

Generic Contractivity for a Class of Nonlinear Mappings

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Abstract: In a previous paper of ours we studied a class of nonlinear self-mappings of a complete metric space endowed with a natural metric. Using the notion of porosity, we showed that most elements of this class possess a unique fixed point which attracts all iterates uniformly. In the present paper we prove a variant of this result by showing that most mappings in this class are quasi-contractive.

Keywords: Complete metric space, fixed point, porosity, quasi-contractive mapping, uniform convergence.

MSC2010: 47H09, 47H10, 54E50

Dedicated to the memory of Professor Francesco S. De Blasi

1 Introduction and Preliminaries

Let K be a nonempty, bounded, closed and convex subset of a Banach space $(X, \|\cdot\|)$. We consider the topological space $K \subset X$ with the relative topology induced by the norm $\|\cdot\|$. Set

$$\text{diam}(K) = \sup\{\|x - y\| : x, y \in K\}.$$

Denote by \mathcal{A} the set of all continuous mappings $A : K \rightarrow K$ which have the following property:

(P1) For each $\epsilon > 0$, there exists a point $x_\epsilon \in K$ such that

$$\|Ax - x_\epsilon\| \leq \|x - x_\epsilon\| + \epsilon \text{ for all } x \in K.$$

For each $A, B \in \mathcal{A}$, set

$$d(A, B) = \sup\{\|A(x) - B(x)\| : x \in K\}.$$

It is clear that the metric space (\mathcal{A}, d) is complete.

The main result of our paper involves the notion of porosity which we now recall [1, 2, 5, 7, 8].

Let (Y, d) be a complete metric space. We denote by $B(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. Fix a real number $q \geq 1$.

We say that a subset $E \subset Y$ is (q) -porous (with respect to the metric d) if there exist numbers $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists $z \in Y$ for which

$$B(z, \alpha r^q) \subset B(y, r) \setminus E.$$

We say that a subset of the space Y is (σ, q) -porous (with respect to d) if it is a countable union of (q) -porous subsets of Y .

Since (q) -porous sets are nowhere dense, all (σ, q) -porous sets are of the first Baire category. To point out the difference between (q) -porous and nowhere dense sets, note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there are a point $z \in Y$ and a number $s > 0$ such that $B(z, s) \subset B(y, r) \setminus E$. If, however, E is also (q) -porous, then for small enough r we can choose $s = \alpha r^q$, where $\alpha \in (0, 1)$ is a constant which only depends on E .

In [6] we established the following result.

Theorem 1.1. *There exists a set $\mathcal{F}_0 \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}_0$ is $(\sigma, 1)$ -porous in (\mathcal{A}, d) and each $A \in \mathcal{F}_0$ has the following three properties:*

(i) *There exists a unique fixed point $x_A \in K$ such that*

$$A^n(x) \rightarrow x_A \text{ as } n \rightarrow \infty, \text{ uniformly for all } x \in K.$$

(ii)

$$\|A(x) - x_A\| \leq \|x - x_A\| \text{ for all } x \in K.$$

(iii) *For each $\epsilon > 0$, there exist a natural number n and a real number $\delta > 0$ such that for each integer $p \geq n$, each point $x \in K$ and each mapping $B \in \mathcal{A}$ satisfying $d(B, A) \leq \delta$, we have $\|B^p(x) - x_A\| \leq \epsilon$.*

In the proof of Theorem 1.1 in [6] we used the following result.

Proposition 1.2. *Let $A \in \mathcal{A}$ and $\epsilon \in (0, 1)$ be given. Then there exist a point $\bar{x} \in K$ and a mapping $B \in \mathcal{A}$ such that*

$$d(A, B) \leq \epsilon$$

and

$$\|\bar{x} - B(x)\| \leq \|\bar{x} - x\| \text{ for all } x \in K.$$

2 Main Result

We are now ready to state the main result of our note. Its proof will be given in the next section.

Theorem 2.1. *There exists a set $\mathcal{F} \subset \mathcal{A}$ such that its complement $\mathcal{A} \setminus \mathcal{F}$ is a $(\sigma, 2)$ -porous set in (\mathcal{A}, d) and such that for each mapping $B \in \mathcal{F}$, there exist a unique point $x_B \in K$ satisfying*

$$B(x_B) = x_B,$$

and a decreasing function $\phi : [0, \infty) \rightarrow [0, 1]$ for which

$$\phi(t) < 1 \text{ for all } t > 0$$

and

$$\|B(x) - x_B\| \leq \phi(\|x - x_B\|)\|x - x_B\| \text{ for all } x \in K. \quad (2.1)$$

Note that each mapping B satisfying (2.1) also satisfies all three assertions of Theorem 1.1 (cf. [3, 4]). We say the such mappings are quasi-contractive.

3 Proof of Theorem 2.1

Let the set \mathcal{F}_0 be as guaranteed by Theorem 1.1.

Let n be a natural number and set

$$\alpha_n = 96^{-1}n^{-1}(\text{diam}(K) + 1)^{-3}. \quad (3.1)$$

Denote by \mathcal{F}_n the set of all $B \in \mathcal{A}$ for which the following property holds:

(P2) For each $z \in K$ satisfying $B(z) = z$,

$$\sup\{\|B(x) - z\|\|z - x\|^{-1} : x \in K \text{ and } \|x - z\| \geq n^{-1}(\text{diam}(K) + 1)^{-1}\} < 1.$$

It is not difficult to see that the following result holds.

Proposition 3.1. *Let $B \in \bigcap_{n=0}^{\infty} \mathcal{F}_n$. Then there exists a unique fixed point $x_B \in K$ such that*

$$B(x_B) = x_B,$$

and a decreasing function $\phi : [0, \infty) \rightarrow [0, 1]$ such that

$$\phi(t) < 1 \text{ for all } t > 0$$

and

$$\|B(x) - x_B\| \leq \phi(\|x - x_B\|)\|x - x_B\| \text{ for all } x \in K.$$

Proposition 3.2. *For each natural number n , the set $\mathcal{A} \setminus \mathcal{F}_n$ is (2) -porous.*

Proof. Assume that $A \in \mathcal{A}$ and $r \in (0, 1]$ are given. By Proposition 1.2, there exist a mapping $A_0 \in \mathcal{A}$ and a point $x_* \in K$ such that

$$d(A, A_0) \leq r/8, \quad (3.2)$$

$$A_0(x_*) = x_* \quad (3.3)$$

and

$$\|A_0(x) - x_*\| \leq \|x - x_*\| \text{ for all } x \in K. \quad (3.4)$$

Set

$$\gamma = (r/4)(\text{diam}(K) + 1)^{-1} \quad (3.5)$$

and

$$A_\gamma(x) = (1 - \gamma)A_0(x) + \gamma x_*, \quad x \in K. \quad (3.6)$$

By (3.3) and (3.6),

$$A_\gamma(x_*) = x_*. \quad (3.7)$$

By (3.4), (3.6) and (3.7), for all $x \in X$,

$$\|A_\gamma(x) - x_*\| = \|(1 - \gamma)A_0(x) - (1 - \gamma)x_*\| \leq (1 - \gamma)\|x - x_*\|. \quad (3.8)$$

Thus $A_\gamma \in \mathcal{A}$. By (3.2) and (3.6),

$$\begin{aligned} d(A_\gamma, A) &\leq d(A_\gamma, A_0) + d(A_0, A) \\ &\leq \sup\{\gamma\|x_* - A_0(x)\| : x \in K\} + r/8 \leq \gamma \text{diam}(K) + r/8. \end{aligned} \quad (3.9)$$

Assume now that a mapping $B \in \mathcal{A}$ satisfies

$$d(A_\gamma, B) \leq \alpha_n r^2. \quad (3.10)$$

By (3.10), (3.9), (3.5) and (3.1),

$$d(B, A) \leq d(B, A_\gamma) + d(A_\gamma, A) \leq \alpha r + r/4 + r/8 < r. \quad (3.11)$$

Assume that

$$x_B \in K \text{ and } B(x_B) = x_B. \quad (3.12)$$

By (3.8), (3.10) and (3.12),

$$\begin{aligned} \|x_* - x_B\| &= \|x_* - B(x_B)\| \leq \|x_* - A_\gamma(x_B)\| + \|A_\gamma(x_B) - B(x_B)\| \\ &\leq (1 - \gamma)\|x_* - x_B\| + \alpha_n r^2, \end{aligned}$$

$$\gamma \|x_* - x_B\| \leq \alpha_n r^2.$$

When combined with (3.5), this implies that

$$\|x_* - x_B\| \leq \alpha_n r^2 \gamma^{-1} \leq 4\alpha_n (\text{diam}(K) + 1)r. \quad (3.13)$$

Let $x \in K$. By (3.8), (3.10) and (3.13),

$$\begin{aligned} \|x_B - B(x)\| &\leq \|x_B - x_*\| + \|x_* - A_\gamma(x)\| + \|A_\gamma x - B(x)\| \\ &\leq 4(\text{diam}(K) + 1)\alpha_n r + (1 - \gamma)\|x_* - x\| + \alpha_n r^2 \\ &\leq 8(\text{diam}(K) + 1)\alpha_n r + (1 - \gamma)(\|x_* - x_B\| + \|x_B - x\|) \\ &\leq 12(\text{diam}(K) + 1)\alpha_n r + (1 - \gamma)\|x_B - x\|. \end{aligned} \quad (3.14)$$

Assume that

$$\|x - x_B\| \geq n^{-1}(\text{diam}(K) + 1)^{-1}. \quad (3.15)$$

By (3.1), (3.5), (3.14) and (3.15),

$$\begin{aligned} \|x - x_B\|^{-1} \|x_B - B(x)\| &\leq 12(\text{diam}(K) + 1)\alpha_n r (n(\text{diam}(K) + 1)) + 1 - \gamma \\ &\leq 12n(\text{diam}(K) + 1)^2 \alpha_n r + 1 - r(4(\text{diam}(K) + 1))^{-1} \\ &= 1 - r((4(\text{diam}(K) + 1))^{-1} - 12n(\text{diam}(K) + 1)^2 \alpha_n) \\ &= 1 - r8^{-1}(\text{diam}(K) + 1)^{-1}. \end{aligned} \quad (3.16)$$

Since (3.16) holds for any $x_B \in K$ satisfying (3.12) and any $x \in K$ satisfying (3.15), we have

$$B \in \mathcal{F}_n.$$

Together with (3.10) and (3.11) this implies that

$$\{B \in \mathcal{A} : d(B, A_\gamma) \leq \alpha_n r^2\} \subset \{B \in \mathcal{A} : d(B, A) \leq r\} \cap \mathcal{F}_n.$$

Proposition 3.2 is proved. \square

Now set

$$\mathcal{F} = \bigcap_{n=0}^{\infty} \mathcal{F}_n.$$

With this set \mathcal{F} , Theorem 2.1 is now seen to follow from Theorem 1.1 and Propositions 3.1 and 3.2.

Acknowledgement. The research of the first author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion General Research Fund.

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