

# Nonlinear periodic problems superlinear at $+\infty$ and sublinear at $-\infty$

Sergiu Aizicovici, Nikolaos S. Papageorgiou and Vasile Staicu

**Abstract:** We consider a nonlinear periodic problem driven by a nonlinear, nonhomogeneous differential operator with a reaction which exhibits an asymmetric growth at  $+\infty$  and at  $-\infty$ . It is  $(p - 1)$ -superlinear near  $+\infty$  and  $(p - 1)$ -sublinear near  $-\infty$ . A particular case of our problem is that of periodic equations with the scalar  $p$ -Laplacian and an asymmetric nonlinearity.

Using variational methods and Morse theory, we prove the existence of at least three nontrivial solutions.

**Keywords:** Asymmetric reaction, nonhomogeneous differential operator, C-condition, critical groups, homotopy equivalent, mountain pass theorem.

**MSC2010:** 34B15, 34B18, 34C25, 58E05.

*Dedicated to the memory of Professor Francesco S. De Blasi*

## 1 Introduction

In this paper we examine the following nonlinear periodic problem

$$\begin{cases} -(a(|u'(t)|)u'(t))' = f(t, u(t)) \text{ a.e. on } T := [0, b] \\ u(0) = u(b), u'(0) = u'(b). \end{cases} \quad (1.1)$$

In the above problem, the differential operator is in general nonhomogeneous and incorporates as a special case the scalar  $p$ -Laplacian. The reaction  $f(t, x)$  is a Carathéodory function (i.e., for all  $x \in \mathbb{R}$ ,  $t \rightarrow f(t, x)$  is measurable and for a.a.  $t \in T$ ,  $x \rightarrow f(t, x)$  is continuous).

Our aim is to study the existence and multiplicity of solutions when the reaction  $f(t, x)$  exhibits an asymmetric behavior near  $+\infty$  and near  $-\infty$  and it is  $(p - 1)$ -superlinear in the positive direction (i.e., as  $x \rightarrow +\infty$ ) and  $(p - 1)$ -sublinear in the negative direction (i.e., as  $x \rightarrow -\infty$ ).

Multiplicity results for nonlinear periodic problems driven by the scalar  $p$ -Laplacian were proved by Aizicovici-Papageorgiou-Staicu [1], [3], [4], Del Pino-Manasevich-Murua [8], Gasinski [11], Gasinski-Papageorgiou [13], and Yang [20].

In all the above mentioned papers, the reaction of the problem exhibits a similar growth near  $+\infty$  and  $-\infty$ . Recently, Aizicovici-Papageorgiou-Staicu [5], [6], studied periodic eigenvalue problems driven by a nonhomogeneous differential operator.

Equations with an asymmetric reaction were studied in the context of semilinear (i.e.,  $p = 2$ ) Neumann problems. We mention the works of Dong [9], de Figueiredo-Ruf [7], Perera [17] and Villegas [19]. Of these, only Perera [17] proves a multiplicity result.

Our approach uses variational methods based on critical point theory together with suitable truncation techniques and Morse theory (critical groups).

## 2 Mathematical Background and Hypotheses

Let  $(X, \|\cdot\|)$  be a Banach space and  $(X^*, \|\cdot\|_*)$  its topological dual. By  $\langle \cdot, \cdot \rangle$  we denote the duality brackets for the pair  $(X^*, X)$ , and  $\xrightarrow{w}$  denotes weak convergence in  $X$ .

Let  $\varphi \in C^1(X)$ . A real number  $c$  is said to be a critical value of  $\varphi$  if there exists  $x^* \in X$  such that  $\varphi'(x^*) = 0$  and  $\varphi(x^*) = c$ . We say that  $\varphi \in C^1(X)$  satisfies the *C-condition*, if the following is true:

”every sequence  $\{x_n\}_{n \geq 1} \subseteq X$  such that  $\{\varphi(x_n)\}_{n \geq 1}$  is bounded in  $\mathbb{R}$  and

$$(1 + \|x_n\|) \varphi'(x_n) \rightarrow 0 \text{ in } X^* \text{ as } n \rightarrow \infty$$

admits a strongly convergent subsequence.”

This is in general weaker than the more common Palais-Smale condition. Nevertheless, the *C-condition* suffices to prove a deformation theorem and from it derive the minimax theory of certain critical values of  $\varphi \in C^1(X)$ . One such minimax theorem, which we recall for future use, is the so called ”mountain pass theorem”.

**Theorem 2.1.** *If  $\varphi \in C^1(X)$  satisfies the C-condition,  $x_0, x_1 \in X$ ,  $\rho > 0$ ,  $\|x_1 - x_0\| > \rho$ ,  $\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} = \eta_\rho$ , and  $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t))$  where*

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\},$$

*then  $c \geq \eta_\rho$  and  $c$  is a critical value of  $\varphi$ .*

Let  $(Y_1, Y_2)$  be a topological pair such that  $Y_2 \subseteq Y_1 \subseteq X$ . For every integer  $k \geq 0$ , by  $H_k(Y_1, Y_2)$  we denote the  $k^{\text{th}}$ -relative singular homology group with integer coefficients for the pair  $(Y_1, Y_2)$ . Recall that  $H_k(Y_1, Y_2) = 0$  for all integers  $k < 0$ .

Let  $\varphi \in C^1(X)$  and  $c \in \mathbb{R}$ . We introduce the following sets:

$$\begin{aligned} \varphi^c &= \{x \in X : \varphi(x) \leq c\}, \\ \dot{\varphi}^c &= \{x \in X : \varphi(x) < c\} \\ K_\varphi &= \{x \in X : \varphi'(x) = 0\}, \\ K_\varphi^c &= \{x \in K_\varphi : \varphi(x) = c\}. \end{aligned}$$

The critical groups of  $\varphi$  at an isolated critical point  $x \in X$  with  $\varphi(x) = c$  (i.e.,  $x \in K_\varphi^c$ ) are defined by

$$C_k(\varphi, x) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x\}) \text{ for all } k \geq 0,$$

where  $U$  is a neighborhood of  $x$  such that  $K_\varphi \cap \varphi^c \cap U = \{x\}$ . The excision property of singular homology implies that the above definition of critical groups is independent of the particular choice of the neighborhood  $U$ . Suppose that  $\varphi \in C^1(X)$  satisfies the  $C$ -condition and  $\inf \varphi(K_\varphi) > -\infty$ . Let  $c < \inf \varphi(K_\varphi)$ . The critical groups of  $\varphi$  at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \geq 0.$$

The second deformation theorem (see for example, Gasinski-Papageorgiou [12], p. 628) implies that the above definition of critical groups of  $\varphi$  at infinity is independent of the choice of the level  $c < \inf \varphi(K_\varphi)$ .

Suppose that  $K_\varphi$  is finite. We define

$$\begin{aligned} M(t, x) &= \sum_{k \geq 0} \text{rank } C_k(\varphi, x) t^k \text{ for all } t \in \mathbb{R}, x \in K_\varphi, \\ P(t, \infty) &= \sum_{k \geq 0} \text{rank } C_k(\varphi, \infty) t^k \text{ for all } t \in \mathbb{R}. \end{aligned}$$

The Morse relation says that

$$\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1+t)Q(t) \tag{2.1}$$

where  $Q(t) = \sum_{k \geq 0} \beta_k t^k$  is a formal series in  $t \in \mathbb{R}$  with nonnegative integer coefficients.

In the analysis of problem (1.1), we will use the Sobolev space

$$W_{per}^{1,p}(T) = \{u \in W^{1,p}(T) : u(0) = u(b)\},$$

with  $1 < p < \infty$ . The space  $W_{per}^{1,p}(T)$  is embedded compactly into  $C(T)$ , and so, the evaluations at  $t = 0$  and  $t = b$  make sense.

In the sequel, for notational economy we set

$$W := W_{per}^{1,p}(T).$$

In addition to the Sobolev space  $W$  we will also use the Banach space

$$\widehat{C}^1(T) = C^1(T) \cap W.$$

This is an ordered Banach space with positive cone

$$\widehat{C}_+ = \{u \in \widehat{C}^1(T) : u(t) \geq 0 \text{ for all } t \in T\}.$$

This cone has a nonempty interior, given by

$$\text{int } \widehat{C}_+ = \{u \in \widehat{C}_+ : u(t) > 0 \text{ for all } t \in T\}.$$

Throughout this paper, the norm of the Banach space  $W$  will be denoted by  $\|\cdot\|$ , i.e.,

$$\|u\| = \left( \|u\|_p^p + \|u'\|_p^p \right)^{\frac{1}{p}} \text{ for all } u \in W,$$

with  $\|\cdot\|_p$  being the norm of  $L^p(T)$ .

Given  $x \in \mathbb{R}$ , we set  $x^\pm = \max\{\pm x, 0\}$ . We have

$$x = x^+ - x^-, \text{ and } |x| = x^+ + x^-.$$

Then for every  $u \in W$ , we define  $u^\pm(\cdot) = u(\cdot)^\pm$  and we have

$$u = u^+ - u^-, \quad |u| = u^+ + u^- \text{ and } u^\pm \in W.$$

Also, if  $h : T \times \mathbb{R} \rightarrow \mathbb{R}$  is a measurable function, then we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in W$$

(the Nemytskii map corresponding to  $h(t, x)$ ).

Finally, by  $|\cdot|_1$  we denote the Lebesgue measure on  $\mathbb{R}$ .

Our hypotheses on the map  $a$  in problem (1.1) are the following:

**H**( $a$ ):  $a : (0, \infty) \rightarrow (0, \infty)$  is a  $C^1$ -function such that:

(i)  $x \rightarrow a(x)x$  is strictly increasing on  $(0, \infty)$ ,  $a(x)x \rightarrow 0$  as  $x \rightarrow 0^+$  and

$$\frac{a'(x)x}{a(x)} \rightarrow C > -1 \text{ as } x \rightarrow 0^+;$$

(ii) there exists  $\widehat{C} > 0$  and  $1 < p < \infty$  such that

$$|a(x)x| \leq \widehat{C} \left(1 + |x|^{p-1}\right) \text{ for all } x \in \mathbb{R};$$

(iii) there exists  $C_0 > 0$  such that

$$a'(x)x^2 \geq C_0x^{p-1} \text{ for all } x > 0;$$

(iv) if  $G_0(x) = \int_0^x a(s) s ds$  for all  $x \geq 0$ , then

$$pG_0(x) - a(x)x^2 \geq 0 \text{ for all } x > 0.$$

Evidently  $G_0(\cdot)$  is strictly convex and strictly increasing on  $(0, \infty)$ . We set

$$G(x) = G_0(|x|) \text{ for all } x \in \mathbb{R}.$$

Then  $G(\cdot)$  is strictly convex and for  $x \neq 0$  we have

$$G'(x) = G'_0(|x|) \frac{x}{|x|} = a(|x|)x.$$

So  $G(\cdot)$  is the primitive of the function  $x \rightarrow a(|x|)x$ ,  $x \in \mathbb{R}$ . Since  $G_0(\cdot)$  is convex and  $G_0(0) = 0$ , we have

$$G_0(x) \leq a(x)x^2 \text{ for all } x > 0. \tag{2.2}$$

Using (2.1) and hypotheses **H**( $a$ ) (ii), (iii), we obtain

$$\frac{C_0}{p} |x|^p \leq G(x) \leq C_1 (1 + |x|^p) \text{ for all } x \in \mathbb{R} \text{ and some } C_1 > 0. \tag{2.3}$$

**Examples:** The following functions satisfy the above hypotheses:

$$\begin{aligned} a_1(x)x &= |x|^{p-2}x \text{ with } 1 < p < \infty, \\ a_2(x)x &= |x|^{p-2}x + |x|^{q-2}x \text{ with } 1 < q < p < \infty, \\ a_3(x)x &= (1+x^2)^{\frac{p-2}{2}}x \text{ with } 1 < p < \infty, \\ a_4(x)x &= |x|^{p-2}x + \frac{|x|^{p-2}x}{1+|x|^p} \text{ with } 1 < p < \infty. \end{aligned}$$

The corresponding potential (primitive) functions are:

$$\begin{aligned} G_1(x) &= \frac{1}{p}|x|^p, \\ G_2(x) &= \frac{1}{p}|x|^p + \frac{1}{q}|x|^q, \\ G_3(x) &= \frac{1}{p} \left[ (1+x^2)^{\frac{p}{2}} - 1 \right], \\ G_4(x) &= \frac{1}{p}|x|^p + \ln(1+|x|^p). \end{aligned}$$

Note that  $a_1(x)x$  corresponds to the scalar  $p$ -Laplacian,  $a_2(x)x$  corresponds to the scalar  $(p, q)$ -Laplacian, and  $a_3(x)x$  corresponds to the generalized scalar  $p$ -mean curvature operator.

Let  $A : W \rightarrow W^*$  be the nonlinear map defined by

$$\langle A(u), v \rangle = \int_0^b a(|u'(t)|) u'(t) v'(t) dt \text{ for all } u, v \in W. \quad (2.4)$$

The following result can be found in Papageorgiou-Rocha-Staicu [16].

**Proposition 2.2.** *If hypotheses  $\mathbf{H}(a)$  hold, then the nonlinear operator  $A : W \rightarrow W^*$  defined by (2.4) is bounded (i.e., it maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone, too), and of type  $(S)_+$ , i.e., if  $u_n \xrightarrow{w} u$  in  $W$  and*

$$\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0,$$

*then  $u_n \rightarrow u$  in  $W_0^{1,p}(\Omega)$ .*

Let  $f_0 : T \times \mathbb{R} \rightarrow \mathbb{R}$  be a Carathéodory function such that

$$|f_0(t, x)| \leq a(t) \left( 1 + |x|^{r-1} \right) \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R},$$

with  $a \in L^1(T)_+$  and  $1 < r < \infty$ . We set

$$F_0(t, x) = \int_0^x f_0(t, s) ds$$

and consider the  $C^1$ -functional  $\psi_0 : W \rightarrow \mathbb{R}$  defined by

$$\psi_0(u) = \int_0^b G(u'(t)) dt - \int_0^b F_0(t, u(t)) dt \text{ for all } u \in W.$$

The following result can be found in Aizicovici-Papageorgiou-Staicu [6]).

**Proposition 2.3.** *If hypotheses  $\mathbf{H(a)}$  hold and  $u_0 \in W$  is a local  $\widehat{C^1}(T)$ -minimizer of  $\psi_0$  (i.e., there exists  $\rho_0 > 0$  such that  $\psi_0(u_0) \leq \psi_0(u_0 + h)$  for all  $h \in \widehat{C^1}(T)$  with  $\|h\|_{\widehat{C^1}(T)} \leq \rho_0$ ), then  $u_0 \in \widehat{C^1}(T)$  and it is a local  $W$ -minimizer of  $\psi_0$ , (i.e., there exists  $\rho_1 > 0$  such that  $\psi_0(u_0) \leq \psi_0(u_0 + h)$  for all  $h \in W$  with  $\|h\| \leq \rho_1$ ).*

The hypotheses on the reaction  $f(t, x)$  are the following:

$\mathbf{H}(f)$ :  $f : T \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function such that  $f(t, 0) = 0$  a.e. on  $T$  and

(i) there exist  $a \in L^1(T)_+$  and  $p < r < \infty$  such that

$$|f(t, x)| \leq a(t) \left(1 + |x|^{r-1}\right) \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R};$$

(ii)  $\lim_{x \rightarrow +\infty} \frac{F(t, x)}{x^p} = +\infty$  uniformly for a.a.  $t \in T$ , and there exist  $\tau > r - p$  and  $\beta_0 > 0$  such that

$$\beta_0 \leq \liminf_{x \rightarrow +\infty} \frac{f(t, x) - pF(t, x)}{x^\tau} \text{ uniformly for a.a. } t \in T,$$

where

$$F(t, x) := \int_0^x f(t, s) ds;$$

(iii) there exist functions  $\widehat{\theta}, \theta \in L^\infty(T)$  such that:

$$\begin{aligned} \widehat{\theta}(t) &\leq \theta(t) \leq 0 \text{ for a.a. } t \in T, \theta \neq 0, \\ \widehat{\theta}(t) &\leq \liminf_{x \rightarrow -\infty} \frac{pF(t,x)}{|x|^p} \leq \limsup_{x \rightarrow +\infty} \frac{pF(t,x)}{x^p} \leq \theta(t) \\ &\text{uniformly for a.a. } t \in T, \\ \limsup_{x \rightarrow -\infty} [pF(t,x) - f(t,x)x] &< +\infty \\ &\text{uniformly for a.a. } t \in T; \end{aligned}$$

(iv) there exist constants  $\widetilde{\xi}_0, \delta_0 > 0$  such that

$$F(t,x) \leq 0 \text{ for a.a. } t \in T, \text{ all } |x| \leq \delta_0, \int F(t, -\widetilde{\xi}_0) dt < 0$$

and for every  $\rho > 0$ , there exists  $\widehat{\xi}_\rho > 0$  such that for a.a.  $t \in T$ ,

$$x \rightarrow f(t,x) + \widehat{\xi}_\rho |x|^{p-2} x$$

is nondecreasing on  $[-\rho, \rho]$ .

**Remarks:** Hypotheses  $\mathbf{H}(f)$  (ii), (iii) reveal the asymmetric character of the nonlinearity  $f(t, \cdot)$ . By virtue of hypothesis  $\mathbf{H}(f)$  (ii), near  $+\infty$ ,  $x \rightarrow f(t,x)$  is  $(p-1)$ -superlinear. However, note that here we do not use the usual in such cases Ambrosetti-Rabinowitz condition (unilateral version). Instead, we employ a weaker requirement.

Hypothesis  $\mathbf{H}(f)$  (iii) implies that for a.a.  $t \in T$ , near  $-\infty$ ,  $x \rightarrow f(t,x)$  is  $(p-1)$ -sublinear. So, we have, a different growth for  $f(t, \cdot)$  in the positive and negative direction, respectively.

### 3 Three Solutions Theorem

In this section, we establish the existence of three nontrivial solutions for problem (1.1). To this end, let  $\varphi : W \rightarrow \mathbb{R}$  be the energy functional for problem (1.1) defined by

$$\varphi(u) = \int_0^b G(u'(t)) dt - \int_0^b F(t, u(t)) dt, \text{ for all } u \in W.$$

Evidently  $\varphi \in C^1(W)$ . Also, we consider the following perturbations-truncations of  $f(t, \cdot)$ :

$$\begin{aligned} \widehat{f}_+(t, x) &= \begin{cases} 0 & \text{if } x \leq 0, \\ f(t, x) + x^{p-1} & \text{if } x > 0, \end{cases} \\ \widehat{f}_-(t, x) &= \begin{cases} f(t, x) + |x|^{p-2}x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0. \end{cases} \end{aligned} \tag{3.1}$$

Both are Carathéodory functions. We set

$$\widehat{F}_\pm(t, x) = \int_0^x \widehat{f}_\pm(t, s) ds$$

and introduce the  $C^1$ -functionals  $\widehat{\varphi}_\pm : W \rightarrow \mathbb{R}$  by

$$\widehat{\varphi}_\pm(u) = \int_0^b G(u'(t)) dt + \frac{1}{p} \|u\|_p^p - \int_0^b \widehat{F}_\pm(t, u(t)) dt, \text{ for all } u \in W.$$

**Proposition 3.1.** *If hypotheses  $\mathbf{H}(a)$  and  $\mathbf{H}(f)$  hold, then the functional  $\varphi$  satisfies the  $C$ -condition.*

*Proof.* Let  $\{u_n\}_{n \geq 1}$  be a sequence in  $W$  such that

$$|\varphi(u_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1 \tag{3.2}$$

and

$$(1 + \|u_n\|) \varphi'(u_n) \rightarrow 0 \text{ in } W^* \text{ as } n \rightarrow \infty. \tag{3.3}$$

From (3.3), we have

$$\left| \langle A(u_n), h \rangle - \int_0^b f(t, u_n) h dt \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \text{ for all } h \in W, \tag{3.4}$$

with  $\varepsilon_n \rightarrow 0^+$ . In (3.4), we choose  $h = u_n^+ \in W$ . Then

$$-\int_0^b a\left(\left|(u_n^+)'\right|\right) \left((u_n^+)'\right)^2 dt + \int_0^b f(t, u_n^+) u_n^+ dt \leq \varepsilon_n \text{ for all } n \geq 1. \tag{3.5}$$

From (3.2) we have

$$\begin{aligned}
& \int_0^b pG\left((u_n^+)' \right) dt + \int_0^b pG\left(- (u_n^-)' \right) dt \\
& - \int_0^b pF\left((u_n^+)' \right) dt - \int_0^b pF\left(t, - (u_n^-)' \right) dt \\
& \leq pM_1 \text{ for all } n \geq 1.
\end{aligned} \tag{3.6}$$

Adding (3.5) and (3.6), we obtain

$$\begin{aligned}
& \int_0^b \left[ pG\left((u_n^+)' \right) - a\left(\left|(u_n^+)' \right|\right) \left((u_n^+)' \right)^2 \right] dt \\
& + \int_0^b [f(t, u_n^+) u_n^+ - pF(t, u_n^+)] dt \\
& + \int_0^b pG\left(- (u_n^-)' \right) dt - \int_0^b pF\left(t, -u_n^- \right) dt \\
& \leq pM_1 \text{ for all } n \geq 1.
\end{aligned} \tag{3.7}$$

By virtue of hypothesis  $\mathbf{H}(f)(i), (iii)$ , given  $\varepsilon > 0$ , we can find  $a_\varepsilon \in L^1(T)$  such that

$$F(t, x) \leq \frac{1}{p} (\theta(t) + \varepsilon) |x|^p + a_\varepsilon(t) \text{ for a.a. } t \in T, \text{ all } x \leq 0. \tag{3.8}$$

Using (3.8) in (3.7), we obtain

$$\begin{aligned}
& \int_0^b \left[ pG\left((u_n^+)' \right) - a\left(\left|(u_n^+)' \right|\right) \left((u_n^+)' \right)^2 \right] dt \\
& + \int_0^b [f(t, u_n^+) u_n^+ - pF(t, u_n^+)] dt \\
& + C_0 \left\| (u_n^-)' \right\|_p^p - \int_0^b \theta(t) |u_n^-|^p dt - \varepsilon \|u_n^-\|^p - C_2 \\
& \leq pM_1 \text{ for some } C_2 > 0, \text{ all } n \geq 1,
\end{aligned}$$

(see (3.8) and (2.3)), hence

$$\begin{aligned} & \int_0^b \left[ pG \left( (u_n^+)' \right) - a \left( \left| (u_n^+)' \right| \right) \left( (u_n^+)' \right)^2 \right] dt \\ & + \int_0^b [f(t, u_n^+) u_n^+ - pF(t, u_n^+)] dt + \xi_0 \|u_n^-\|^p \\ & \leq pM_1 + C_2 =: C_3 \text{ for all } n \geq 1, \text{ some } \xi_0 > 0 \end{aligned}$$

(see Aizicovici-Papageorgiou-Staicu [6], Lemma 2.1), therefore

$$\int_0^b [f(t, u_n^+) u_n^+ - pF(t, u_n^+)] dt \leq C_3 \text{ for all } n \geq 1 \text{ (see } \mathbf{H}(a)(iv) \text{)}. \quad (3.9)$$

Hypotheses  $\mathbf{H}(f)(i)$ ,  $(ii)$  imply that we can find  $\beta_1 \in (0, \beta_0)$  and  $a_1 \in L^1(T)_+$  such that

$$\beta_1 x^\tau - a_1(t) \leq f(t, x)x - pF(t, x) \text{ for a.a. } t \in T, \text{ all } x \geq 0. \quad (3.10)$$

Returning to (3.9) and using (3.10), we obtain

$$\beta_1 \|u_n^+\|_\tau^\tau \leq M_2 \text{ for some } M_2 > 0, \text{ all } n \geq 1,$$

hence

$$\{u_n^+\}_{n \geq 1} \subset L^\tau(T) \text{ is bounded.} \quad (3.11)$$

In (3.4) we choose  $h = u_n^+ \in W$  and use hypothesis  $\mathbf{H}(a)(iii)$  to arrive at

$$C_0 \left\| (u_n^+)' \right\|_p^p - \int_0^b f(t, u_n^+) u_n^+ dt \leq \varepsilon_n \text{ for all } n \geq 1, \text{ (see (2.3)).} \quad (3.12)$$

By virtue of  $\mathbf{H}(f)(i)$  we have

$$f(t, u_n^+(t)) u_n^+(t) \leq a(t) (u_n^+(t) + u_n^+(t)^\tau) \text{ for a.a. } t \in T, \text{ all } n \geq 1. \quad (3.13)$$

Using (3.13) in (3.12) we obtain

$$\left\| (u_n^+)' \right\|_p^p \leq \varepsilon_n + C_5 (\|u_n^+\| + \|u_n^+\|_r^\tau) \text{ for some } C_5 > 0, \text{ all } n \geq 1. \quad (3.14)$$

From hypothesis  $\mathbf{H}(f)$  (ii) it is clear that we can always assume that  $\tau \leq r < \infty$ . So, we can find  $t \in [0, 1)$  such that

$$\frac{1}{r} = \frac{1-t}{\tau}. \quad (3.15)$$

Invoking the interpolation inequality (see, for example, Gasinski-Papageorgiou [12], p. 905), we have

$$\|u_n^+\|_r \leq \|u_n^+\|_\tau^{1-t} \|u_n^+\|_\infty^t \text{ for all } n \geq 1,$$

hence

$$\|u_n^+\|_r^r \leq C_6 \|u_n^+\|_\tau^{tr} \text{ for some } C_6 > 0, \text{ all } n \geq 1 \quad (3.16)$$

(see (3.11)). Using (3.16) in (3.14) we have

$$\|(u_n^+)'\|_p^p \leq C_7 \left(1 + \|u_n^+\| + \|u_n^+\|^{tr}\right) \text{ for some } C_7 > 0, \text{ all } n \geq 1,$$

therefore

$$\|u_n^+\|^p \leq C_8 \left(1 + \|u_n^+\| + \|u_n^+\|^{tr}\right) \text{ for some } C_8 > 0, \text{ all } n \geq 1 \quad (3.17)$$

(see (3.11) and Gasinski-Papageorgiou [12], p. 227)). From (3.15) we have

$$tr = r - \tau < p \text{ (see hypothesis } \mathbf{H}(f) \text{ (ii))}.$$

So, from (3.17) it follows that

$$\{u_n^+\}_{n \geq 1} \subset W \text{ is bounded.} \quad (3.18)$$

Then from (3.7), (3.18) and hypotheses  $\mathbf{H}(a)$  (iv),  $\mathbf{H}(f)$  (i), we have

$$\int_0^b pG\left(-(u_n^-)'\right) dt - \int_0^b F(t, -u_n^-) dt \leq M_3 \text{ for some } M_3 > 0, \text{ all } n \geq 1.$$

Using (3.18) and Lemma 2.1 of Aizicovici-Papageorgiou-Staicu [6], we have

$$\xi_0 \|u_n^-\|^p \leq M_4 \text{ for some } M_4, \xi_0 > 0, \text{ all } n \geq 1,$$

hence

$$\{u_n^-\}_{n \geq 1} \subset W \text{ is bounded.} \quad (3.19)$$

From (3.18) and (3.19) it follows that  $\{u_n\}_{n \geq 1} \subset W$  is bounded and so, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W \text{ and } u_n \rightarrow u \text{ in } C(T). \quad (3.20)$$

In (3.4) we choose  $h = u_n - u \in W$ , pass to the limit as  $n \rightarrow \infty$  and use (3.20). Then

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = 0,$$

therefore

$$u_n \rightarrow u \text{ in } W$$

(see Proposition 2.2). This proves that the functional  $\varphi$  satisfies the  $C$ -condition.  $\square$

**Proposition 3.2.** *If hypotheses  $\mathbf{H}(a)$  and  $\mathbf{H}(f)$  hold, then the functional  $\widehat{\varphi}_+$  satisfies the  $C$ -condition.*

*Proof.* Let  $\{u_n\}_{n \geq 1}$  be a sequence in  $W$  such that

$$|\widehat{\varphi}_+(u_n)| \leq M_5 \text{ for some } M_5 > 0, \text{ all } n \geq 1. \quad (3.21)$$

and

$$(1 + \|u_n\|) \widehat{\varphi}'_+(u_n) \rightarrow 0 \text{ in } W^* \text{ as } n \rightarrow \infty. \quad (3.22)$$

From (3.22) we have

$$\begin{aligned} & \left| \langle A(u_n), h \rangle + \int_0^b |u_n|^{p-2} u_n h dt - \int_0^b \widehat{f}_+(t, u_n) h dt \right| \\ & \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \text{ for all } h \in W, \text{ with } \varepsilon_n \rightarrow 0^+. \end{aligned} \quad (3.23)$$

In (3.23), we choose  $h = -u_n^- \in W$ . Then from hypothesis  $\mathbf{H}(a)$  (iii) and (3.1), we have

$$C_0 \left\| (u_n^-)' \right\|_p^p + \|u_n^-\|_p^p \leq \varepsilon_n \text{ for all } n \geq 1$$

hence

$$u_n^- \rightarrow 0 \text{ in } W \text{ as } n \rightarrow \infty. \quad (3.24)$$

From (3.21) and (3.24) we have

$$\int_0^b pG\left((u_n^+)' \right) dt + \|u_n^+\|_p^p - \int_0^b p\widehat{F}(t, u_n^+) dt \leq M_6 \quad (3.25)$$

for some  $M_6 > 0$ , all  $n \geq 1$ .

Also, if in (3.23), we choose  $h = u_n^+ \in W$ , then

$$-\int_0^b a\left(\left|(u_n^+)'\right|\right)\left((u_n^+)'\right)^2 dt - \|u_n^+\|_p^p + \int_0^b \widehat{f}_+(t, u_n^+) dt \leq \varepsilon_n \text{ for all } n \geq 1. \quad (3.26)$$

We add (3.25) and (3.26). Using hypothesis  $\mathbf{H}(a)$  (iv) and (3.1), we obtain

$$\int_0^b [f(t, u_n^+) u_n^+ - pF(t, u_n^+)] dt \leq M_7 \text{ for some } M_7 > 0, \text{ all } n \geq 1. \quad (3.27)$$

Using (3.10), we infer that  $\{u_n^+\}_{n \geq 1} \subset L^r(T)$  is bounded. By virtue of hypothesis  $\mathbf{H}(f)$  (i), we have

$$|f(t, x)x| \leq a(t)[|x| + |x|^r] \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R}. \quad (3.28)$$

In (3.23), we choose  $h = u_n^+ \in W$  and obtain

$$\begin{aligned} C_0 \left\| (u_n^+)'\right\|_p^p + \|u_n^+\|_p^p &\leq \varepsilon_n + \int_0^b f(t, u_n^+) u_n^+ dt \\ &\leq C_9 (1 + \|u_n^+\|_r^r) \text{ for some } C_9 > 0, \text{ all } n \geq 1 \end{aligned}$$

(see (3.26) and (2.3)). Using (3.15), the interpolation inequality and the boundedness of  $\{u_n^+\}_{n \geq 1} \subset L^r(T)$ , as in the proof of Proposition 3.1 (see (3.16) and (3.17)), we obtain

$$\|u_n^+\|^p \leq C_{10} \left(1 + \|u_n^+\|^{tr}\right) \text{ for some } C_{10} > 0, \text{ all } n \geq 1,$$

hence

$$\{u_n^+\}_{n \geq 1} \subset W \text{ is bounded} \quad (3.29)$$

(since  $tr = r - \tau < p$ , see  $\mathbf{H}(f)$  (ii)). From (3.24) and (3.29) it follows that  $\{u_n\}_{n \geq 1} \subset W$  is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W \text{ and } u_n \rightarrow u \text{ in } C(T). \quad (3.30)$$

In (3.23) we choose  $h = u_n - u \in W$ , pass to the limit as  $n \rightarrow \infty$  and use (3.30). Then

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = 0,$$

therefore

$$u_n \rightarrow u \text{ in } W$$

(see Proposition 2.2). This proves that the functional  $\widehat{\varphi}_+$  satisfies the  $C$ -condition.  $\square$

**Proposition 3.3.** *If hypotheses  $\mathbf{H}(a)$  and  $\mathbf{H}(f)$  hold, then the functional  $\widehat{\varphi}_-$  is coercive.*

*Proof.* By virtue of hypotheses  $\mathbf{H}(f)$  (i) and (iii), given  $\varepsilon > 0$  we can find  $a_2 \in L^1(T)$  such that

$$F(t, x) \leq \frac{1}{p} (\theta(t) + \varepsilon) |x|^p + a_2(t) \text{ for a.a. } t \in T, \text{ all } x \leq 0. \quad (3.31)$$

Then for all  $u \in W$ , we have

$$\begin{aligned} \widehat{\varphi}_-(u) &= \int_0^b G(u'(t)) dt + \frac{1}{p} \|u\|_p^p - \int_0^b \widehat{F}_-(t, u(t)) dt \\ &\geq \frac{C_0}{p} \|u'\|_p^p - \frac{1}{p} \int_0^b \theta(t) |u(t)|^p dt - \frac{\varepsilon}{p} \|u\|_p^p - \|a_2\|_1 \\ &\quad \text{(see (2.2) and (3.31))} \\ &\geq \frac{\xi_0 - \varepsilon}{p} \|u\|_p^p - \|a_2\|_1 \text{ with } \xi_0 > 0 \\ &\quad \text{(see [6]), Lemma 2.1).} \end{aligned}$$

Choosing  $\varepsilon \in (0, \xi_0)$  in the last inequality, we conclude that  $\widehat{\varphi}_-$  is coercive.  $\square$

Now we are ready to produce two constant sign solutions.

**Proposition 3.4.** *If hypotheses  $\mathbf{H}(a)$  and  $\mathbf{H}(f)$  hold, then problem (1.1) has at least two constant sign solutions  $u_0 \in \text{int } \widehat{C}_+$  and  $v_0 \in -\text{int } \widehat{C}_+$ . Moreover,  $v_0$  is a local minimizer of  $\varphi$ .*

*Proof.* First we show that  $u = 0$  is a local minimizer of the functional  $\widehat{\varphi}_+$ . So, let  $u \in \widehat{C}^1(T)$  with  $\|u\|_{\widehat{C}^1(T)} \leq \delta_0$ , where  $\delta_0 > 0$  is as postulated by hypothesis  $\mathbf{H}(f)$  (iv). Then

$$\begin{aligned} \widehat{\varphi}_+(u) &= \int_0^b G(u'(t)) dt + \frac{1}{p} \|u\|_p^p - \int_0^b \widehat{F}_+(t, u(t)) dt \\ &\geq \frac{C_0}{p} \|u'\|_p^p \text{ (see (2.2) and } \mathbf{H}(f) \text{ (iv))}, \end{aligned}$$

hence  $u = 0$  is a local  $\widehat{C}^1(T)$ -minimizer of the functional  $\widehat{\varphi}_+$ , therefore  $u = 0$  is a local  $W$ -minimizer of the functional  $\widehat{\varphi}_+$  (see Proposition 2.3). This implies that we can find  $\rho \in (0, 1)$  small, such that

$$\widehat{\varphi}_+(0) = 0 < \inf \{ \widehat{\varphi}_+(u) : \|u\| = \rho \} =: \widehat{\eta}_\rho^+ \quad (3.32)$$

(see Aizicovici-Papageorgiou-Staicu [2], p. 57). For  $\xi \in (0, \infty)$  we have

$$\widehat{\varphi}_+(\xi) = - \int_0^b F_+(t, \xi) dt \quad (\text{see (3.1)}),$$

hence

$$\widehat{\varphi}_+(\xi) \rightarrow -\infty \text{ as } \xi \rightarrow +\infty \quad (\text{see } \mathbf{H}(f)(ii)). \quad (3.33)$$

From Proposition 3.2 we know that  $\widehat{\varphi}_+$  satisfies the  $C$ -condition. This fact together with (3.32) and (3.33) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find  $u_0 \in W$  such that

$$\widehat{\varphi}_+(0) = 0 < \widehat{\eta}_\rho^+ \leq \widehat{\varphi}_+(u_0) \quad (3.34)$$

and

$$\widehat{\varphi}'_+(u_0) = 0. \quad (3.35)$$

From (3.34) we see that  $u_0 \neq 0$ . From (3.35) we have

$$A(u_0) + |u_0|^{p-2} u_0 = N_{\widehat{f}_+}(u_0). \quad (3.36)$$

On (3.36) we act with  $-u_0^- \in W$ , and use (3.1) and Proposition 3.2 to obtain

$$C_0 \left\| (u_0^-)' \right\|_p^p + \|u_0^-\|_p^p \leq 0,$$

therefore

$$u_0 \geq 0, \quad u_0 \neq 0.$$

Hence, (3.36) becomes

$$A(u_0) = N_f(u_0) \quad (\text{see (3.1)}),$$

and we get

$$\begin{cases} -(a(|u'_0(t)|) u'_0(t))' = f(t, u_0(t)) \quad \text{a.e. on } T, \\ u_0(0) = u_0(b), \quad u'_0(0) = u'_0(b). \end{cases} \quad (3.37)$$

Then  $u_0 \in \widehat{C}^1(T)$ . Let  $\rho = \|u_0\|_\infty$  and  $\xi_\rho > 0$  be as postulated by hypothesis  $\mathbf{H}(f)(iv)$ . From (3.37) we have

$$\begin{aligned} - (a(|u'_0(t)|) u'_0(t))' + \xi_\rho u_0(t)^{p-1} &= f(t, u_0(t)) + \xi_\rho u_0(t)^{p-1} \\ &\geq 0 \text{ a.e. on } T, \end{aligned}$$

hence

$$(a(|u'_0(t)|) u'_0(t))' \leq \xi_\rho u_0(t)^{p-1} \text{ a.e. on } T. \quad (3.38)$$

From (3.38) and the strong maximum principle of Pucci-Serrin ([18], p.111) it follows that  $u_0(t) > 0$  for all  $t \in (0, b)$ .

Then the boundary point theorem of Pucci-Serrin ([18], p.120) implies that  $u_0 \in \text{int } C_+$ . From Proposition 3.3 we know that  $\widehat{\varphi}_-$  is coercive. Also, using the Sobolev embedding theorem, we see that  $\widehat{\varphi}_-$  is sequentially lower semicontinuous.

So, by the Weierstrass theorem we can find  $v_0 \in W$  such that

$$\widehat{\varphi}_-(v_0) = \inf \{ \widehat{\varphi}_-(u) : u \in W \}. \quad (3.39)$$

Let  $v = -\widetilde{\xi}_0 \in -\text{int } \widehat{C}_+$  be as in hypothesis  $\mathbf{H}(f)(iv)$ . We have

$$\widehat{\varphi}_-(-\widetilde{\xi}_0) = - \int_0^b F(t, -\widetilde{\xi}_0) dt < 0$$

(see (3.1) and hypothesis  $\mathbf{H}(f)(iv)$ ), hence

$$\widehat{\varphi}_-(v_0) < 0 = \widehat{\varphi}_-(0)$$

(see (3.39)), therefore

$$v_0 \neq 0.$$

From (3.39) we have

$$\widehat{\varphi}'_-(v_0) = 0.$$

which implies

$$A(v_0) + |v_0|^{p-2} v_0 = N_{\widehat{f}_-}(v_0). \quad (3.40)$$

Acting on (3.40) with  $v_0^+ \in W$  and using (3.1) and Proposition 3.2, we obtain

$$C_0 \left\| (v_0^+)' \right\|_p^p + \|v_0^+\|_p^p \leq 0,$$

hence

$$v_0 \leq 0, \quad v_0 \neq 0.$$

Then (3.40) becomes

$$A(v_0) = N_f(v_0),$$

and we get

$$\begin{cases} -(a(|v'_0(t)|)v'_0(t))' = f(t, v_0(t)) \text{ a.e. on } T, \\ v_0(0) = v_0(b), v'_0(0) = v'_0(b). \end{cases}$$

Hence  $v_0 \in \widehat{C}^1(T)$ , and as before, using the results of Pucci-Serrin ([18], pp. 111, 120), we obtain  $v_0 \in -\text{int } \widehat{C}_+$ . Note that

$$\varphi|_{-\widehat{C}_+} = \widehat{\varphi}_-|_{-\widehat{C}_+}.$$

Therefore  $v_0 \in -\text{int } \widehat{C}_+$  is a local  $\widehat{C}(T)$ -minimizer of  $\varphi$ , and from Proposition 2.3 it follows that  $v_0 \in -\text{int } \widehat{C}_+$  is a local  $W$ -minimizer of  $\varphi$ .  $\square$

Next, using Morse theory, we will produce a third nontrivial solution for problem (1.1). To this end, we start by computing the critical groups of  $\varphi$  at infinity.

**Proposition 3.5.** *If hypotheses  $\mathbf{H}(a)$  and  $\mathbf{H}(f)$  hold, then  $C_k(\varphi, \infty) = 0$  for all  $k \geq 0$ .*

*Proof.* Let  $\psi := \varphi|_{\widehat{C}^1(T)}$ . The regularity properties of solutions of (1.1) imply that

$$K_\psi = K_\varphi = K.$$

Since  $\widehat{C}^1(T) \hookrightarrow W$  densely, from Palais [15], we have

$$H_k(W, \dot{\varphi}^\alpha) = H_k(\widehat{C}^1(T), \dot{\varphi}^\alpha) \text{ for all } \alpha \in \mathbb{R}, \text{ all } k \geq 0. \quad (3.41)$$

Let  $\alpha < \inf \varphi(K) = \inf \psi(K)$ . We have

$$C_k(\varphi, \infty) = H_k(W, \varphi^\alpha) = H_k(W, \dot{\varphi}^\alpha) \text{ for all } k \geq 0, \quad (3.42)$$

$$C_k(\psi, \infty) = H_k(\widehat{C}^1(T), \psi^\alpha) = H_k(\widehat{C}^1(T), \dot{\psi}^\alpha) \text{ for all } k \geq 0, \quad (3.43)$$

(see Granas-Dugundji [14], p.407). Then from (3.41), (3.42), (3.43), it follows that in order to prove the proposition, it suffices to show that

$$H_k(\widehat{C}^1(T), \psi^\alpha) = 0 \text{ for all } k \geq 0, \alpha < 0 \text{ with } |\alpha| \text{ big.}$$

To this end, we introduce the following sets

$$\partial B_1^C = \left\{ u \in \widehat{C}^1(T) : \|u\|_{\widehat{C}^1(T)} = 1 \right\},$$

$$\partial B_{1,+}^C = \{u \in \partial B_1^C : u(t) > 0 \text{ for some } t \in (0, b)\}.$$

Let  $h_+ : [0, 1] \times \partial B_{1,+}^C \rightarrow \partial B_{1,+}^C$  be the homotopy defined by

$$h_+(t, u) = \frac{(1-t)u + t\widehat{u}_0}{\|(1-t)u + t\widehat{u}_0\|_{\widehat{C}^1(T)}} \text{ for all } (t, u) \in [0, 1] \times \partial B_{1,+}^C,$$

where  $\widehat{u}_0 \in \text{int } \widehat{C}_+$  with  $\|\widehat{u}_0\|_{\widehat{C}^1(T)} = 1$ . We have

$$h_+(0, u) = u, \quad h_+(1, u) = \widehat{u}_0,$$

hence  $\partial B_{1,+}^C$  is contractible in itself. For  $u \in \partial B_{1,+}^C$  and  $\lambda > 0$  we have

$$\begin{aligned} \varphi(\lambda u) &= \int_0^b G(\lambda u'(t)) dt - \int_0^b F(t, \lambda u(t)) dt \\ &\leq C_{11} \left( 1 + \lambda^p \|u'\|_p^p \right) - \int_0^b F(t, \lambda u(t)) dt \text{ for some } C_{11} > 0 \text{ (see (2.3))} \\ &= C_{11} \left( 1 + \lambda^p \|u'\|_p^p \right) - \int_0^b F(t, \lambda u^+(t)) dt - \int_0^b F(t, -\lambda u^-(t)) dt. \end{aligned} \quad (3.44)$$

By virtue of **H**( $f$ )( $i$ ), ( $ii$ ), given any  $\xi > 0$ , we can find  $a_3 \in L^1(T)$  such that

$$F(t, x) \geq \xi x^p - a_3(t) \text{ for a.a. } t \in T, \text{ all } x \geq 0. \quad (3.45)$$

On the other hand, hypotheses **H**( $f$ )( $i$ ), ( $iii$ ) imply that there exist  $C_{12} > 0$  and  $a_4 \in L^1(T)$  such that

$$F(t, x) \geq -C_{12} |x|^p - a_4(t) \text{ for a.a. } t \in T, \text{ all } x \leq 0. \quad (3.46)$$

Returning to (3.44) and using (3.45) and (3.46), we have

$$\begin{aligned} \varphi(\lambda u) &\leq \lambda^p C_{11} \|u'\|_p^p - \lambda^p \xi \|u^+\|_p^p + \lambda^p C_{12} \|u^-\|_p^p + C_{13} \\ &\quad \text{for some } C_{13} > 0 \\ &\leq \lambda^p \left[ C_{11} \|u'\|_p^p + C_{12} \|u^-\|_p^p - \xi \|u^+\|_p^p \right] + C_{13}. \end{aligned} \quad (3.47)$$

Since  $\xi > 0$  is arbitrary, we choose  $\xi > 0$  big such that

$$C_{11} \|u'\|_p^p + C_{12} \|u^-\|_p^p < \xi \|u^+\|_p^p.$$

Then from (3.47) it follows that

$$\varphi(\lambda u) \rightarrow -\infty \text{ as } \lambda \rightarrow +\infty. \quad (3.48)$$

Hypotheses  $\mathbf{H}(f)$  (ii), (iii) imply that there exist  $\beta_2 \in (0, \beta_0)$ ,  $M_8 > 0$  and  $\xi^* > 0$  such that

$$pF(t, x) - f(t, x)x \leq -\beta_2 x^\tau \text{ for a.a. } t \in T, \text{ all } x \geq M_8, \quad (3.49)$$

$$pF(t, x) - f(t, x)x \leq \xi^* \text{ for a.a. } t \in T, \text{ all } x \leq 0. \quad (3.50)$$

By (3.49) and (3.50) and hypothesis  $\mathbf{H}(f)$  (i), for every  $u \in W$ , we have

$$\begin{aligned} \int_0^b [pF(t, u) - f(t, u)u] dt &= \int_{\{u \leq 0\}} [pF(t, u) - f(t, u)u] dt \\ &+ \int_{\{u \geq M_8\}} [pF(t, u) - f(t, u)u] dt + \int_{\{0 < u < M_8\}} [pF(t, u) - f(t, u)u] dt \\ &\leq C_{14} - \beta_2 \int_{\{u \geq M_8\}} u^\tau dt. \end{aligned} \quad (3.51)$$

Let  $i : \widehat{C^1}(T) \rightarrow W$  be the embedding map. Hence  $i \in \mathcal{L}(\widehat{C^1}(T), W)$ . We see that

$$\psi = \varphi \circ i.$$

From the chain rule, we have

$$\psi' = i^* \varphi'(u) \text{ for all } u \in W. \quad (3.52)$$

Let  $\langle \cdot, \cdot \rangle_C$  denote the duality brackets for the pair  $(\widehat{C}^1(T)^*, \widehat{C}^1(T))$ . We have

$$\begin{aligned}
 \frac{d}{d\lambda} \psi(\lambda u) &= \langle \psi'(\lambda u), u \rangle_C \\
 &= \langle i^* \varphi'(\lambda u), u \rangle_C \quad (\text{see (3.52)}) \\
 &= \langle \varphi'(\lambda u), u \rangle_C \\
 &= \frac{1}{\lambda} \left[ \int_0^b a(|\lambda u'|) (\lambda u')^2 dt - \int_0^b f(t, \lambda u) \lambda u dt \right] \\
 &\leq \frac{1}{\lambda} \left[ \int_0^b pG(\lambda u') dt - \int_0^b pF(t, \lambda u) dt + C_{14} \right] \\
 &\quad (\text{see } \mathbf{H}(a)(iv) \text{ and (3.51)}) \\
 &= \frac{1}{\lambda} [p\varphi(\lambda u) + C_{14}]
 \end{aligned} \tag{3.53}$$

From (3.48) and (3.53), we see that for  $\lambda > 0$  big (such that  $\varphi(\lambda u) < -\frac{C_{14}}{p}$ ), we have

$$\frac{d}{d\lambda} \psi(\lambda u) < 0. \tag{3.54}$$

From Proposition 3.3, we have

$$\inf_{-\widehat{C}_+} \psi = \inf_{-\widehat{C}_+} \varphi > -C_{15} \text{ for some } C_{15} > 0.$$

Let

$$\alpha < \min \left\{ -C_{15}, -\frac{C_{14}}{p}, \inf_{\overline{B}_1^C} \psi \right\}$$

where

$$\overline{B}_1^C = \left\{ u \in \widehat{C}^1(T) : \|u\|_{\widehat{C}^1(T)} \leq 1 \right\}.$$

From (3.54) we see that we can find a unique  $\gamma(u) \geq 1$  such that

$$\begin{aligned}
 \psi(\lambda u) &> \alpha \text{ if } \lambda < \gamma(u), \\
 \psi(\lambda u) &= \alpha \text{ if } \lambda = \gamma(u), \\
 \psi(\lambda u) &< \alpha \text{ if } \lambda > \gamma(u)
 \end{aligned}$$

and

$$\psi^\alpha = \left\{ \lambda u : u \in \partial B_{1,+}^C, \lambda \geq \gamma(u) \right\}. \tag{3.55}$$

By virtue of the implicit function theorem, we have  $\gamma \in C(\partial B_{1,+}^C, [1, \infty))$ . Let

$$V_+ = \{\lambda u : u \in \partial B_{1,+}^C, \lambda \geq 1\}.$$

It is easily seen that  $\partial B_{1,+}^C$  is a retract of  $V_+$  and  $V_+$  is deformable onto  $\partial B_{1,+}^C$  in  $\widehat{C}^1(T)$ . Then invoking Dugundji [10] (Theorem 6.5, p.325), we infer that  $\partial B_{1,+}^C$  is a deformation retract of  $V_+$ . Therefore

$$V_+ \text{ and } \partial B_{1,+}^C \text{ are homotopy equivalent.} \quad (3.56)$$

We introduce the homotopy  $\widehat{h}_+ : [0, 1] \times V_+ \rightarrow V_+$  by

$$\widehat{h}_+(t, \lambda u) = \begin{cases} (1-t)\lambda u + t\gamma(u)u & \text{if } \lambda \in [1, \gamma(u)], \\ \lambda u & \text{if } \lambda \geq \gamma(u). \end{cases}$$

Then (cf. (3.55))

$$\begin{aligned} \widehat{h}_+(0, \cdot) &= Id, \quad \widehat{h}_+(1, \lambda u) \in \psi^\alpha \text{ for all } \lambda u \in V_+, \\ \widehat{h}_+(t, \cdot) |_{\psi^\alpha} &= Id |_{\psi^\alpha}. \end{aligned}$$

This means that  $\psi^\alpha$  is a strong deformation retract of  $V_+$ . Therefore

$$V_+ \text{ and } \psi^\alpha \text{ are homotopy equivalent.} \quad (3.57)$$

From (3.56) and (3.57) it follows that

$$\psi^\alpha \text{ and } \partial B_{1,+}^C \text{ are homotopy equivalent,}$$

hence

$$H_k(\widehat{C}^1(T), \psi^\alpha) = H_k(\widehat{C}^1(T), \partial B_{1,+}^C) \text{ for all } k \geq 0 \quad (3.58)$$

(see Granas-Dugundji [14], p. 387). Recall that  $\partial B_{1,+}^C$  is contractible in itself. So,

$$H_k(\widehat{C}^1(T), \partial B_{1,+}^C) = 0 \text{ for all } k \geq 0 \quad (3.59)$$

(see Granas-Dugundji [14], p. 389). From (3.58) and (3.59) it follows that

$$H_k(\widehat{C}^1(T), \psi^\alpha) = 0 \text{ for all } k \geq 0,$$

therefore

$$C_k(\varphi, \infty) = 0 \text{ for all } k \geq 0$$

(see the first part of the proof). □

Also, we compute the critical groups of  $\widehat{\varphi}_+$  at infinity.

**Proposition 3.6.** *If hypotheses  $\mathbf{H}(a)$  and  $\mathbf{H}(f)$  hold, then  $C_k(\widehat{\varphi}_+, \infty) = 0$  for all  $k \geq 0$ .*

*Proof.* By virtue of hypotheses  $\mathbf{H}(f)(i), (ii)$ , given any  $\xi > 0$ , we can find  $a_5 \in L^1(T)$  such that

$$F(t, x) \geq \xi x^p - a_5(t) \text{ for a.a. } t \in T, \text{ all } x \geq 0.$$

We introduce the set

$$E_+ = \{u \in W : \|u\| = 1, u^+ \neq 0\}.$$

For  $u \in E_+$  and  $\lambda > 0$ , we have

$$\begin{aligned} \widehat{\varphi}_+(\lambda u) &= \int_0^b G(\lambda u'(t)) dt + \frac{\lambda^p}{p} \|u\|_p^p - \int_0^b \widehat{F}_+(t, u) dt \\ &\leq C_{16} \left(1 + \lambda^p \|u'\|_p^p\right) + \frac{\lambda^p}{p} \|u\|_p^p - \xi \lambda^p \|u^+\|_p^p + C_{17} \\ &\quad \text{for some } C_{16}, C_{16} > 0 \\ &= \lambda^p \left[\|u'\|_p^p + \frac{1}{p} \|u\|_p^p - \xi \|u^+\|_p^p\right] + C_{18}, \\ &\quad \text{with } C_{18} = C_{16} + C_{17} > 0. \end{aligned} \tag{3.60}$$

Since  $\xi > 0$  is arbitrary, we choose  $\xi > 0$  big such that

$$\|u'\|_p^p + \frac{1}{p} \|u\|_p^p < \xi \|u^+\|_p^p.$$

So, from (3.60) it is clear that

$$\widehat{\varphi}_+(\lambda u) \rightarrow -\infty \text{ as } \lambda \rightarrow +\infty. \tag{3.61}$$

Similarly, as in the proof of Proposition 3.5 (see (3.51)), for all  $u \in W$  we have

$$\begin{aligned} &\int_0^b \left[ p\widehat{F}_+(t, u) - \widehat{f}_+(t, u)u \right] dt = \int_{\{u>0\}} [pF(t, u) - f(t, u)u] dt \\ &= \int_{\{0<u<M_8\}} [pF(t, u) - f(t, u)u] dt \\ &+ \int_{\{u \geq M_8\}} [pF(t, u) - f(t, u)u] dt, \text{ for some } M_8 > 0 \text{ big (see (3.49))} \\ &\leq C_{19} - \beta_2 \int_{\{u \geq M_8\}} u^\tau dt, \text{ for some } C_{19} > 0 \end{aligned} \tag{3.62}$$

Using (3.62), we have

$$\begin{aligned}
\frac{d}{d\lambda} \widehat{\varphi}_+(\lambda u) &= \langle \widehat{\varphi}'_+(\lambda u), u \rangle \\
&= \int_0^b a(|\lambda u'|) \lambda (u')^2 dt + \frac{\lambda^{p-1}}{p} \|u\|_p^p - \int_0^b \widehat{f}_+(t, \lambda u) u dt \\
&= \frac{1}{\lambda} \left[ \int_0^b a(|\lambda u'|) (\lambda u')^2 dt + \frac{\lambda^p}{p} \|u\|_p^p - \int_0^b \widehat{f}_+(t, \lambda u) \lambda u dt \right] \\
&\leq \frac{1}{\lambda} \left[ \int_0^b pG(\lambda u') dt + \lambda^p \|u\|_p^p - \int_0^b p\widehat{F}_+(t, \lambda u) \lambda u dt + C_{19} \right] \\
&\quad \text{(see } \mathbf{H}(a)(iv) \text{ and (3.62))} \\
&= \frac{1}{\lambda} [p\widehat{\varphi}_+(\lambda u) + C_{19}].
\end{aligned}$$

So, for  $\lambda > 0$  big (such that  $\widehat{\varphi}_+(\lambda u) < -\frac{C_{19}}{p}$ ), we have

$$\frac{d}{d\lambda} \widehat{\varphi}_+(\lambda u) < 0.$$

Let  $d < -\frac{C_{19}}{p}$ . We can find a unique  $\gamma_+(u) > 0$ ,  $\gamma_+ \in C(E_+)$  (by the implicit function theorem) such that

$$\widehat{\varphi}_+(\gamma_+(u)u) = d \text{ for all } u \in E_+. \quad (3.63)$$

Let  $D_+ := \{u \in W : u^+ \neq 0\}$  and define

$$\widehat{\gamma}_+(u) = \frac{1}{\|u\|} \gamma_+\left(\frac{u}{\|u\|}\right) \text{ for all } u \in D_+.$$

We have  $\widehat{\gamma}_+ \in C(D_+)$ , and from (3.63) it follows

$$\widehat{\varphi}_+(\widehat{\gamma}_+(u)u) = d \text{ for all } u \in D_+. \quad (3.64)$$

Moreover, if  $\widehat{\varphi}_+(u) = d$ , then  $\widehat{\gamma}_+(u) = 1$ . We set

$$\widetilde{\gamma}_+(u) = \begin{cases} 1 & \text{if } \widehat{\varphi}_+(u) \leq d \\ \widehat{\gamma}_+(u) & \text{if } \widehat{\varphi}_+(u) > d. \end{cases} \quad (3.65)$$

for all  $u \in D_+$ . We see that  $\widetilde{\gamma}_+ \in C(D_+)$ .

We introduce the homotopy  $\tilde{h}_+ : [0, 1] \times D_+ \rightarrow D_+$  by

$$\tilde{h}_+(t, u) = (1 - t)u + t\tilde{\gamma}_+(u)u \text{ for all } (t, u) \in [0, 1] \times D_+.$$

We have

$$\begin{aligned} \tilde{h}_+(0, u) &= u, \quad \tilde{h}_+(1, u) \in \widehat{\varphi}_+^d \text{ and } \tilde{h}_+(t, \cdot) |_{\widehat{\varphi}_+^d} = Id |_{\widehat{\varphi}_+^d} \\ &\text{for all } (t, u) \in [0, 1] \times D_+. \end{aligned}$$

This shows that  $\widehat{\varphi}_+^d$  is a strong deformation retract of  $D_+$ . Hence

$$D_+ \text{ and } \widehat{\varphi}_+^d \text{ are homotopy equivalent,}$$

therefore

$$H_k(W, D_+) = H_k(W, \widehat{\varphi}_+^d) \text{ for all } k \geq 0. \quad (3.66)$$

Without any loss of generality we assume that  $K_{\widehat{\varphi}_+}$  is finite. (Otherwise we already have a sequence of distinct positive solutions of (1.1).) We choose

$$d < \min \left\{ \inf \widehat{\varphi}_+(K_{\widehat{\varphi}_+}), -\frac{C_{19}}{p} \right\}.$$

Then we have

$$H_k(W, \widehat{\varphi}_+^d) = C_k(\widehat{\varphi}_+, \infty) \text{ for all } k \geq 0. \quad (3.67)$$

Consider the homotopy  $h^* : [0, 1] \times D_+ \rightarrow D_+$  defined by

$$h^*(t, u) = \frac{(1 - t)u + t\widehat{u}_0}{\|(1 - t)u + t\widehat{u}_0\|}$$

where  $\widehat{u}_0 \in \text{int } \widehat{C}_+$ . Then  $h^*(1, u) = \frac{\widehat{u}_0}{\|\widehat{u}_0\|}$ , which proves that  $D_+$  is contractible in itself. Therefore

$$H_k(W, D_+) = 0 \text{ for all } k \geq 0 \quad (3.68)$$

(see Granas-Dugundji [14], p.389). From (3.66), (3.67) and (3.68) we conclude that

$$C_k(\widehat{\varphi}_+, \infty) = 0 \text{ for all } k \geq 0.$$

□

Now we are ready to produce a third nontrivial solution and have the complete multiplicity theorem (three solution theorem) for problem (1.1).

**Theorem 3.7.** *If hypotheses  $\mathbf{H}(a)$  and  $\mathbf{H}(f)$  hold, then problem (1.1) has at least three nontrivial solutions  $u_0 \in \text{int } \widehat{C}_+$ ,  $v_0 \in -\text{int } \widehat{C}_+$  and  $y_0 \in \widehat{C}^1(T)$ .*

*Proof.* From Proposition 3.4 we already have two constant sign solutions  $u_0 \in \text{int } \widehat{C}_+$  and  $v_0 \in -\text{int } \widehat{C}_+$ . We may assume that  $K_{\widehat{\varphi}_+} = \{0, u_0\}$ , or otherwise we already have a third solution of (1.1) which in fact is positive.

**Claim:**  $C_k(\widehat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z}$  for all  $k \geq 0$ . Here and in what follows  $\delta_{k,j}$  ( $k, j \in \mathbb{Z}_+$ ) denotes the Kronecker delta. Let

$$d < 0 = \widehat{\varphi}_+(0) < \eta < \widehat{\eta}_\rho^+ \leq \widehat{\varphi}_+(u_0)$$

(see (3.32) and (3.34)). We consider the following triple of sets

$$\widehat{\varphi}_+^d \subseteq \widehat{\varphi}_+^\eta \subseteq W.$$

For this triple, we consider the corresponding long exact sequence of homology groups

$$\dots \rightarrow H_k(W, \widehat{\varphi}_+^d) \xrightarrow{i_*} H_k(W, \widehat{\varphi}_+^\eta) \xrightarrow{\partial_*} H_{k-1}(\widehat{\varphi}_+^\eta, \widehat{\varphi}_+^d) \rightarrow \dots, \quad (3.69)$$

where  $i_*$  is the group homomorphism induced by the inclusion  $(W, \widehat{\varphi}_+^d) \xhookrightarrow{i} (W, \widehat{\varphi}_+^\eta)$  and  $\partial_*$  is the boundary homomorphism. We have

$$H_k(W, \widehat{\varphi}_+^d) = C_k(\widehat{\varphi}_+, \infty) \text{ for all } k \geq 0 \text{ (see Proposition 3.6)}, \quad (3.70)$$

$$H_k(W, \widehat{\varphi}_+^\eta) = C_k(\widehat{\varphi}_+, u_0) \text{ for all } k \geq 0 \text{ (recall that } K_{\widehat{\varphi}_+} = \{0, u_0\}\text{)}, \quad (3.71)$$

$$H_{k-1}(\widehat{\varphi}_+^\eta, \widehat{\varphi}_+^d) = C_{k-1}(\widehat{\varphi}_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \quad (3.72)$$

(see the proof of Proposition 3.4)).

From (3.70), (3.71), (3.72) and the exactness of (3.69), we see that in (3.69) only the tail of a long sequence (i.e.,  $k = 1$ ), is nontrivial. So, we focus on the tail and use the rank theorem to obtain

$$\begin{aligned} \text{rank } C_1(\widehat{\varphi}_+, u_0) &= \text{rank } H_1(W, \widehat{\varphi}_+^\eta) \\ &= \text{rank } \ker \partial_* + \text{rank } \text{Im } \partial_* \\ &= \text{rank } \text{Im } i_* + \text{rank } \text{Im } \partial_* \text{ (by the exactness of (3.69))} \\ &\leq 0 + 1 \text{ (see (3.70), (3.72))}. \end{aligned} \quad (3.73)$$

On the other hand, from the proof of Proposition 3.4, we know that  $u_0 \in \text{int } \widehat{C}_+$  is a critical point of  $\widehat{\varphi}_+$  of mountain pass type. Hence

$$\text{rank } C_1(\widehat{\varphi}_+, u_0) \geq 1. \quad (3.74)$$

From (3.73) and (3.74), it follows that

$$\text{rank } C_1(\widehat{\varphi}_+, u_0) = 1,$$

hence

$$C_k(\widehat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0, \quad (3.75)$$

as claimed.

Since  $\varphi|_{\widehat{C}_+} = \widehat{\varphi}_+|_{\widehat{C}_+}$  and  $u_0 \in \text{int } \widehat{C}_+$ , we have

$$C_k(\varphi|_{\widehat{C}_+}, u_0) = C_k(\widehat{\varphi}_+|_{\widehat{C}_+}, u_0) \text{ for all } k \geq 0,$$

hence

$$C_k(\varphi, u_0) = C_k(\widehat{\varphi}_+, u_0) \text{ for all } k \geq 0 \text{ (see Palais [15])},$$

therefore

$$C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see (3.75))}. \quad (3.76)$$

Recall that  $v_0$  and  $0$  are local minimizers of  $\varphi$  (see Proposition 3.4 and its proof). Therefore, we have

$$C_k(\varphi, v_0) = C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0. \quad (3.77)$$

Finally, from Proposition 3.5, we have

$$C_k(\varphi, \infty) = 0 \text{ for all } k \geq 0. \quad (3.78)$$

Suppose that  $K_\varphi = \{0, v_0, u_0\}$ . From (3.76), (3.77), (3.78) and the Morse relation with  $t = -1$  (see (2.1)), we infer that

$$2(-1)^0 + (-1)^1 = 0,$$

which is a contradiction.

So, there exists  $y_0 \in K_\varphi$ ,  $y_0 \notin \{0, v_0, u_0\}$ .

Therefore  $y_0$  is the third nontrivial solution of (1.1) and  $y_0 \in \widehat{C}_+^1(T)$ .  $\square$

**Acknowledgement.** The third author gratefully acknowledges the partial support by *FEDER* funds through *COMPETE - Operational Programme Factors of Competitiveness* and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* and the Portuguese Foundation for Science and Technology (FCT), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690

## References

- [1] S. Aizicovici, N. S. Papageorgiou, and V. Staicu. Multiple nontrivial solutions for nonlinear periodic problems with the  $p$ -Laplacian. *J. Differential Equations*, 243:504–535, 2007.
- [2] S. Aizicovici, N. S. Papageorgiou, and V. Staicu. Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints. *Mem. Amer. Math. Soc.*, 196(915), 2008.
- [3] S. Aizicovici, N. S. Papageorgiou, and V. Staicu. Nonlinear resonant periodic problems with concave terms. *J. Math. Anal. Appl.*, 375:342–364, 2011.
- [4] S. Aizicovici, N. S. Papageorgiou, and V. Staicu. Positive solutions for nonlinear periodic problems with concave terms. *J. Math. Anal. Appl.*, 381:866–883, 2011.
- [5] S. Aizicovici, N. S. Papageorgiou, and V. Staicu. Positive solutions for nonlinear nonhomogeneous periodic eigenvalue problems. *Ann. Univ. Buch. Math. Ser.*, 3(LXI):129–144, 2012.
- [6] S. Aizicovici, N. S. Papageorgiou, and V. Staicu. Nodal and multiple solutions for nonlinear periodic problems with competing nonlinearities. *Commun. Contemp. Math.*, pp. 1–29, published online 12/03/2013. DOI:10.1142/S0219199713500016.
- [7] D. de Figueiredo and B. Ruf. On a superlinear Sturm-Liouville problem with a related bouncing problem. *J. Reine Angew. Math.*, 421:1–22, 1991.
- [8] M. Del Pino, R. Manasevich, and A. Murua. Existence and multiplicity of solutions with prescribed period for a second order quasilinear ODE. *Nonlinear Anal.*, 18:79–92, 1992.
- [9] Y. Dong. A Neumann problem at resonance with the nonlinearity restricted in one direction. *Nonlinear Anal.*, 51:739–747, 2002.
- [10] J. Dugundji. *Topology*. Allyn and Bacon Inc., Boston, 1966.
- [11] L. Gasinski. Positive solutions for resonant boundary value problems with the scalar  $p$ -Laplacian and a nonsmooth potential. *Discrete Contin. Dyn. Syst.*, 17:143–158, 2007.
- [12] L. Gasinski and N. S. Papageorgiou. *Nonlinear Analysis*. Chapman & Hall/CRC Press, Boca Raton, 2006.

- [13] L. Gasinski and N. S. Papageorgiou. Three nontrivial solutions for periodic problems with the p-Laplacian and a p-superlinear nonlinearity. *Commun. Pure Appl. Anal.*, 8:1421–1437, 2009.
- [14] A. Granas and J. Dugundji. *Fixed Point Theory*. Springer, New York, 2005.
- [15] R. Palais. Homotopy theory of infinite dimensional manifolds. *Topology*, 5:1–16, 1966.
- [16] N. S. Papageorgiou, E. M. Rocha, and V. Staicu. A multiplicity theorem for hemivariational inequalities with a p-Laplacian-like differential operator. *Nonlinear Anal.*, 69:1150–1163, 2008.
- [17] K. Perera. Existence and multiplicity results for Sturm-Liouville equations asymptotically linear at  $-\infty$  and superlinear at  $+\infty$ . *Nonlinear Anal.*, 39:669–684, 2000.
- [18] P. Pucci and J. Serrin. *The Maximum Principle*. Birkhauser, Basel, 2007.
- [19] S. Villegas. A Neumann problem with asymmetric nonlinearity and a related minimizing problem. *J. Differential Equations*, 145:145–155, 1998.
- [20] X. Yang. Multiple periodic solutions of a class of p-Laplacian. *J. Math. Anal. Appl.*, 314:17–29, 2006.

S. Aizicovici  
Department of Mathematics,  
Ohio University,  
Athens, OH 45701,  
USA  
E-mail: [aizicovs@ohio.edu](mailto:aizicovs@ohio.edu)

N. S. Papageorgiou  
Department of Mathematics,  
National Technical University,  
Zografou Campus,  
Athens 15780,  
Greece  
E-mail: [npapg@math.ntua.gr](mailto:npapg@math.ntua.gr)

V. Staicu  
CIDMA and Department of Mathematics,  
University of Aveiro,  
3810-193 Aveiro,  
Portugal  
E-mail: [vasile@ua.pt](mailto:vasile@ua.pt)