

Dual Compatibility for Measure-Driven Function Space Scales

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Abstract: We address a basic but recurring functional-analytic issue: given a continuous linear operator acting between nested scales of spaces, its dual and the dual of its restriction typically act on non-comparable abstract dual spaces, so that a naive “restriction” statement is ill-posed. While one can, if appropriate conditions are met, resolve this difficulty on a case-by-case basis by introducing concrete identifications, we propose a global and systematic approach based on embedding dual spaces into common distribution spaces via anchor spaces. This method is classical for Lebesgue measure on $\mathbb{R}^n$, where the Schwartz space anchors the embedding of many function spaces into the tempered distributions. Here we extend the same philosophy to a generic Borel measure μ on $\mathbb{R}^n$: with $\mathcal{S}|_\Gamma$ denoting the restricted Schwartz space on $\Gamma = \text{supp}(\mu)$, we obtain canonical realizations of dual spaces in $(\mathcal{S}|_\Gamma)'$ under an appropriate density condition and a mild local finiteness assumption on μ . Our abstract compatibility theorem then applies, in particular, to operators mapping from the classical scale on $\mathbb{R}^n$ (anchored by $\mathcal{S}(\mathbb{R}^n)$) into such a measure-driven scale on Γ (anchored by $\mathcal{S}|_\Gamma$), showing that realized duals agree on their common domain. As a concrete application, we consider the trace operator tr_Γ^s for Bessel potential spaces $H_p^s(\mathbb{R}^n)$ on d -sets.

Keywords: Dual compatibility, anchor spaces, realized duals, restricted Schwartz space, L_p spaces over Borel measures, trace operator, Bessel potential spaces, d -sets.

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1 Introduction and Motivation

Let T be a continuous linear operator acting between two nested scales of spaces,

$$\begin{array}{ccc} F_1 & \hookrightarrow & G_1 \\ T|_{F_1} \downarrow & & \downarrow T \\ F_2 & \hookrightarrow & G_2 \end{array}$$

where $F_1 \hookrightarrow G_1$ and $F_2 \hookrightarrow G_2$ and $T|_{F_1} : F_1 \rightarrow F_2$ denotes the restriction of T to F_1 . Here and throughout, $X \hookrightarrow Y$ means that $X \subset Y$ with continuous inclusion (i.e. the identity map on X is continuous from X into Y).

A common difficulty arises upon taking duals:

$$T' : G'_2 \rightarrow G'_1, \quad (T|_{F_1})' : F'_2 \rightarrow F'_1.$$

These maps have different codomains, and in general it does not even make sense to state that $(T|_{F_1})'$ extends T' or that one can restrict $(T|_{F_1})'$ to G'_2 , because G'_2 and F'_2 are in general non-comparable abstract spaces.

In concrete situations one can sometimes circumvent this by ad hoc identifications, e.g. by realizing dual spaces through explicit pairings or by exploiting additional structure (Hilbert pivots, explicit representation theorems, etc.). However, such case-by-case arguments tend to obscure the functional-analytic mechanism behind the compatibility and must be repeated for each new choice of spaces.

The purpose of this paper is to propose a systematic alternative: embed all relevant dual spaces into common ambient distribution spaces via anchor spaces so that the restriction comparison becomes meaningful and automatic. This strategy is classical for Lebesgue measure on $\mathbb{R}^n$, where the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ provides an anchor and the tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ provide an ambient space in which many duals can be realized. Here we show that, under appropriate hypotheses, the same idea can be carried out for a generic Borel measure μ on $\mathbb{R}^n$ by using the restricted Schwartz space $\mathcal{S}|_\Gamma$ on $\Gamma = \text{supp}(\mu)$ as anchor and its continuous dual $(\mathcal{S}|_\Gamma)'$ as the ambient distribution space on Γ .

Our abstract compatibility theorem is given in Section 2. The measure-theoretic construction of the anchor embedding $\mathcal{S}|_\Gamma \hookrightarrow L_p(\mu)$ is given in Section 3. In Section 4 we apply the abstract scheme to the trace operator mapping Bessel potential spaces $H^s_p(\mathbb{R}^n)$ into function spaces on a d -set Γ : there the key insight is that the realized dual $(\text{tr}^s_\Gamma)^*$ reduces to the restriction of a single, s -independent operator $r' : (\mathcal{S}|_\Gamma)' \rightarrow \mathcal{S}'(\mathbb{R}^n)$, making compatibility across different regularity levels trivial. Additional remarks connecting $(\mathcal{S}|_\Gamma)'$ with the classical identification operator id_Γ of Triebel are collected in Appendix A.

2 Abstract Framework via Anchor Spaces and Realizations

Throughout this paper, for any topological vector space X we write X' for its *continuous dual*, i.e. the space of continuous linear functionals from X into $\mathbb{C}$. If $T : X \rightarrow Y$ is a continuous linear operator between topological vector spaces, we denote by

$$T' : Y' \rightarrow X', \quad T'(y') := y' \circ T,$$

the induced operator on continuous duals. We refer to T' as the *dual operator* (or, equivalently, the *transpose* on continuous duals).

We recall the following basic result and proof.

Proposition 2.1. *Let X, Y be topological vector spaces and let $E : X \hookrightarrow Y$ with dense range, i.e. $\overline{E(X)} = Y$. Then the dual map*

$$E' : Y' \rightarrow X', \quad E'(y') = y' \circ E,$$

is injective. Equivalently, one may identify Y' with a linear subspace of X' via restriction.

Proof. If $E'(y') = 0$, then y' vanishes on $E(X)$. Since $\ker(y')$ is closed in Y and $E(X)$ is dense, y' vanishes on Y , hence $y' = 0$. $\square$

This justifies the following definition and identification:

Definition 2.2. Let Φ and X be topological vector spaces and let $i : \Phi \hookrightarrow X$ with dense range. For $\ell \in X'$, define its *realization in Φ'* by

$$\ell^* := i'(\ell) = \ell|_{\Phi} \in \Phi'.$$

We denote by $X^* \subset \Phi'$ the image $i'(X')$ and identify X' with X^* . And we also call X^* the *realization of X' in Φ'* .

A space Φ as above will be thought of as the *anchor* space for a scale of linear superspaces of it, in the sense that the duals of all those superspaces can be identified with linear subspaces of Φ' , in accordance with the definition above.

Next we state and prove our abstract compatibility theorem for dual operators between two scales as above. The proof uses only standard arguments, but it is useful to have both the statement and the proof for reference.

Theorem 2.3. *Let $\Phi_1, \Phi_2, F_1, F_2, G_1, G_2$ be topological vector spaces and*

$$\Phi_1 \hookrightarrow F_1 \hookrightarrow G_1, \quad \Phi_2 \hookrightarrow F_2 \hookrightarrow G_2.$$

Assume that Φ_1 is dense in both F_1 and G_1 , and that Φ_2 is dense in both F_2 and G_2 .

Let $T : G_1 \rightarrow G_2$ be continuous linear, and assume that its restriction $T|_{F_1} : F_1 \rightarrow F_2$ is also continuous.

Let $F_1^*, G_1^* \subset \Phi_1'$ and $F_2^*, G_2^* \subset \Phi_2'$ be the realizations of F_1', G_1' and F_2', G_2' , respectively, in the sense of Definition 2.2. Define the realized dual operators

$$T^* : G_2^* \rightarrow \Phi_1', \quad (T|_{F_1})^* : F_2^* \rightarrow \Phi_1'$$

by

$$T^*(g_2^*) := (T'(g_2'))^*, \quad (T|_{F_1})^*(f_2^*) := ((T|_{F_1})'(f_2'))^*,$$

where $g_2' \in G_2'$ and $f_2' \in F_2'$ are the unique elements whose realizations are g_2^* and f_2^* , respectively.

Then the restriction identity holds:

$$T^* = (T|_{F_1})^*|_{G_2^*}.$$

Proof. Step 1: G_2^* sits inside F_2^* . Since Φ_2 is dense in G_2 and $\Phi_2 \hookrightarrow F_2 \hookrightarrow G_2$, the image of F_2 in G_2 contains the dense subset Φ_2 , hence F_2 is dense in G_2 . By Proposition 2.1 applied to the embedding $F_2 \hookrightarrow G_2$, the transpose map identifies G_2' with a subspace of F_2' (by restriction). Passing to realizations in Φ_2' , this yields a canonical inclusion

$$G_2^* \subset F_2^*.$$

Thus the expression $(T|_{F_1})^*|_{G_2^*}$ is meaningful.

Step 2: Well-definedness of T^* and $(T|_{F_1})^*$. Take $g_2^* \in G_2^*$. By definition of realization and the density of Φ_2 in G_2 , there is a unique $g_2' \in G_2'$ such that $g_2^* = (g_2')^*$; then $T'(g_2') \in G_1'$ and, since Φ_1 is dense in G_1 , its realization $(T'(g_2'))^* \in \Phi_1'$ is well-defined. This defines $T^*(g_2^*)$.

Similarly, for $f_2^* \in F_2^*$ there is a unique $f_2' \in F_2'$ with $f_2^* = (f_2')^*$ (by density of Φ_2 in F_2), then $(T|_{F_1})'(f_2') \in F_1'$ and, since Φ_1 is dense in F_1 , its realization lies in Φ_1' . This defines $(T|_{F_1})^*(f_2^*)$.

Step 3: Equality on G_2^* . Let $g_2^* \in G_2^*$ and let $g_2' \in G_2'$ be the unique element with $g_2^* = (g_2')^*$. We show that $T^*(g_2^*) = (T|_{F_1})^*(g_2^*)$ in Φ_1' by comparing their values on $\varphi \in \Phi_1$.

On the one hand, by definition of realization,

$$T^*(g_2^*)(\varphi) = (T'(g_2'))^*(\varphi) = T'(g_2')(\varphi) = g_2'(\varphi),$$

where φ is viewed in G_1 via $\Phi_1 \hookrightarrow G_1$. On the other hand,

$$(T|_{F_1})^*(g_2^*)(\varphi) = ((T|_{F_1})'(g_2'))^*(\varphi) = (T|_{F_1})'(g_2')(\varphi) = g_2'((T|_{F_1})\varphi),$$

where now φ is viewed in F_1 via $\Phi_1 \hookrightarrow F_1$. Since $T|_{F_1}$ is the restriction of T to F_1 and $\Phi_1 \subset F_1 \subset G_1$, we have $(T|_{F_1})\varphi = T\varphi$, hence the two expressions coincide for all $\varphi \in \Phi_1$. Therefore the two realized functionals in Φ'_1 are equal, i.e. $T^*(g_2^*) = (T|_{F_1})^*(g_2^*)$. $\square$

Remark 2.4. Theorem 2.3 does not require choosing *pivot* spaces. In applications one may introduce *pivot dual pairs* (e.g. $(L_p(\mu), L_{p'}(\mu))$ for $1 < p < \infty$, or just the pivot space $L_2(\mu)$ when $p = 2$ and then consider a corresponding so-called “Gelfand triple”) to obtain explicit identifications of certain duals and explicit formulae for their action. This is additional structure that can be useful for interpretation but is not logically required for the abstract compatibility mechanism.

3 Embeddings Associated with a Uniformly Locally Finite Measure

Let μ be a Borel measure on $\mathbb{R}^n$ and let $\Gamma = \text{supp}(\mu)$. We follow Triebel [4, Ch. 1] regarding conventions in measure theory. In particular, in calling a set function $\mu : \mathcal{P}(\mathbb{R}^n) \rightarrow [0, \infty]$ satisfying $\mu(\emptyset) = 0$, monotonicity, and countable subadditivity a *measure* (often called an *outer measure* in other texts) on $\mathbb{R}^n$. And then a *Borel measure* stands for a measure for which all Borel sets are μ -measurable (in the sense of Carathéodory).

Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space and define

$$\mathcal{S}|_{\Gamma} := \{\varphi|_{\Gamma} : \varphi \in \mathcal{S}(\mathbb{R}^n)\}.$$

We equip $\mathcal{S}|_{\Gamma}$ with the locally convex topology generated by the seminorms $p_N|_{\Gamma}$, $N \in \mathbb{N}_0$, with

$$p_N|_{\Gamma}(\varphi|_{\Gamma}) := \inf\{p_N(\psi) : \psi \in \mathcal{S}(\mathbb{R}^n), \psi|_{\Gamma} = \varphi|_{\Gamma}\}, \quad \varphi|_{\Gamma} \in \mathcal{S}|_{\Gamma},$$

where, as usual,

$$p_N(\psi) := \sup_{x \in \mathbb{R}^n} (1 + |x|)^N \sum_{|\alpha| \leq N} |\partial^{\alpha} \psi(x)|, \quad \psi \in \mathcal{S}(\mathbb{R}^n).$$

We shall assume a uniform local finiteness hypothesis on our measure μ , so for ease of reference let us state it as follows:

Hypothesis 3.1 (uniformly locally finite measure). There exist $r_0 > 0$ and $c > 0$ such that for all $x \in \Gamma$,

$$\mu(\Gamma \cap B(x, r_0)) \leq c.$$

Of course, assuming such an hypothesis is equivalent of assuming it with “all $x \in \mathbb{R}^n$ ” instead of “all $x \in \Gamma$ ”, for possible different constants r_0 or c .

Lemma 3.2. *Assume Hypothesis 3.1. Let $1 \leq p < \infty$. Then we have for any $N > \frac{n}{p}$ that*

$$\int_{\Gamma} (1 + |x|)^{-Np} d\mu(x) < \infty.$$

Proof. Step 1: Fix $R > 0$ and set

$$s_0 := \frac{r_0}{\sqrt{n}}.$$

Consider the cube $Q := [-R, R]^n$. Each coordinate interval $[-R, R]$ can be covered by

$$m_R := \left\lfloor \frac{2R}{s_0} \right\rfloor + 1$$

intervals of length s_0 , hence Q can be covered by at most $M_R := m_R^n$ axis-parallel cubes $Q_1, \dots, Q_{M_R}$ of side length s_0 . Moreover,

$$M_R = m_R^n \leq \left(\frac{2R}{s_0} + 1 \right)^n \leq C_0(1 + R)^n,$$

for a constant C_0 depending only on n and r_0 (since $s_0 = r_0/\sqrt{n}$).

If $\Gamma \cap Q_j = \emptyset$, then $\mu(\Gamma \cap Q_j) = 0$. Otherwise, choose $\gamma_j \in \Gamma \cap Q_j$. For any $x \in Q_j$ we have

$$|x - \gamma_j| \leq \text{diam}(Q_j) = \sqrt{n} s_0 = r_0,$$

so that $Q_j \subset B(\gamma_j, r_0)$. Hence, by Hypothesis 3.1,

$$\mu(\Gamma \cap Q_j) \leq \mu(\Gamma \cap B(\gamma_j, r_0)) \leq c.$$

Summing over j and using $B(0, R) \subset Q$ gives

$$\mu(\Gamma \cap B(0, R)) \leq \mu(\Gamma \cap Q) \leq \sum_{j=1}^{M_R} \mu(\Gamma \cap Q_j) \leq M_R c \leq C_1(1 + R)^n,$$

with $C_1 := c C_0$.

Step 2: Decompose Γ into dyadic regions:

$$\Gamma = (\Gamma \cap B(0, 1)) \cup \bigcup_{k=0}^{\infty} \left(\Gamma \cap \{2^k \leq |x| < 2^{k+1}\} \right).$$

Then

$$\int_{\Gamma} (1 + |x|)^{-Np} d\mu(x) \leq \mu(\Gamma \cap B(0, 1)) + \sum_{k=0}^{\infty} \int_{\Gamma \cap \{2^k \leq |x| < 2^{k+1}\}} (1 + |x|)^{-Np} d\mu(x).$$

On the region $\{2^k \leq |x| < 2^{k+1}\}$ we have $(1 + |x|)^{-Np} \leq (2^k)^{-Np}$, hence

$$\int_{\Gamma \cap \{2^k \leq |x| < 2^{k+1}\}} (1 + |x|)^{-Np} d\mu(x) \leq 2^{-kNp} \mu(\Gamma \cap B(0, 2^{k+1})).$$

Using Step 1,

$$\mu(\Gamma \cap B(0, 2^{k+1})) \leq C_1 (1 + 2^{k+1})^n \leq C_2 2^{kn}$$

for a constant C_2 depending only on n , r_0 and c . Therefore

$$\int_{\Gamma} (1 + |x|)^{-Np} d\mu(x) \leq \mu(\Gamma \cap B(0, 1)) + C_2 \sum_{k=0}^{\infty} 2^{-k(Np-n)}.$$

The geometric series converges whenever $Np > n$, from which we conclude. $\square$

Proposition 3.3. *Assume Hypothesis 3.1. Let $1 \leq p < \infty$. Then $\mathcal{S}|_{\Gamma} \hookrightarrow L_p(\mu)$.*

Proof. Let $\varphi|_{\Gamma} \in \mathcal{S}|_{\Gamma}$, and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be any extension with $\psi|_{\Gamma} = \varphi|_{\Gamma}$. Then $|\psi(x)| \leq p_N(\psi)(1 + |x|)^{-N}$ for all $x \in \mathbb{R}^n$, hence

$$\|\varphi|_{\Gamma}\|_{L_p(\mu)}^p = \int_{\Gamma} |\psi(x)|^p d\mu(x) \leq p_N(\psi)^p \int_{\Gamma} (1 + |x|)^{-Np} d\mu(x), \quad N \in \mathbb{N}_0.$$

Choosing the N greater than $\frac{n}{p}$ and taking p -th roots and the infimum over all extensions ψ yields, with the help of Lemma 3.2,

$$\|\varphi|_{\Gamma}\|_{L_p(\mu)} \leq C^{1/p} p_N|_{\Gamma}(\varphi|_{\Gamma}), \quad \text{with } C := \int_{\Gamma} (1 + |x|)^{-Np} d\mu(x) < \infty,$$

which proves embedding and continuity. $\square$

Corollary 3.4. *Assume Hypothesis 3.1, let $1 \leq p < \infty$, and assume also that $\mathcal{S}|_{\Gamma}$ is dense in $L_p(\mu)$. Then $(L_p(\mu))'$ can be realized within $(\mathcal{S}|_{\Gamma})'$ in the sense explained in Definition 2.2.*

Proof. By Proposition 3.3, $\mathcal{S}|_\Gamma \hookrightarrow L_p(\mu)$. By assumption, $\mathcal{S}|_\Gamma$ is dense in $L_p(\mu)$. Therefore we have all that is needed to apply Definition 2.2. $\square$

Remark 3.5. In the setting above, the Borel measure μ is assumed to be *uniformly locally finite* (Hypothesis 3.1) on $\mathbb{R}^n$, therefore also *locally finite* there. If, in addition, μ is *Borel regular* in Triebel's sense, meaning that for every $A \subset \mathbb{R}^n$ there exists a Borel set $B \subset \mathbb{R}^n$ with

$$A \subset B \quad \text{and} \quad \mu(A) = \mu(B),$$

then Triebel's Proposition 1.4 in [4] yields that μ is a *Radon measure* on $\mathbb{R}^n$.

For Radon measures, one has the standard approximation of measurable sets by compact and open sets. This allows one to repeat the density argument used by Triebel (see the proof of [4, Theorem 3.8]) to show that $\mathcal{D}(\mathbb{R}^n)|_\Gamma$ (and hence $\mathcal{S}|_\Gamma$) is dense in $L_p(\mu)$.

That is to say that Corollary 3.4 applies in the case of Borel regular and uniformly locally finite measures μ on $\mathbb{R}^n$. This includes, for example, the classical Lebesgue measure on $\mathbb{R}^n$. It also includes, more generally, Borel regular measures μ such that there exist $d \in [0, n]$ and $c_2 > 0$ with

$$\mu(\Gamma \cap B(\gamma, r)) \leq c_2 r^d, \quad \gamma \in \Gamma, \quad 0 < r < 1.$$

Notice that if, in addition, the corresponding lower estimate holds, we recover the case of measures supported on d -sets in the sense of Triebel [3] (see also [2]). In such cases those measures are essentially the same as $\mathcal{H}^d|_\Gamma$, see [4, Theorem 3.4].

Remark 3.6 (Integral representation of the realized dual). By Corollary 3.4, every $\Lambda \in (L_p(\mu))'$ gives rise to a continuous linear functional on $\mathcal{S}|_\Gamma$, namely $\Lambda_{\mathcal{S}|_\Gamma}$. Thus $(L_p(\mu))'$ is realized abstractly inside $(\mathcal{S}|_\Gamma)'$ by restriction.

Writing p' for the conjugate exponent of p , with the convention $p' = \infty$ when $p = 1$, since μ is σ -finite under the standing assumptions, the classical duality theorem yields that for each $\Lambda \in (L_p(\mu))'$ there exists $g \in L_{p'}(\mu)$ such that

$$\Lambda(f) = \int_\Gamma f g \, d\mu, \quad f \in L_p(\mu),$$

this g being uniquely determined as an element of $L_{p'}(\mu)$. In particular, the restriction of Λ to $\mathcal{S}|_\Gamma$ can be written concretely as

$$\varphi \longmapsto \int_\Gamma \varphi g \, d\mu, \quad \varphi \in \mathcal{S}|_\Gamma.$$

Further connections with the classical identification operator viewpoint in $\mathcal{S}'(\mathbb{R}^n)$ are collected in Appendix A, part I.

4 Application: Trace of Bessel Potentials on d -Sets

We now apply the abstract compatibility theorem to trace operators on d -sets, using trace spaces defined directly via the trace operator following [1].

4.1 d -sets and the trace operator

We recall from [4, Definition 3.1]:

Definition 4.1 (d -set). Let $n \in \mathbb{N}$, let Γ be a set in $\mathbb{R}^n$, and let $0 \leq d \leq n$. Then Γ is called a d -set if there exists a Borel measure μ in $\mathbb{R}^n$ with the following two properties:

- (i) $\text{supp } \mu = \Gamma$;
- (ii) there exist constants $c_1 > 0$ and $c_2 > 0$ such that for all $\gamma \in \Gamma$ and all r with $0 < r < 1$,

$$c_1 r^d \leq \mu(B(\gamma, r) \cap \Gamma) \leq c_2 r^d.$$

As noted in Remark 3.5, a d -set measure μ satisfies Hypothesis 3.1 and is Radon, so that all the conclusions of Section 3 apply.

Throughout this section we fix a d -set $\Gamma \subset \mathbb{R}^n$ with $0 < d < n$, $1 < p < \infty$, and write $L_p(\Gamma)$ with the same meaning of $L_p(\mu)$, for the corresponding measure μ . Moreover, spaces denoted by a letter B or a letter H will stand for Besov spaces and Bessel-potential spaces, respectively, for the appropriate parameters visible in each case (cf. [3], where $F_{p,2}^s(\mathbb{R}^n)$ there coincides, up to equivalence of norms, with our $H_p^s(\mathbb{R}^n)$ here).

By [1, Proposition 6.2] (based on [4, §18]), the restriction map $\varphi \mapsto \varphi|_\Gamma$ on $\mathcal{S}(\mathbb{R}^n)$ extends to a continuous surjective operator

$$\text{tr}_\Gamma : B_{p,1}^{\frac{n-d}{p}}(\mathbb{R}^n) \rightarrow L_p(\Gamma), \quad (4.1)$$

and there exist constants $c_1, c_2 > 0$ such that

$$c_1 \|f\|_{L_p(\Gamma)} \leq \inf \left\{ \|g\|_{B_{p,1}^{\frac{n-d}{p}}(\mathbb{R}^n)} : \text{tr}_\Gamma g = f \right\} \leq c_2 \|f\|_{L_p(\Gamma)}, \quad f \in L_p(\Gamma).$$

In other words, tr_Γ induces a linear and topological isomorphism between the quotient space $B_{p,1}^{\frac{n-d}{p}}(\mathbb{R}^n) / \ker(\text{tr}_\Gamma)$ and $L_p(\Gamma)$.

4.2 Trace spaces on Γ

For $s > 0$ we have $H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n) \hookrightarrow B_{p,1}^{\frac{n-d}{p}}(\mathbb{R}^n)$ (see [3, 2.3.2]), so tr_Γ is well defined and continuous on $H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n)$.

Definition 4.2 (Trace space on Γ). For $s > 0$, define

$$\mathbb{H}_p^s(\Gamma) := \text{tr}_\Gamma(H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n)),$$

endowed with the norm

$$\|f\|_{\mathbb{H}_p^s(\Gamma)} := \inf \left\{ \|g\|_{H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n)} : g \in H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n), \text{tr}_\Gamma g = f \right\}.$$

inherited from the quotient norm in $H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n) / \ker(\text{tr}_\Gamma|_{H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n)})$, to which $\mathbb{H}_p^s(\Gamma)$ is linearly isomorphic.

This is the space $\mathbb{H}_{p,0}^s(\Gamma)$ of [1, Definition 6.3]. With this norm, $\mathbb{H}_p^s(\Gamma)$ is a Banach space and the restricted trace

$$\text{tr}_\Gamma^s := \text{tr}_\Gamma|_{H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n)} : H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n) \rightarrow \mathbb{H}_p^s(\Gamma)$$

is continuous and surjective by construction.

4.3 Embedding chains and density

Fix $s' > s > 0$. On the source side,

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow H_p^{s'+\frac{n-d}{p}}(\mathbb{R}^n) \hookrightarrow H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n) \hookrightarrow B_{p,1}^{\frac{n-d}{p}}(\mathbb{R}^n),$$

with $\mathcal{S}(\mathbb{R}^n)$ dense in the other three spaces of this chain (see [3, 2.3.3]).

On the target side,

$$\mathcal{S}|_\Gamma \hookrightarrow \mathbb{H}_p^{s'}(\Gamma) \hookrightarrow \mathbb{H}_p^s(\Gamma) \hookrightarrow L_p(\Gamma). \quad (4.2)$$

These follow from the corresponding embeddings on $\mathbb{R}^n$ via the quotient-norm constructions: for instance, $H_p^{s'+\frac{n-d}{p}}(\mathbb{R}^n) \hookrightarrow H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n)$ implies $\mathbb{H}_p^{s'}(\Gamma) \hookrightarrow \mathbb{H}_p^s(\Gamma)$; and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H_p^{s'+\frac{n-d}{p}}(\mathbb{R}^n)$ implies $\mathcal{S}|_\Gamma \hookrightarrow \mathbb{H}_p^{s'}(\Gamma)$, since for any $\varphi|_\Gamma \in \mathcal{S}|_\Gamma$,

$$\|\varphi|_\Gamma\|_{\mathbb{H}_p^{s'}(\Gamma)} \leq \inf_{\substack{\psi|_{\Gamma=\varphi|_\Gamma} \\ \psi \in \mathcal{S}(\mathbb{R}^n)}} \|\psi\|_{H_p^{s'+\frac{n-d}{p}}(\mathbb{R}^n)} \leq C p_N|_\Gamma(\varphi|_\Gamma)$$

for an appropriate seminorm $p_N|_\Gamma$ of $\mathcal{S}|_\Gamma$; for the last embedding $\mathbb{H}_p^s(\Gamma) \hookrightarrow L_p(\Gamma)$, besides $H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n) \hookrightarrow B_{p,1}^{\frac{n-d}{p}}(\mathbb{R}^n)$ we have also to use the left estimate stated after (4.1).

We also easily get from the density properties on the source side and the surjectivity and continuity of $\text{tr}_\Gamma^{s'}$, tr_Γ^s and tr_Γ that $\mathcal{S}|_\Gamma$ is dense in all the other spaces shown in (4.2): e.g., for any $f \in \mathbb{H}_p^s(\Gamma)$ there exists $g \in H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n)$ with $\text{tr}_\Gamma^s(g) = f$; approximating g by a sequence $(\varphi_k)_k \subset \mathcal{S}(\mathbb{R}^n)$ in $H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n)$ immediately gives that $\varphi_k|_\Gamma \rightarrow f$ in $\mathbb{H}_p^s(\Gamma)$.

4.4 Compatibility of duals for the trace

The key observation is that the restriction of tr_Γ^s to $\mathcal{S}(\mathbb{R}^n)$ is simply the pointwise restriction map $r : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}|_\Gamma$, $r(\varphi) = \varphi|_\Gamma$, regardless of $s > 0$. More precisely, denoting the embedding from (4.2) by $j : \mathcal{S}|_\Gamma \hookrightarrow \mathbb{H}_p^s(\Gamma)$, one has $\text{tr}_\Gamma^s(\varphi) = j(\varphi|_\Gamma)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

We apply Theorem 2.3 with the choice

$$\begin{aligned}\Phi_1 &= F_1 = \mathcal{S}(\mathbb{R}^n), & G_1 &= H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n), \\ \Phi_2 &= F_2 = \mathcal{S}|_\Gamma, & G_2 &= \mathbb{H}_p^s(\Gamma),\end{aligned}$$

and $T = \text{tr}_\Gamma^s : G_1 \rightarrow G_2$. The restriction $T|_{F_1} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}|_\Gamma$ coincides with r , which is continuous since $p_N|_\Gamma(r(\varphi)) \leq p_N(\varphi)$ by definition. All density and embedding conditions have been verified in the previous subsection. Since $\Phi_1 = F_1$ and $\Phi_2 = F_2$, the realized dual $(T|_{F_1})^*$ reduces to

$$r' : (\mathcal{S}|_\Gamma)' \rightarrow \mathcal{S}'(\mathbb{R}^n),$$

and Theorem 2.3 yields:

Corollary 4.3. *Let Γ be a d -set in $\mathbb{R}^n$ with $0 < d < n$, let $1 < p < \infty$, and let $s > 0$. Then*

$$(\text{tr}_\Gamma^s)^* = r'|_{(\mathbb{H}_p^s(\Gamma))^*},$$

where $(\mathbb{H}_p^s(\Gamma))^* \subset (\mathcal{S}|_\Gamma)'$ is the realization of $(\mathbb{H}_p^s(\Gamma))'$ in the sense of Definition 2.2.

Since r' is independent of s , the compatibility between different regularity levels is immediate:

Corollary 4.4. *Under the same hypotheses of the previous corollary, for any $s' > s > 0$,*

$$(\text{tr}_\Gamma^s)^* = (\text{tr}_\Gamma^{s'})^*|_{(\mathbb{H}_p^s(\Gamma))^*}.$$

Proof. By (4.2), $\mathbb{H}_p^{s'}(\Gamma) \hookrightarrow \mathbb{H}_p^s(\Gamma)$ with $\mathcal{S}|_\Gamma$ dense in both, hence $(\mathbb{H}_p^s(\Gamma))^* \subset (\mathbb{H}_p^{s'}(\Gamma))^*$ (cf. Step 1 in the proof of Theorem 2.3). Both $(\text{tr}_\Gamma^s)^*$ and $(\text{tr}_\Gamma^{s'})^*$ equal r' restricted to their respective domains by Corollary 4.3, so the restriction identity follows. $\square$

Remark 4.5. Since $\mathbb{H}_p^s(\Gamma) \hookrightarrow L_p(\Gamma)$ with dense range, Proposition 2.1 gives $(L_p(\Gamma))^* \subset (\mathbb{H}_p^s(\Gamma))^*$. In Appendix A, Part II, we show that, restricted to elements of $(L_p(\Gamma))^*$, $(\text{tr}_\Gamma^s)^*$ coincides with the classical identification operator id_Γ of Triebel (see the Appendix for the precise formulation).

Appendices

A The dual of the trace and the identification operator

This appendix connects the intrinsic distribution space $(\mathcal{S}|_\Gamma)'$ to the classical identification operator id_Γ of Triebel (cf. [5, §9.2] and [4, §18, Eqn. (18.6)]), and then shows how the dual of the trace extends id_Γ in the sense of [1, Remark 6.16].

Part I: Factorization of id_Γ (general setting)

We assume the hypotheses of Corollary 3.4: μ is a Borel measure on $\mathbb{R}^n$ satisfying Hypothesis 3.1, $\Gamma = \text{supp}(\mu)$, $1 \leq p < \infty$, and $\mathcal{S}|_\Gamma$ is dense in $L_p(\mu)$. We write p' for the conjugate exponent of p ($p' = \infty$ when $p = 1$).

Consider the two canonical maps:

- The inclusion $i : \mathcal{S}|_\Gamma \hookrightarrow L_p(\mu)$ with dense range (Proposition 3.3 and the density assumption). Its dual

$$i' : (L_p(\mu))' \rightarrow (\mathcal{S}|_\Gamma)'$$

is injective by Proposition 2.1.

- The restriction map $r : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}|_\Gamma$, $r(\varphi) = \varphi|_\Gamma$, which is continuous and surjective. Its dual

$$r' : (\mathcal{S}|_\Gamma)' \rightarrow \mathcal{S}'(\mathbb{R}^n), \quad r'(u) = u \circ r,$$

is injective by Proposition 2.1.

By Remark 3.6, every $\Lambda \in (L_p(\mu))'$ is represented by a unique $g \in L_{p'}(\mu)$ via $\Lambda(f) = \int_\Gamma f g d\mu$. We write Λ_g for this functional and $I_{p'} : L_{p'}(\mu) \rightarrow (L_p(\mu))'$ for the canonical isometric isomorphism with $I_{p'}(g) = \Lambda_g$.

Proposition A.1. *Under the assumptions above, for any $g \in L_{p'}(\mu)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$,*

$$((r' \circ i')(\Lambda_g))(\varphi) = \int_{\Gamma} \varphi(\gamma) g(\gamma) \mu(d\gamma).$$

Proof. We compute:

$$((r' \circ i')(\Lambda_g))(\varphi) = i'(\Lambda_g)(r(\varphi)) = i'(\Lambda_g)(\varphi|_{\Gamma}) = \Lambda_g(i(\varphi|_{\Gamma})) = \int_{\Gamma} \varphi(\gamma) g(\gamma) \mu(d\gamma),$$

where the last step uses that i is the inclusion map. $\square$

The right-hand side is precisely how one defines the classical *identification operator*:

$$(\text{id}_{\Gamma} g)(\varphi) := \int_{\Gamma} \varphi(\gamma) g(\gamma) \mu(d\gamma), \quad g \in L_{p'}(\mu), \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Hence Proposition A.1 gives the factorization

$$\text{id}_{\Gamma} = r' \circ i' \circ I_{p'}.$$

The introduction of $(\mathcal{S}|_{\Gamma})'$ thus separates the identification operator into two natural steps: an intrinsic realization on Γ (via i') and an injection into $\mathcal{S}'(\mathbb{R}^n)$ (via r').

Part II: The dual of the trace extends id_{Γ} (specialization to d -sets)

We now place ourselves in the setting of Section 4: Γ is a d -set with $0 < d < n$, $1 < p < \infty$, and $s > 0$.

Since $\mathbb{H}_p^s(\Gamma) \hookrightarrow L_p(\Gamma)$ with $\mathcal{S}|_{\Gamma}$ dense in both (cf. Subsection 4.3), Proposition 2.1 yields the canonical inclusion $(L_p(\Gamma))^* \subset (\mathbb{H}_p^s(\Gamma))^* \subset (\mathcal{S}|_{\Gamma})'$.

Let $g \in L_{p'}(\Gamma)$ and let $\Lambda_g = I_{p'}(g) \in (L_p(\Gamma))'$ be the corresponding functional. Its realization in $(\mathcal{S}|_{\Gamma})'$ is $\Lambda_g^* = i'(\Lambda_g)$, where $i : \mathcal{S}|_{\Gamma} \hookrightarrow L_p(\Gamma)$ is the inclusion from Part I. Since $\Lambda_g^* \in (L_p(\Gamma))^* \subset (\mathbb{H}_p^s(\Gamma))^*$, Corollary 4.3 gives

$$(\text{tr}_{\Gamma}^s)^*(\Lambda_g^*) = r'(\Lambda_g^*) = r'(i'(\Lambda_g)) = (r' \circ i' \circ I_{p'})(g) = \text{id}_{\Gamma}(g),$$

the last equality being the factorization from Part I. This confirms [1, Remark 6.16]: the dual of the trace extends the identification operator.

Since the computation is manifestly independent of s , the identity $(\text{tr}_{\Gamma}^s)^*(\Lambda_g^*) = \text{id}_{\Gamma}(g)$ holds for every $s > 0$. Note that the “for every $s > 0$ ” aspect of this identity is also consistent with Corollary 4.4, which guarantees that once the identity is established for a single value of s , it automatically extends to all others.

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