent operators.

Remark 2.3. It follows from Theorem 2.1 that, under condition (2.1), the operator A is invertible. Moreover, as proved in [20, p.23], its inverse $A^{-1}\colon X \to X$ is a Lipschitz continuous operator, with Lipschitz constant $\frac{1}{m_A}$. Therefore,

$$\|A^{-1}u - A^{-1}v\|_X \le \frac{1}{m_A} \|u - v\|_X \qquad \forall\, u,\, v \in X. \tag{2.5}$$

We now proceed with the following elementary result which will be useful in Sections 3 and 5 of this manuscript.

Lemma 2.4. *Let $u \in X$ and $\theta > 0$. Then, the following hold:*

$$(u,v)_X + \theta\|v\|_X \geq 0 \quad \forall\, v \in X \quad \Longleftrightarrow \quad \|u\|_X \leq \theta. \tag{2.6}$$

$$(u,v)_X + \theta(\|v\|_X + 1) \geq 0 \quad \forall\, v \in X \quad \Longrightarrow \quad \|u\|_X \leq \theta + \sqrt{\theta}. \tag{2.7}$$

Proof. Assume that $(u,v)_X + \theta\|v\|_X \geq 0$ for any $v \in X$. We take $v = -u$ in this inequality to deduce that $\|u\|_X^2 \leq \theta\|u\|_X$, which implies that $\|u\|_X \leq \theta$. Conversely, if $\|u\|_X \leq \theta$ then $(u,v)_X \geq -\|u\|_X\|v\|_X \geq -\theta\|v\|_X$ for any $v \in X$, which implies that $(u,v)_X + \theta\|v\|_X \geq 0$, for any $v \in X$. This concludes the proof of the equivalence (2.6).

Assume now that $(u,v)_X + \theta(\|v\|_X + 1) \geq 0$ for any $v \in X$. We take $v = -u$ in this inequality to see that $\|u\|_X^2 \leq \theta\|u\|_X + \theta$ and, using the elementary inequality

$$x^2 \leq ax + b \quad \Longrightarrow \quad x \leq a + \sqrt{b} \quad \forall\, x,\, a,\, b \geq 0,$$

we deduce that $\|u\|_X \leq \theta + \sqrt{\theta}$. This concludes the proof of the implication (2.7). □

In the next sections we shall underline the link of problems $\mathcal{P}$ and $\mathcal{Q}$ with some variational inequalities. To this end, we consider a normed space $(Y, \|\cdot\|_Y)$ and two functions $j : X \to \mathbb{R}$, $\varphi : Y \times X \to \mathbb{R}$, assumed to satisfy the following conditions.

$$j : X \to \mathbb{R} \text{ is convex and lower semicontinuous.} \tag{2.8}$$

$$\begin{cases} \text{(a) } \varphi(y,\cdot)\colon X \to \mathbb{R} \text{ is convex and lower semicontinuous on } X, \\ \qquad \forall\, y \in Y. \\ \text{(b) There exists } \alpha_\varphi \geq 0 \text{ such that} \\ \qquad \varphi(y_1, v_2) - \varphi(y_1, v_1) + \varphi(y_1, v_1) - \varphi(y_2, v_2) \\ \qquad\quad \leq \alpha_\varphi \|y_1 - y_2\|_Y\, \|v_1 - v_2\|_X \quad \forall\, y_1,\, y_2 \in Y,\ v_1,\, v_2 \in X. \end{cases} \tag{2.9}$$

We also denote by $C([0,T];Y)$ the space of continuous functions defined on $[0,T]$ with values in Y and let $\mathcal{T} : C([0,T];X) \to C([0,T];Y)$ be a history-dependent operator, that is,

$$\begin{cases} \text{There exists } L_\mathcal{T} > 0 \text{ such that} \\ \quad \|\mathcal{T}u(t) - \mathcal{T}v(t)\|_Y \leq L_\mathcal{T} \displaystyle\int_0^t \|u(s) - v(s)\|_X\, ds \\ \quad\ \forall\, u,\ v \in C([0,T];X),\ t \in [0,T]. \end{cases} \tag{2.10}$$

We now recall the following existence and uniqueness results.

Theorem 2.5. *Assume* (2.1), (2.4) *and* (2.8). *Then, there exists a unique element* $u \in X$ *such that*

$$(Au, v - u)_X + j(v) - j(u) \geq (f, v - u)_X \qquad \forall\, v \in X. \tag{2.11}$$

Theorem 2.6. *Assume* (2.1), (2.3), (2.9) *and* (2.10). *Then, there exists a unque function* $u \in C([0,T]; X)$ *such that*

$$\begin{aligned}(Au(t), v - u(t))_X + \varphi(\mathcal{T}u(t), v) - \varphi(\mathcal{T}u(t), u(t))\\ \geq (f(t), v - u(t))_X \qquad \forall\, v \in X,\ t \in [0,T].\end{aligned} \tag{2.12}$$

Theorem 2.5 represents a standard result for elliptic variational inequalities. Its proof can be found in [20, p.40], for instance. We also note that Theorem 2.1 is a particular of Theorem 2.5, obtained when j vanishes. We refer to inequalities of the form (2.12) as history-dependent variational inequalities. A proof of Theorem 2.6 can be found in [21, p.62], based on a fixed point argument for history-dependent operators. Moreover, it is easy to see that Theorem 2.2 represents a particular case of Theorem 2.6, obtained when $Y = X$, $\mathcal{T} = \mathcal{S}$ and $\varphi(u, v) = (u, v)_X$, for all u, $v \in V$. Additional results on variational inequalities can be found in [9, 11, 13, 17], for instance.

3 Elliptic equations

In this section we state and prove convergence criteria for Problem $\mathcal{P}$. To this end, we assume in what follows that (2.1) and (2.4) hold, even if we do not mention it explicitely, and we denote by u the solution of Problem $\mathcal{P}$ provided by Theorem 2.1. Moreover, we consider an arbitrary sequence $\{u_n\} \subset X$ and for any sequence $\{\theta_n\} \subset \mathbb{R}_+$ which converges to zero we use the short hand notation $0 \leq \theta_n \to 0$. Here and below all the limits are considered as $n \to \infty$ and, in addition, 0_X will represent the zero element of the space X.

Our first result in this section is the following.

Theorem 3.1. *Assume* (2.1) *and* (2.4). *Then, the following statements are equivalent.*

$$u_n \to u \quad \text{in} \quad X. \tag{3.1}$$

$$\begin{aligned}&\textit{There exists a sequence } 0 \leq \theta_n \to 0 \textit{ such that}\\ &\qquad \|Au_n - f\|_X \leq \theta_n \qquad \forall\, n \in \mathbb{N}.\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\textit{There exists a sequence } 0 \le \theta_n \to 0 \textit{ such that}\\&\qquad \|Au_n - f\|_X \le \theta_n(\|u_n\|_X + 1) \quad \forall\, n \in \mathbb{N}.\end{aligned} \tag{3.3}$$

$$\begin{aligned}&\textit{There exists a sequence } 0 \le \theta'_n \to 0 \textit{ such that}\\&\qquad (Au_n, v)_X + \theta'_n\|v\|_X \ge (f, v)_X \quad \forall\, v \in X,\ n \in \mathbb{N}.\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\textit{There exists a sequence } 0 \le \theta''_n \to 0 \textit{ such that}\\&\qquad (Au_n, v)_X + \theta''_n(\|v\|_X + 1) \ge (f, v)_X \quad \forall\, v \in X,\ n \in \mathbb{N}.\end{aligned} \tag{3.5}$$

Proof. The proof is structured in several steps, as follows.

(3.1) $\Longrightarrow$ (3.2). Assume (3.1) and let $n \in \mathbb{N}$. We use condition (2.1) (a) and equality $Au = f$ to see that

$$\|Au_n - f\|_X = \|Au_n - Au\|_X \le M_A\|u_n - u\|_X$$

and, therefore, (3.2) holds, with $\theta_n = M_A\|u_n - u\|_X \to 0$.

(3.2) $\Longrightarrow$ (3.3). This implication is obvious.

(3.3) $\Longrightarrow$ (3.4). Assume (3.3) and let $n \in \mathbb{N}$. We use the equivalence (2.6) with $u = Au_n - f$ and $\theta = \theta_n(\|u_n\|_X + 1)$ to see that

$$(Au_n - f, v)_X + \theta_n(\|u_n\|_X + 1)\|v\|_X \ge 0 \quad \forall\, v \in X \tag{3.6}$$

and, taking $v = -u_n$ we deduce that

$$(Au_n - A0_X, u_n)_X \le \theta_n(\|u_n\|_X + 1)\|u_n\|_X + (f - A0_X, u_n)_X.$$

This inequality combined with assumption (2.1)(b) yields

$$(m_A - \theta_n)\|u_n\|_X \le \theta_n + \|f - A0_X\|_X$$

and, since $\theta_n \to 0$, we deduce that there exists $M > 0$ which depends on A and f but does not depend on n, such that

$$\|u_n\|_X \le M. \tag{3.7}$$

We now combine inequalities (3.6) and (3.7) to see that (3.4) holds with $\theta'_n = \theta_n(M + 1)$.

(3.4) $\Longrightarrow$ (3.5). This implication is obvious.

(3.5) $\Longrightarrow$ (3.1). Assume (3.5) and let $n \in \mathbb{N}$. We use the implication (2.7) with $u = Au - f$ and $\theta = \theta''_n$ to see that

$$\|Au_n - f\|_X \le \theta''_n + \sqrt{\theta''_n} \to 0,$$

which shows that $Au_n \to Au$ in X. Therefore, using the continuity of the operator A^{-1}, guaranteed by Remark 2.3, we deduce that (3.1) holds, which concludes the proof. □

We now use Theorem 3.1 to study the dependence of the solution with respect to the data. To this end, for each $n \in \mathbb{N}$ we consider a Lipschitz continuous strongly monotone operator $A_n : X \to X$ as well as an element $f_n \in X$. Then, using Theorem 2.1 it follows that there exists a unique element $u_n \in X$ such that $A_n u_n = f_n$. In addition, we denote by m_n the strong monotonicity constant of the operator A_n and we consider the folowing assumptions.

$$\left\{\begin{array}{l}\text{There exists } 0 \leq a_n \to 0 \text{ such that}\\ \|A_n v - Av\|_X \leq a_n(\|v\|_X + 1) \quad \forall\, v \in X,\ n \in \mathbb{N}.\end{array}\right. \tag{3.8}$$

$$\text{There exists } m_0 > 0 \text{ such that } m_n \geq m_0 \quad \forall\, n \in \mathbb{N}. \tag{3.9}$$

$$f_n \to f \quad \text{in} \quad X. \tag{3.10}$$

Our second result in the section is the following.

Theorem 3.2. *Assume* (2.1), (2.4) *and* (3.8)–(3.10). *Then,* $u_n \to u$ *in* X.

Proof. We start by proving that the sequence $\{u_n\}$ is bounded in X. To this end, we fix $n \in \mathbb{N}$ and note that equality $A_n u_n = f_n$ implies that

$$(A_n u_n - A_n 0_X, u_n)_X = (f_n, u_n)_X - (A_n 0_X, u_n)_X \leq (\|f_n\|_X + \|A_n 0_X\|_X)\|u_n\|_X.$$

Therefore, the strong monotonicity of A_n combined with assumptions (3.9) and (3.8) implies that

$$\begin{aligned} m_0\|u_n\|_X &\leq \|f_n\|_X + \|A_n 0_X\|_X \leq \|f_n\|_X + \|A_n 0_X - A0_X\|_X + \|A0_X\|_X\\ &\leq \|f_n\|_X + a_n + \|A0_X\|_X. \end{aligned}$$

Using now the convergences (3.10) and $a_n \to 0$ we deduce that there exists $M > 0$ which does not depend on n such that

$$\|u_n\|_X \leq M. \tag{3.11}$$

We now use the equality $A_n u_n = f_n$, assumptions (3.8) and the bound (3.11) to write

$$\|Au_n - f\|_X = \|Au_n - A_n u_n\|_X + \|f_n - f\|_X \leq a_n(M+1) + \|f_n - f\|_X.$$

This inequality shows that condition (3.2) is satisfied with

$$\theta_n = a_n(M+1) + \|f_n - f\|_X \to 0.$$

We now use Theorem 3.1 to conclude the proof. □

Remark 3.3. Note that Theorem 3.2 provides a continuous dependence result of the solution of Problem $\mathcal{P}$ with respect the operator A and the element f.

4 Applications to elasticity

We now illustrate the use of Theorems 2.1 and 3.2 in the study of some nonlinear problems arising in elasticity. To this end, we recall that the canonical inner products and the corresponding norms on the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{1/2} \quad \forall\, \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \quad \forall\, \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

respectively. Moreover, for simplicity, we denote by $\mathbf{0}$ the zero element of these spaces. Consider now an operator $\mathcal{F}$ and a subset K such that

$$\mathcal{F} : \mathbb{S}^d \to \mathbb{S}^d \text{ is strongly monotone and Lipschitz continuous.} \tag{4.1}$$

$$K \text{ is a closed convex subset of } \mathbb{S}^d \text{ such that } \mathbf{0} \in K. \tag{4.2}$$

We have the following existence, uniqueness and convergence result.

Theorem 4.1. *Assume* (4.1) *and* (4.2)*. Then, for any stress tensor* $\boldsymbol{\sigma} \in \mathbb{S}^d$ *there exists a unique strain tensor* $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ *such that* (1.1) *holds. Moreover, for any stress tensor* $\boldsymbol{\sigma}_n \in \mathbb{S}^d$ *and any elasticity coefficient* $\alpha_n \geq 0$ *there exists a unique strain tensor* $\boldsymbol{\varepsilon}_n \in \mathbb{S}^d$ *such that*

$$\boldsymbol{\sigma}_n = \mathcal{F}\boldsymbol{\varepsilon}_n + \alpha_n\,(\boldsymbol{\varepsilon}_n - P_K\boldsymbol{\varepsilon}_n). \tag{4.3}$$

In addition, if $\boldsymbol{\sigma}_n \to \boldsymbol{\sigma}$ *in* $\mathbb{S}^d$ *and* $\alpha_n \to 0$*, then* $\boldsymbol{\varepsilon}_n \to \boldsymbol{\varepsilon}$ *in* $\mathbb{S}^d$*.*

Proof. The unique solvability of equation (1.1) is a direct consequence of assumption (4.1) which allows us to apply Theorem 2.1 in the space $X = \mathbb{S}^d$ with $A = \mathcal{F}$ and $f = \boldsymbol{\sigma}$.

Denote by $G : \mathbb{S}^d \to \mathbb{S}^d$ the operator defined by

$$G\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} - P_K\boldsymbol{\varepsilon} \quad \forall\, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \tag{4.4}$$

Let ε_1, $\varepsilon_2 \in \mathbb{S}^d$. Then, the nonexpansivity of the projection operator implies that

$$\|P_K\varepsilon_1 - P_K\varepsilon_2\| \le \|\varepsilon_1 - \varepsilon_2\| \tag{4.5}$$

and, therefore,

$$\|G\varepsilon_1 - G\varepsilon_2\| \le 2\|\varepsilon_1 - \varepsilon_2\|,$$

$$\begin{aligned}(G\varepsilon_1 - G\varepsilon_2)\cdot(\varepsilon_1 - \varepsilon_2) &= \|\varepsilon_1 - \varepsilon_2\|^2 - (P_K\varepsilon_1 - P_K\varepsilon_2)\cdot(\varepsilon_1 - \varepsilon_2)\\ &\ge \|\varepsilon_1 - \varepsilon_2\|^2 - \|P_K\varepsilon_1 - P_K\varepsilon_2\|\|\varepsilon_1 - \varepsilon_2\| \ge 0.\end{aligned}$$

These inequalities show that

$$G : \mathbb{S}^d \to \mathbb{S}^d \text{ is a Lipschitz continuous monotone operator.} \tag{4.6}$$

Let $n \in \mathbb{N}$, $\alpha_n \ge 0$ and let $\mathcal{F}_n : \mathbb{S}^d \to \mathbb{S}^d$ be the operator defined by

$$\mathcal{F}_n\varepsilon = \mathcal{F}\varepsilon + \alpha_n(\varepsilon - P_K\varepsilon) \quad \forall\, \varepsilon \in \mathbb{S}^d. \tag{4.7}$$

We use assumption (4.1) and the properties (4.6) to see that $\mathcal{F}_n$ is a Lipschitz continuous strongly monotone operator on the space $\mathbb{S}^d$, with constant $m_{\mathcal{F}_n} = m_{\mathcal{F}}$, $m_{\mathcal{F}}$ being the constant of strong monotonicity of $\mathcal{F}$. The unique solvability of equation (4.3) is now a direct consequence on Theorem 2.1, applied with $X = \mathbb{S}^d$, $A = \mathcal{F}_n$ and $f = \boldsymbol{\sigma}_n$.

Let $\boldsymbol{w} \in \mathbb{S}^d$ and note that assumption (4.2) implies that $P_K\mathbf{0} = \mathbf{0}$. Therefore, using (4.7) and (4.5) with $\varepsilon_1 = \boldsymbol{w}$ and $\varepsilon_2 = \mathbf{0}$ we deduce that

$$\|\mathcal{F}_n\boldsymbol{w} - \mathcal{F}\boldsymbol{w}\| = \alpha_n\|\boldsymbol{w} - P_K\boldsymbol{w}\| \le \alpha_n(\|\boldsymbol{w}\| + \|P_K\boldsymbol{w}\|) \le 2\alpha_n\|\boldsymbol{w}\|.$$

We conclude from here that the operators $A_n = \mathcal{F}_n$ and $A = \mathcal{F}$ satisfy condition (3.8) with $X = \mathbb{S}^d$ and $a_n = \alpha_n$. Moreover, condition (3.9) is satisfied, too, with $m_0 = m_{\mathcal{F}}$. The convergence part in Theorem 4.1 is now a direct consequence of Theorem 3.2. □

Remark 4.2. In addition to the mathematical interest in the convergence result in Theorem 4.1, it is important from the mechanical point of view, since it shows that for a small elasticity coefficient α, a small perturbation on the stress applied to materials of the form (1.2) give rise to a strain close to that strain obtained when the corresponding stress is applied to materials of the form (1.1).

We now present an aplication of our abstract results in the study of a contact problem with elastic materials and, to this, end, we need additional notation. We assume in what follows that Ω is a bounded domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$ with smooth boundary Γ composed of three sets $\overline{\Gamma}_1$, and $\overline{\Gamma}_2$, $\overline{\Gamma}_3$ with the mutually disjoint relatively open sets Γ_1, Γ_2, Γ_3, such that $\mathrm{meas}\,(\Gamma_1) > 0$. We denote by $\overline{\Omega} = \Omega \cup \Gamma$ the closure of Ω in $\mathbb{R}^d$. Moreover, we use boldface letters for vectors and tensors, such as the outward unit normal on Γ, denoted by $\boldsymbol{\nu}$. A typical point in $\mathbb{R}^d$ is denoted by $\boldsymbol{x} = (x_i)$. The indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. Also, the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$.

Everywhere below we use standard notation for Lebesgue and Sobolev spaces of real-valued functions defined on Ω and Γ. For a function $\boldsymbol{v} \in H^1(\Omega)^d$ we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ to Γ and v_ν for the normal trace to Γ, that is $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$. Moreover, we use $\boldsymbol{v}_\tau$ for the tangential trace of $\boldsymbol{v}$, i.e., $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu\boldsymbol{\nu}$, as well as the notation

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = (\varepsilon_{ij}(\boldsymbol{v})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\,(v_{i,j} + v_{j,i}) \quad \forall\, \boldsymbol{v} = (v_i) \in H^1(\Omega)^d.$$

Next, we introduce the spaces

$$V = \big\{\, \boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = \boldsymbol{0} \;\text{ a.e. on } \Gamma_1,\; v_\nu = 0 \;\text{ a.e. on } \Gamma_3 \,\big\}, \tag{4.8}$$

$$Q = \big\{\, \boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega),\; 1 \le i,\, j \le d \,\big\}, \tag{4.9}$$

which are real Hilbert spaces with the canonical inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx,$$

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}\, \tau_{ij}\, dx = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx.$$

The associated norms on these spaces will be denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. This implies that

$$\|\boldsymbol{u}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_Q \quad \forall\, \boldsymbol{u} \in V \tag{4.10}$$

which, in particular, shows that the deformation operator $\boldsymbol{\varepsilon}\colon V \to Q$ is continuous. Moreover, recall that the trace operator is a linear continuous operator from V with values in $L^2(\Gamma)^d$. Therefore, there exists $c_0 > 0$ such that

$$\|\boldsymbol{u}\|_{L^2(\Gamma_2)^d} \le c_0 \|\boldsymbol{u}\|_V \quad \forall\, \boldsymbol{u} \in V. \tag{4.11}$$

Consider now the data $\mathcal{F}$, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$ and g which satisfy the following conditions.

$$\left\{\begin{array}{l}\mathcal{F}\colon \Omega\times\mathbb{S}^d\to\mathbb{S}^d \text{ is such that}\\ \text{(a) there exists } L_{\mathcal{F}}>0 \text{ such that}\\ \qquad \|\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\|\le L_{\mathcal{F}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|\\ \qquad\quad \text{for all } \ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, \ \text{a.e. } \boldsymbol{x}\in\Omega,\\ \text{(b) there exists } m_{\mathcal{F}}>0 \text{ such that}\\ \qquad (\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_1)-\mathcal{F}(\boldsymbol{x},\boldsymbol{\varepsilon}_2))\cdot(\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\ge m_{\mathcal{F}}\,\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2\\ \qquad\quad \text{for all } \ \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2\in\mathbb{S}^d, \ \text{a.e. } \boldsymbol{x}\in\Omega,\\ \text{(c) } \mathcal{F}(\cdot,\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d,\\ \text{(d) } \mathcal{F}(\boldsymbol{x},\boldsymbol{0})=\boldsymbol{0} \quad \text{a.e. } \boldsymbol{x}\in\Omega.\end{array}\right. \tag{4.12}$$

$$\boldsymbol{f}_0\in L^2(\Omega)^d, \quad \boldsymbol{f}_2\in L^2(\Gamma_2)^d. \tag{4.13}$$

$$g\in L^2(\Gamma_3), \quad g(\boldsymbol{x})\ge 0 \quad \text{a.e. } \boldsymbol{x}\in\Gamma_3. \tag{4.14}$$

We now introduce the operator $A: V\to V$, the element $\boldsymbol{f}\in V$ and the function j_g defined as follows:

$$(A\boldsymbol{u},\boldsymbol{v})_V=\int_\Omega \mathcal{F}\boldsymbol{\varepsilon}(\boldsymbol{u})\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx, \tag{4.15}$$

$$(\boldsymbol{f},\boldsymbol{v})_V=\int_\Omega \boldsymbol{f}_0\cdot\boldsymbol{v}\,dx+\int_{\Gamma_2}\boldsymbol{f}_2\cdot\boldsymbol{v}\,d\Gamma, \tag{4.16}$$

$$j_g(\boldsymbol{v})=\int_{\Gamma_3} g\|\boldsymbol{v}_\tau\|\,d\Gamma, \tag{4.17}$$

for all $\boldsymbol{u},\boldsymbol{v}\in V$. With these notation we consider the following variational problems.

Problem $\mathcal{V}$. *Find a displacement field $\boldsymbol{u}\in V$ such that $A\boldsymbol{u}=\boldsymbol{f}$* or, equivalantly,

$$(A\boldsymbol{u},\boldsymbol{v})_V=(\boldsymbol{f},\boldsymbol{v})_V \qquad \forall\,\boldsymbol{v}\in V.$$

Problem $\mathcal{V}_g$. *Find a displacement field $\boldsymbol{u}_g\in V$ such that*

$$(A\boldsymbol{u}_g,\boldsymbol{v}-\boldsymbol{u}_g)_V+j_g(\boldsymbol{v})-j_g(\boldsymbol{u}_g)\ge(\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u}_g)_V \qquad \forall\,\boldsymbol{v}\in V. \tag{4.18}$$

Note that Problem $\mathcal{V}$ represents the variational formulation of a mathematical model which describes the equilibrium of an elastic body in contact with an obstacle, the so-called foundation. The main ingredients of the physical setting are the following: an elastic body occupies in the reference configuration the domain Ω, is fixed on the part Γ_1 of its boundary, is submitted to body forces of density $\boldsymbol{f}_0$ and surface tractions of density $\boldsymbol{f}_2$ which act on in Ω and Γ_3, respectively, and is contact with a foundation of the part Γ_3 of its boundary. The process is static, the contact is bilateral, i.e., there is no separation between the body's surface and the foundation, and it is frictionless. Moreover, the material's behaviour is described with the constitutive law (1.1). Problem $\mathcal{V}_g$ has a similar interpretation. The difference is that now the contact is frictional and it is described with the Tresca friction law in which the friction bound is the given function g. More details on the statement of contact problems with elastic materials can be found in [3, 12, 13, 14, 17, 20], for instance.

In the study of Problems $\mathcal{V}$ and $\mathcal{V}_g$ we have the following existence, uniqueness and convergence results.

Theorem 4.3. *Assume* (4.12)–(4.14)*. Then, Problem $\mathcal{V}$ has a unique solution $\boldsymbol{u}$ and Problem $\mathcal{V}_g$ has a unique solution $\boldsymbol{u}_g$. Moreover,*

$$\boldsymbol{u}_g \to \boldsymbol{u} \quad \text{in} \quad V \quad \text{as} \quad g \to 0 \quad \text{in} \quad L^2(\Gamma_2). \tag{4.19}$$

Proof. Using assumption (4.12) it is easy to see that

$$\|A\boldsymbol{u} - A\boldsymbol{v}\|_V \le L_{\mathcal{F}} \|\boldsymbol{u} - \boldsymbol{v}\|_V, \quad (A\boldsymbol{u} - A\boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})_V \ge m_{\mathcal{F}} \|\boldsymbol{u} - \boldsymbol{v}\|_V^2 \tag{4.20}$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$, which shows that $A : V \to V$ is a strongly monotone Lipschitz continuous operator. The unique solvability of Problem $\mathcal{V}$ follows now from Theorem 2.1 with $X = V$ and A, $\boldsymbol{f}$ defined by (4.15), (4.16), respectively.

Next, using assumption (4.14) and definition (4.17) it follows that $j_g : V \to \mathbb{R}_+$ is a continuous seminorm, hence it is convex and lower semicontinuous. Therefore, the unique solvability of Problem $\mathcal{V}_g$ follows from Theorem 2.5.

Assume now that $\{g_n\}$ represents a sequence of functions which satisfies condition (4.14) and, for simplicity, denote $\boldsymbol{u}_{g_n} = \boldsymbol{u}_n$, $j_{g_n} = j_n$. Then, using (4.18) we find that

$$(A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n)_V + j_n(\boldsymbol{v}) - j_n(\boldsymbol{u}_n) \ge (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u}_n)_V \qquad \forall\, \boldsymbol{v} \in V. \tag{4.21}$$

We now use definition (4.17) and assumption (4.14) to see that

$$j_n(\boldsymbol{v}) - j_n(\boldsymbol{u}_n) = \int_{\Gamma_3} g_n(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_{n\tau}\|)\, d\Gamma \le \int_{\Gamma_3} g_n \Big| \|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_{n\tau}\| \Big|\, d\Gamma$$

$$\le \int_{\Gamma_3} g_n \|\boldsymbol{v}_\tau - \boldsymbol{u}_{n\tau}\|\, d\Gamma \le \int_{\Gamma_3} g_n \|\boldsymbol{v} - \boldsymbol{u}_n\|\, d\Gamma \le \|g_n\|_{L^2(\Gamma_3)} \|\boldsymbol{v} - \boldsymbol{u}_n\|_{L^2(\Gamma)^d}$$

for all $\boldsymbol{v} \in V$. Therefore, using the trace inequality (4.11) we find that

$$j_n(\boldsymbol{v}) - j_n(\boldsymbol{u}_n) \leq c_0 \|g_n\|_{L^2(\Gamma_3)} \|\boldsymbol{v} - \boldsymbol{u}_n\|_V \quad \forall\, \boldsymbol{v} \in V. \tag{4.22}$$

We now combine the inequalities (4.21) and (4.22) to obtain that

$$(A\boldsymbol{u}_n, \boldsymbol{v})_V + c_0 \|g_n\|_{L^2(\Gamma_3)} \|\boldsymbol{v}\|_V \geq (\boldsymbol{f}, \boldsymbol{v})_V \qquad \forall\, \boldsymbol{v} \in V. \tag{4.23}$$

Then, since $\|g_n\|_{L^2(\Gamma_3)} \to 0$ we are in a position to use the equivalence between the statements (3.1) and (3.4) in Theorem 3.2 to conclude the proof. □

Remark 4.4. In addition of the mathematical interest in the convergence result in Theorem 4.3, it is important from the mechanical point of view, since it shows that the solution of the frictionless contact problem $\mathcal{V}$ can be approached by the solution of the frictional contact problem $\mathcal{V}_g$ for a small friction bound.

5 History-dependent equations

In this section we state and prove convergence criteria for Problem $\mathcal{Q}$. To this end we assume in what follows that (2.1), (2.2) and (2.3) hold, even if we do not mention it explicitely. We denote by $u \in C([0,T];X)$ the solution of Problem $\mathcal{Q}$ provided by Theorem 2.2 and consider an arbitrary sequence $\{u_n\} \subset C([0,T];X)$. Our first result in this section is the following.

Theorem 5.1. *Assume* (2.1), (2.2) *and* (2.3). *Then, the following statements are equivalent.*

$$u_n \to u \quad \text{in} \quad C([0,T];X). \tag{5.1}$$

There exists a sequence $0 \leq \theta_n \to 0$ such that (5.2)

$$\|Au_n(t) + \mathcal{S}u_n(t) - f(t)\|_X \leq \theta_n \qquad \forall\, t \in [0,T],\ n \in \mathbb{N}.$$

There exists a sequence $0 \leq \theta_n \to 0$ such that (5.3)

$$\|Au_n(t) + \mathcal{S}u_n(t) - f(t)\|_X \leq \theta_n(\|u_n(t)\|_X + 1) \qquad \forall\, t \in [0,T],\ n \in \mathbb{N}.$$

There exists a sequence $0 \leq \theta'_n \to 0$ such that (5.4)

$$(Au_n(t), v)_X + (\mathcal{S}u_n(t), v)_X + \theta'_n \|v\|_X \geq (f(t), v)_X \qquad \forall\, t \in [0,T],\ v \in X,\ n \in \mathbb{N}.$$

There exists a sequence $0 \leq \theta''_n \to 0$ such that (5.5)

$$(Au_n(t), v)_X + (\mathcal{S}u_n(t), v)_X + \theta''_n (\|v\|_X + 1) \geq (f(t), v)_X \qquad \forall\, t \in [0,T],\ v \in X,\ n \in \mathbb{N}.$$

Proof. The proof is structured in several steps, as follows.

(5.1) $\Longrightarrow$ (5.2). Assume (5.1) and let $n \in \mathbb{N}$, $t \in [0,T]$. We use equation (1.7) and assumptions (2.1) (a), (2.2) to see that

$$\|Au_n(t) + \mathcal{S}u_n(t) - f(t)\|_X = \|Au_n(t) + \mathcal{S}u_n(t) - Au(t) - \mathcal{S}u(t)\|_X$$
$$\leq M_A\|u_n(t) - u(t)\|_X + L_{\mathcal{S}} \int_0^t \|u_n(s) - u(s)\|_X ds.$$

Therefore, (5.2) holds with $\theta_n = (M_A + L_{\mathcal{S}}T)\|u_n - u\|_{C([0,T];X)}$.

(5.2) $\Longrightarrow$ (5.3). This implication is obvious.

(5.3) $\Longrightarrow$ (5.4). Assume (5.3) and let $n \in \mathbb{N}$, $t \in [0,T]$. We use (2.6) to see that

$$(Au_n(t) + \mathcal{S}u_n(t) - f(t), v)_X + \theta_n(\|u_n(t)\|_X + 1)\|v\|_X \geq 0 \quad \forall v \in X \tag{5.6}$$

and, taking $v = -u_n(t)$ we deduce that

$$(Au_n(t) - A0_X, u_n(t))_X \leq \theta_n(\|u_n(t)\|_X + 1)\|u_n(t)\|_X$$
$$+(\mathcal{S}0_X(t) - \mathcal{S}u_n(t), u_n(t))_X + (f(t) - A0_X - \mathcal{S}0_X(t), u_n(t))_X.$$

This inequality combined with assumption (2.1)(b) and (2.2) yields

$$(m_A - \theta_n)\|u_n(t)\|_X \leq \theta_n + L_{\mathcal{S}} \int_0^t \|u_n(s)\|_X ds + \|f(t) - A0_X - \mathcal{S}0_X(t)\|_X$$

and, using the convergence $\theta_n \to 0$ and the continuity of the functions $t \mapsto f(t)$, $t \mapsto \mathcal{S}0_X(t)$ we deduce that there exists two constants $C_1 > 0$ and $C_2 > 0$ which do not depend on n and t, such that

$$\|u_n(t)\|_X \leq C_1 + C_2 \int_0^t \|u_n(s)\|_X ds. \tag{5.7}$$

We now use the Gronwall argument to deduce that there exists $M > 0$ which does not depend on n and t such that

$$\|u_n(t)\|_X \leq M. \tag{5.8}$$

Finally, we combine inequalities (5.6) and (5.8) to see that (5.4) holds with $\theta_n' = \theta_n(M+1) \to 0$.

(5.4) $\Longrightarrow$ (5.5). This implication is obvious.

(5.5) $\Longrightarrow$ (5.1). Assume (5.5) and let $n \in \mathbb{N}$, $t \in [0,T]$. We use implication (2.7) to see that

$$\|Au_n(t) + \mathcal{S}u_n(t) - f(t)\|_X \le \theta''_n + \sqrt{\theta''_n}$$

and, using (1.7) we deduce that

$$\|Au_n(t) + \mathcal{S}u_n(t) - Au(t) - \mathcal{S}u(t)\|_X \le \theta''_n + \sqrt{\theta''_n}. \tag{5.9}$$

We now use inequalities (5.9) and (2.2) to find that

$$\|Au_n(t) - Au(t)\|_X \le \|Au_n(t) + \mathcal{S}u_n(t) - Au(t) - \mathcal{S}u(t)\|_X$$

$$+\|\mathcal{S}u(t) - \mathcal{S}u_n(t)\|_X \le \theta''_n + \sqrt{\theta''_n} + L_{\mathcal{S}} \int_0^t \|u_n(s) - u(s)\|_X ds$$

and, since inequality (2.5) implies that

$$\|Au_n(t) - Au(t)\|_X \ge m_A \|u_n(t) - u(t)\|_X,$$

we conclude that

$$m_A \|u_n(t) - u(t)\|_X \le \theta''_n + \sqrt{\theta''_n} + L_{\mathcal{S}} \int_0^t \|u_n(s) - u(s)\|_X ds.$$

Finally, we use the Gronwall lemma to see that

$$\|u_n(t) - u(t)\|_X \le \frac{1}{m_A}(\theta''_n + \sqrt{\theta''_n})\, e^{\frac{L_{\mathcal{S}}}{m_A}t}.$$

Therefore, the convergence $\theta''_n \to 0$ implies (5.1), which concludes the proof. □

We now use Theorem 5.1 to study the dependence of the solution with respect to the data. To this end, for each $n \in \mathbb{N}$ we consider a history-dependent operator $\mathcal{S}_n : C([0,T];X) \to C([0,T];X)$ and an element $f_n \in C([0,T];X)$. Then, using Theorem 3.1 it follows that there exists a unique element $u_n \in C([0,T];X)$ such that $Au_n(t) + \mathcal{S}_n u_n(t) = f_n(t)$, for all $t \in [0,T]$. In addition, assume that

$$\left\{\begin{array}{l} \text{There exists } 0 \le s_n \to 0 \text{ such that} \\ \|\mathcal{S}_n v - \mathcal{S}v\|_X \le s_n\Big(\displaystyle\int_0^t \|v(s)\|_X\, ds + 1\Big) \\ \forall\, t \in [0,T],\ v \in C([0,T];X). \end{array}\right. \tag{5.10}$$

$$f_n \to f \quad \text{in} \quad C([0,T];X). \tag{5.11}$$

Our second result in the section is the following.

Theorem 5.2. *Assume* (2.1)–(2.3), (5.10) *and* (5.11). *Then,* $u_n \to u$ *in* $C([0,T];X)$.

Proof. We start by proving that the sequence $\{u_n\}$ is uniformely bounded. To this end, we fix $n \in \mathbb{N}$ and $t \in [0,T]$. We use equality $Au_n(t) + \mathcal{S}_n u_n(t) = f_n(t)$ to write

$$\begin{aligned}&(Au_n(t) - A0_X, u_n(t))_X\\ &\quad = (\mathcal{S}u_n(t) - \mathcal{S}_n u_n(t), u_n(t))_X + (f_n(t) - f(t), u_n(t))_X\\ &\qquad + (f(t) - A0_X - \mathcal{S}0_X(t), u_n(t))_X + (\mathcal{S}0_X(t) - \mathcal{S}u_n(t), u_n(t))_X\end{aligned}$$

Therefore, using assumptions (2.1) (b) and (2.2) and (5.10) we find that

$$\begin{aligned}m_A\|u_n(t)\|_X \le s_n\Big(\int_0^t \|u_n(s)\|_X\,ds + 1\Big) + \|f_n(t) - f(t)\|_X\\ + \|f(t) - A0_X - \mathcal{S}0_X(t)\|_X + L_{\mathcal{S}}\int_0^t \|u_n(s)\|_X ds.\end{aligned}$$

Next, using the convergence $s_n \to 0$, (5.11) and the continuity of the functions $t \mapsto f(t)$, $t \mapsto \mathcal{S}0_X(t)$, we deduce that there exists two constants $C_1 > 0$ and $C_2 > 0$ which do not depend on n and t, such that (5.7) holds. This inequality implies the bound (5.8) with some $M > 0$ which does not depend on n and t.

We now use use equality $Au_n(t) + \mathcal{S}_n u_n(t) = f_n(t)$, again, to see that

$$\begin{aligned}&\|Au_n(t) + \mathcal{S}u_n(t) - f(t)\|_X\\ &\quad = \|Au_n(t) + \mathcal{S}u_n(t) - f(t) + f_n(t) - Au_n(t) - \mathcal{S}_n u_n(t)\|_X\\ &\quad \le \|\mathcal{S}u_n(t) - \mathcal{S}_n u_n(t)\|_X + \|f_n(t) - f(t)\|_X\end{aligned}$$

and, using assumption (5.10), we find that

$$\begin{aligned}&\|Au_n(t) + \mathcal{S}u_n(t) - f(t)\|_X\\ &\qquad \le s_n\Big(\int_0^t \|u_n(s)\|_X ds + 1\Big) + \|f_n(t) - f(t)\|_X.\end{aligned}$$

We now combine this inequality with the bound (5.8) to see that (5.2) holds with $\theta_n = s_n(MT+1) + \|f_n - f\|_{C([0,T];X)} \to 0$. Finally, we use Theorem 5.1 to conclude the proof. □

Remark 5.3. Note that Theorem 3.2 provides a continuous dependence result for the solution of Problem $\mathcal{Q}$ with respect the operator $\mathcal{S}$ and the element f.

6 Applications in viscoelasticity

Theorems 2.2 and 5.2 can be used in the study of viscoelastic constitutive laws of the form (1.4). Theorem 2.2 provides the unique solvability of this equation and Theorems 5.1, 5.2 are useful to deduce convergence results which could represent the continuous dependence of the solution $\boldsymbol{\varepsilon}$ with respect to the relaxation tensor and the stress function. The arguments are similar to those presented in Section 4, in the study of elastic constitutive laws and, therefore, we skip the details. Nevertheless, below, we illustrate the use of Theorems 2.2 and 5.2 in the variational analysis of a viscoelastic contact problem.

To this end, we use the notation introduced in Section 4 and we consider a time interval $[0,T]$ with $T>0$ given. In addition, besides the spaces (4.8) and (4.9), we need the space of symmetric fourth order tensors defined by

$$\mathbf{Q}_\infty = \{\, \mathcal{E} = (e_{ijkl}) \mid e_{ijkl} = e_{jikl} = e_{klij} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d \,\}. \tag{6.1}$$

It is well-known that $\mathbf{Q}_\infty$ is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{0\le i,j,k,l\le d} \|e_{ijkl}\|_{L^\infty(\Omega)}.$$

In addition, the inequality below holds:

$$\|\mathcal{E}\boldsymbol{\tau}\|_Q \le d\,\|\mathcal{E}\|_{\mathbf{Q}_\infty}\|\boldsymbol{\tau}\|_Q \qquad \forall\, \mathcal{E} \in \mathbf{Q}_\infty,\ \boldsymbol{\tau} \in Q. \tag{6.2}$$

Consider now an elasticity operator $\mathcal{F}$ which satisfy condition (4.12) and, moreover, consider the data $\mathcal{B}$, $\boldsymbol{f}_0$, $\boldsymbol{f}_2$ and μ which satisfy the following conditions.

$$\mathcal{B} \in C([0,T];\mathbf{Q}_\infty). \tag{6.3}$$

$$\boldsymbol{f}_0 \in C([0,T];L^2(\Omega)^d), \qquad \boldsymbol{f}_2 \in C([0,T];L^2(\Gamma_2)^d). \tag{6.4}$$

$$\mu \in L^\infty(\Gamma_3), \qquad \mu(\boldsymbol{x}) \ge 0 \qquad \text{a.e. } \boldsymbol{x} \in \Gamma_3. \tag{6.5}$$

Next, besides the operator (4.15) we need the operators $\mathcal{S} : C([0,T];V) \to C([0,T];V)$, $\mathcal{R} : C([0,T];V) \to C([0,T];L^2(\Gamma_3))$, the function $\boldsymbol{f} : [0,T] \to V$ and the function $j_\mu : L^2(\Gamma_3) \times V \to \mathbb{R}$ defined as follows:

$$(\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v})_V = \int_\Omega \Big(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\,ds\Big)\cdot\boldsymbol{\varepsilon}(\boldsymbol{v})\,dx \tag{6.6}$$

$$\forall\,\boldsymbol{u}\in C([0,T];V),\ \ \boldsymbol{v}\in V,$$

$$\mathcal{R}\boldsymbol{u}(t) = \int_0^t \|\boldsymbol{v}_\tau(s)\|\,ds \qquad \forall\,\boldsymbol{u}\in C([0,T];V),\ \ \boldsymbol{v}\in V, \tag{6.7}$$

$$(\boldsymbol{f}(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,dx + \int_{\Gamma_2}\boldsymbol{f}_2(t)\cdot\boldsymbol{v}\,d\Gamma \quad \forall\,t\in[0,T],\ \ \boldsymbol{v}\in V, \tag{6.8}$$

$$j_\mu(\xi,\boldsymbol{v}) = \int_{\Gamma_3}\mu\xi\|\boldsymbol{v}_\tau\|d\Gamma \qquad \forall\,\xi\in L^2(\Gamma_3),\ \ \boldsymbol{v}\in V. \tag{6.9}$$

With these notation we consider the following variational problems.

Problem $\mathcal{W}$. *Find a displacement field* $\boldsymbol{u}:[0,T]\to V$ *such that* $A\boldsymbol{u}(t)+\mathcal{S}\boldsymbol{u}(t)=\boldsymbol{f}(t)$ for all $t\in[0,T]$ or, equivalantly,

$$(A\boldsymbol{u}(t),\boldsymbol{v})_V + (\mathcal{S}\boldsymbol{u}(t),\boldsymbol{v})_V = (\boldsymbol{f}(t),\boldsymbol{v})_V \qquad \forall\,\boldsymbol{v}\in V,\ \ t\in[0,T].$$

Problem $\mathcal{W}_\mu$. *Find a displacement field* $\boldsymbol{u}_\mu:[0,T]\to V$ *such that*

$$(A\boldsymbol{u}_\mu(t),\boldsymbol{v}-\boldsymbol{u}_\mu(t))_V + (\mathcal{S}\boldsymbol{u}_\mu(t),\boldsymbol{v}-\boldsymbol{u}_\mu(t))_V \tag{6.10}$$

$$+j_\mu(\mathcal{R}\boldsymbol{u}(t),\boldsymbol{v}) - j_\mu(\mathcal{R}\boldsymbol{u}(t),\boldsymbol{u}(t)) \geq (\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}_\mu(t))_V \quad \forall\,\boldsymbol{v}\in V,\ \ t\in[0,T].$$

Note that Problem $\mathcal{W}$ represents the variational formulation of a mathematical model which describes the equilibrium of a viscoelastic body in contact with an obstacle, the so-called foundation. The model is based on ingredients similar to those in Section 4, the difference arising from the fact that now the body is assumed to be viscoelastic and the body forces and surface tractions are time-dependent. The material's behavior is described with the constitutive law (1.4), the contact is bilateral and is frictionless. In contrast, the model in Problem $\mathcal{W}_\mu$ is frictional. There, the friction is described with a total-slip version of Coulomb's law of dry friction in which μ represents the coefficient of friction. More details on the statement of contact problems with viscoelastic materials can be found in [3, 8, 20, 21], for instance.

In the study of Problems $\mathcal{W}$ and $\mathcal{W}_\mu$ we have the following existence, uniqueness and convergence results.

Theorem 6.1. *Assume* (4.12) *and* (6.3)–(6.5). *Then, Problem* $\mathcal{W}$ *has a unique solution* $\boldsymbol{u} \in C([0,T];V)$ *and Problem* $\mathcal{W}_\mu$ *has a unique solution* $\boldsymbol{u}_\mu \in C([0,T];V)$. *Moreover,*

$$\boldsymbol{u}_\mu \to \boldsymbol{u} \quad \text{in} \quad V \quad \text{as} \quad \mu \to 0 \quad \text{in} \quad L^\infty(\Gamma_3). \tag{6.11}$$

Proof. The proof is structured in three steps, as follows.

Step i). Unique solvability of Problem $\mathcal{W}$. We use Theorem 2.2 with $X = V$ and A, $\mathcal{S}$, $\boldsymbol{f}$ defined by (4.15), (6.6), (6.8), respectively. Note that condition (2.1) follows from inequalities (4.20). Assume now that $\boldsymbol{u}$, $\boldsymbol{v} \in C([0,T];V)$, $t \in [0,T]$ and $\boldsymbol{w} \in V$. We use definition (6.6), assumption (6.3) and inequality (6.2) to see that

$$(\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{v}(t), \boldsymbol{w})_V = \int_\Omega \Big(\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) - \boldsymbol{\varepsilon}(\boldsymbol{v}(s)\Big) ds \cdot (\boldsymbol{\varepsilon}(\boldsymbol{w}))\, dx$$

$$\leq d \max_{r\in[0,T]} \|\mathcal{B}(r)\|_{\mathbf{Q}_\infty} \Big(\int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u}(s)) - \boldsymbol{\varepsilon}(\boldsymbol{v}(s))\|_Q ds \Big) \|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_Q$$

$$= d\,\|\mathcal{B}\|_{C([0,T];\mathbf{Q}_\infty)} \Big(\int_0^t \|\boldsymbol{u}(s) - \boldsymbol{v}(s)\|_V ds \Big) \|\boldsymbol{w}\|_V,$$

which implies that

$$\|\mathcal{S}\boldsymbol{u}(t) - \mathcal{S}\boldsymbol{v}(t)\|_V \leq d\,\|\mathcal{B}\|_{C([0,T];\mathbf{Q}_\infty)} \int_0^t \|\boldsymbol{u}(s) - \boldsymbol{v}(s)\|_V ds.$$

We conclude from here that condition (2.2) is satisfied with $L_{\mathcal{S}} = d\,\|\mathcal{B}\|_{C([0,T];\mathbf{Q}_\infty)}$. Next, it is easy to see that assumptions (6.4) and definition (6.8) imply that $\boldsymbol{f} \in C([0,T];V)$ and, therefore, condition (2.4) is satisfied, too. The unique solvability of Problem $\mathcal{W}$ follows now from Theorem 2.2.

Step ii). Unique solvability of Problem $\mathcal{W}_\mu$. First, we introduce the Hilbert space $Y = V \times L^2(\Gamma_3)$ endowed with the inner product

$$(y, z)_Y = (\boldsymbol{u}, \boldsymbol{v})_V + (\xi, \eta)_{L^2(\Gamma_3)} \quad \forall\, y = (\boldsymbol{u}, \xi),\ z = (\boldsymbol{v}, \eta) \in Y.$$

Next, we consider the function $\varphi : Y \times V \to \mathbb{R}$ and the operator $\mathcal{T} : C([0,T];V) \to C([0,T];Y)$ defined by

$$\varphi(y, \boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v})_V + j_\mu(\xi, \boldsymbol{v}) \quad \forall\, y = (\boldsymbol{u}, \xi) \in Y,\ \boldsymbol{v} \in V, \tag{6.12}$$

$$\mathcal{T}\boldsymbol{u}(t) = (\mathcal{S}\boldsymbol{u}(t), \mathcal{R}\boldsymbol{u}(t)) \quad \forall\, \boldsymbol{u} \in C([0,T];V),\ t \in (0,T]. \tag{6.13}$$

Using (4.11) we have

$$
\begin{aligned}
& j_\mu(\xi_1, \boldsymbol{v}_2) - j_\mu(\xi_1, \boldsymbol{v}_1) + j_\mu(\xi_2, \boldsymbol{v}_1) - j_\mu(\xi_1, \boldsymbol{v}_2) \\
&\quad = \int_{\Gamma_3} \mu(\xi_1 - \xi_2)(\|\boldsymbol{v}_{2\tau}\| - \|\boldsymbol{v}_{1\tau}\|)d\Gamma \\
&\quad \le \int_{\Gamma_3} \mu|\xi_1 - \xi_2| \|\boldsymbol{v}_1 - \boldsymbol{v}_2\| d\Gamma \le c_0 \|\mu\|_{L^\infty(\Gamma_3)} \|\xi_1 - \xi_2\|_{L^2(\Gamma_3)} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V
\end{aligned}
$$

for any ξ_1, $\xi_2 \in L^2(\Gamma_3)$ and $\boldsymbol{v}_1$, $\boldsymbol{v}_2 \in V$. This implies that the function φ satisfies condition (2.9) with $X = V$. Moreover, it is easy to see that and the operator $\mathcal{T}$ satisfy condition (2.10) on the spaces $Y = V \times L^2(\Gamma_3)$ and $X = V$. Therefore, we are in a position to use Theorem 2.6 in order to obtain that there exists a unique function $\boldsymbol{u}_\mu \in C([0,T]; V)$ such that

$$
\begin{aligned}
& (A\boldsymbol{u}_\mu(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V + \varphi(\mathcal{T}\boldsymbol{u}_\mu(t), \boldsymbol{v}) - \varphi(\mathcal{T}\boldsymbol{u}_\mu(t), \boldsymbol{u}_\mu(t)) \\
&\qquad \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_\mu(t))_V \quad \forall \boldsymbol{v} \in V,\ t \in [0,T].
\end{aligned} \tag{6.14}
$$

We now substitute the equalities (6.12) and (6.13) in (6.14) to see that $\boldsymbol{u}_\mu$ is the unique solution of Problem $\mathcal{W}_\mu$, which concludes the proof of this step.

Step iii). Proof of the convergence (6.11). Assume now that $\{\mu_n\}$ represents a sequence of functions which satisfies condition (6.5) and, for simplicity, denote $\boldsymbol{u}_{\mu_n} = \boldsymbol{u}_n$, $j_{\mu_n} = j_n$. Let $\boldsymbol{v} \in V$ and $t \in [0,T]$. Then, using (6.10) we find that

$$
\begin{aligned}
& (A\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t))_V + (\mathcal{S}\boldsymbol{u}_n(t), \boldsymbol{v} - \boldsymbol{u}_n(t))_V \\
&\qquad + j_n(\mathcal{R}\boldsymbol{u}_n(t), \boldsymbol{v}) - j_n(\mathcal{R}\boldsymbol{u}_n(t), \boldsymbol{u}_n) \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}_n(t))_V.
\end{aligned} \tag{6.15}
$$

We now use the definitions (6.9), (6.7) to see that

$$
\begin{aligned}
& j_n(\mathcal{R}\boldsymbol{u}_n(t), \boldsymbol{v}) - j_n(\mathcal{R}\boldsymbol{u}_n(t), \boldsymbol{u}_n(t)) \\
&\quad = \int_{\Gamma_3} \mu_n \Big(\int_0^t \|\boldsymbol{u}_{n\tau}(s)\|\, ds \Big) (\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_{n\tau}(t)\|)\, d\Gamma \\
&\quad \le \int_{\Gamma_3} \mu_n \Big(\int_0^t \|\boldsymbol{u}_{n\tau}(s)\|\, ds \Big) \Big| \|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_{n\tau}\| \Big|\, d\Gamma \\
&\quad \le \int_{\Gamma_3} \mu_n \Big(\int_0^t \|\boldsymbol{u}_n(s)\|\, ds \Big) \|\boldsymbol{v} - \boldsymbol{u}_n(t)\|\, d\Gamma \\
&\quad \le \|\mu_n\|_{L^\infty(\Gamma_3)} \Big(\int_0^t \|\boldsymbol{u}_n(s)\|_{L^2(\Gamma_3)^d}\, ds \Big) \|\boldsymbol{v} - \boldsymbol{u}_n(t)\|_{L^2(\Gamma_3)^d}.
\end{aligned}
$$

Therefore, using the trace inequality (4.11) we find that

$$\begin{aligned} & j_n(\mathcal{R}\boldsymbol{u}_n(t),\boldsymbol{v}) - j_n(\mathcal{R}\boldsymbol{u}_n(t),\boldsymbol{u}_n(t)) \\ & \quad \le c_0^2\|\mu_n\|_{L^\infty(\Gamma_3)}\Big(\int_0^t \|\boldsymbol{u}_n(s)\|_V\,ds\Big)\|\boldsymbol{v}-\boldsymbol{u}_n(t)\|_V. \end{aligned} \tag{6.16}$$

We now combine the inequalities (6.15) and (6.16) and denote $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}_n(t)$ to see that

$$\begin{aligned} & (A\boldsymbol{u}_n(t),\boldsymbol{w})_V + (\mathcal{S}\boldsymbol{u}_n(t),\boldsymbol{w})_V \\ & \quad + c_0^2\|\mu_n\|_{L^\infty(\Gamma_3)}\Big(\int_0^t \|\boldsymbol{u}_n(s)\|_V\,ds\Big)\|\boldsymbol{w}\|_V \ge (\boldsymbol{f}(t),\boldsymbol{w})_V \qquad \forall\,\boldsymbol{w}\in V. \end{aligned} \tag{6.17}$$

On the other hand, taking $\boldsymbol{v} = \boldsymbol{0}_V$ in (6.15) and using inequality $j_n(\mathcal{R}\boldsymbol{u}_n(t),\boldsymbol{u}_n(t)) \ge 0$ we obtain that

$$(A\boldsymbol{u}_n(t),\boldsymbol{u}_n(t))_X \le (\mathcal{S}\boldsymbol{u}_n(t),\boldsymbol{u}_n(t))_X + (\boldsymbol{f}(t),\boldsymbol{u}_n(t))_X.$$

Next, since $A\boldsymbol{0}_V = \boldsymbol{0}_V$, $\mathcal{S}\boldsymbol{0}_V(t) = \boldsymbol{0}_V$, using the properties of the operators A and $\mathcal{S}$ we deduce that

$$m_{\mathcal{F}}\|\boldsymbol{u}_n(t)\|_V \le L_{\mathcal{S}}\int_0^t \|\boldsymbol{u}_n(s)\|_V\,ds + \|\boldsymbol{f}(t)\|_V$$

and, after using the Gronwall argument, we obtain that there exists a constant $M > 0$ which does not depend on n and t such that

$$\|\boldsymbol{u}_n(t)\|_V \le M. \tag{6.18}$$

We now combine inequalities (6.17) and (6.18) to see that

$$\begin{aligned} & (A\boldsymbol{u}_n(t),\boldsymbol{w})_V + (\mathcal{S}\boldsymbol{u}_n(t),\boldsymbol{w})_V \\ & \quad + c_0^2 MT\|\mu_n\|_{L^\infty(\Gamma_3)}\|\boldsymbol{w}\|_V \ge (\boldsymbol{f}(t),\boldsymbol{w})_V \qquad \forall\,\boldsymbol{w}\in V. \end{aligned}$$

Then, since $\|\mu_n\|_{L^\infty(\Gamma_3)} \to 0$ we are in a position to use the equivalence between the statements (5.1) and (5.4) in Theorem 5.1 to conclude the proof. □

Remark 6.2. In addition of the mathematical interest in the convergence result in Theorem 6.1 it is important from the mechanical point of view, since it shows that the solution of the frictionless contact problem $\mathcal{W}$ can be approached by the solution of the frictional contact problem $\mathcal{W}_\mu$ for a small coefficient of friction.

7 Conclusions

In this paper we obtained convergence criteria to the solution of two nonlinear equations in a real Hilbert spaces X. Our main results are Theorems 3.1 and 5.1 which characterize the convergence of a sequence to the unique solution of the corresponding equations, in the spaces X and $C([0,T];X)$, respectively. We exploited this theorem to deduce various convergence results. Then, we provided some applications in Solid and Contact Mechanics. These applications show the continuous dependence of the solution with respect to the data and, in addition, they show the link between problems with a different mathematical structure and a different physical meaning. For instance, we proved that the solution of frictional contact problem (which is in a form of a variational inequality) converges to the solution of a frictionless contact problem (which is in a form of a nonlinear equation) as the friction bound vanishes.

The research presented in this manuscript can be developped in further direction. The first one would be to obtain convergence criteria to the solution by relaxing the assumptions (2.1) and (2.2) on the operators A and $\mathcal{S}$, respectively. For instance, it should be interesting to consider the case when A is a pseudomonotone coercive operator and $\mathcal{S}$ is an almost history-dependent operator. Another direction would be to extend Theorems 3.1 and 5.1 to evolutionary equations. The corresponding results could be applied in the sensitivity analysis of various mathematical models which describe the evolution of the mechanical state of an elastic, viscoelastic or viscoplastic body in contact with a foundation. For such models the history-dependent operator appears either in the constitutive law and/or in the boundary conditions, as we already shown in the exemples in Section 6. In this way various convergence results can be obtained and the link between various mathematical models of contact could be established. Finally, it would be interesting to provide computer simulations which validate the corresponding convergence results.

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Mircea Sofonea
Laboratoire de Modélisation Pluridisciplinaire et Simulations,
University of Perpignan Via Domitia,
52 Avenue Paul Alduy,
66860 Perpignan, France
E-mail: `sofonea@univ-perp.fr`