Hyers-Ulam-Rassias stability of Volterra integral equations within weighted spaces

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Abstract: We obtain weak conditions to guarantee the Hyers-Ulam-Rassias stability of (nonlinear) Volterra integral equations with delay. In particular, this leads to a generalization of some results previously known. Basically, this is done by using certain weight functions in the framework of the space of continuous functions. Indeed, the method consists in a convenient combination of the classical Banach fixed point theorem together with a consideration of a weighted metric. Therefore, we avoid the use of the strict successive approximation method and also the consideration of generalized metrics (which need to be typically combined with a consequent fixed point alternative theorem). Some concrete examples are presented at the end of the paper.

Keywords: Hyers-Ulam-Rassias stability, Volterra integral equation, fixed point, weight function.

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1 Introduction and related techniques

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability issue in functional equations is to understand and compare the behavior of the solutions of the inequality with those of the initially given functional equation. For the case of the additive Cauchy equation

\[ f(x + y) = f(x) + f(y) \]

the corresponding stability problem was considered already in 1925 by Gy. Pólya and G. Szegö in their book [21] (Teil I, Aufgabe 99) in the context of natural numbers (see [13]).

Much later, in 1941, Hyers [15] proved the following more general result by answering a problem of Ulam affirmatively (cf. [23] and [24]):
Let $S_1$ and $S_2$ be two (real) Banach spaces and assume that a mapping $h : S_1 \to S_2$ satisfies the inequality

$$\|h(x + y) - h(x) - h(y)\| \leq \epsilon, \quad x, y \in S_1,$$

for some nonnegative $\epsilon$. Then there is a (unique) additive mapping $A : S_1 \to S_2$ such that

$$\|A(x) - h(x)\| \leq \epsilon, \quad x \in S_1,$$

holds.

In addition, it was also proved in [15] that $A(x) = \lim_{n \to \infty} h(2^n x)/2^n$ ($x \in S_1$).

The last result is nowadays called the Hyers-Ulam Stability Theorem of the additive Cauchy equation. Since Hyers’ result, a great number of papers on the subject have been published, extending and generalizing Ulam’s problem and Hyers’ theorem in various directions. One of these new directions was introduced by Th. M. Rassias [22] by considering unbounded right-hand sides in (1.1) which depend on certain functions of $x$ and $y$ (instead of considering only bounded Cauchy differences $f(x+y) - f(x) - f(y)$ as in the Hyers case). Moreover, during the last fifteen years a considerable attention has been given to the study of Hyers-Ulam and Hyers-Ulam-Rassias stability of a great variety of functional equations (see, e.g., [5, 6, 10, 16, 17, 18]).

The formal definition of the above mentioned Hyers-Ulam-Rassias stability is now introduced for the type of (Volterra) integral equations with delay which we will be considering in here. Namely, for each function $y$ satisfying

$$\left| y(x) - g(x) - \Psi \left( \int_a^x f(x, t, y(t), y(\alpha(t))) \, dt \right) \right| \leq \sigma(x), \quad x \in [a, b]$$

(where $\sigma$ is a non-negative function and $\Psi$ a fixed given function), there is a solution $y_0$ of the Volterra integral equation

$$y(x) = g(x) + \Psi \left( \int_a^x f(x, t, y(t), y(\alpha(t))) \, dt \right), \quad x \in [a, b],$$

(1.2)

where $a$ and $b$ are fixed real numbers (such that $a < b$), $g : [a, b] \to \mathbb{C}$ and $f : [a, b] \times [a, b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are continuous functions and $\alpha : [a, b] \to [a, b]$ is a continuous delay function which therefore fulfils $\alpha(x) \leq x$, for all $x \in [a, b]$. Moreover, if there is a constant $C_1 > 0$ independent of $y$ and $y_0$ such that

$$|y(x) - y_0(x)| \leq C_1 \sigma(x),$$
for all \( x \), then we say that the integral equation (1.2) has the \textit{Hyers-Ulam-Rassias stability}.

In contrast to the direct method, some of the present techniques to obtain Hyers-Ulam-Rassias stability of functional equations use a combination of a fixed point alternative result with a generalized metric in an appropriate setting. More precisely, considering some set \( X \), we recall that a function \( d : X \times X \to [0, \infty) \) is called a \textit{generalized metric} on \( X \) if it satisfies:

- \( d(x_1, x_2) = 0 \) if and only if \( x_1 = x_2 \);
- \( d(x_1, x_2) = d(x_2, x_1) \) for all \( x_1, x_2 \in X \);
- \( d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3) \) for all \( x_1, x_2, x_3 \in X \).

Then, under suitable conditions, it is possible to use the following fixed point alternative result which we were mentioning above.

\textbf{Theorem 1.1} ([9]). Let \((X, d)\) be a complete generalized metric space and let \( T : X \to X \) be a strictly contractive operator with Lipschitz constant \( L < 1 \). Then, for each given element \( x \in X \), either

(i) \( d(T^n x, T^{n+1} x) = \infty \) for all nonnegative integers \( n \), or

(ii) there exists a positive integer \( n_0 \) such that

(1) \( d(T^n x, T^{n+1} x) < \infty \), for all \( n \geq n_0 \);

(2) the sequence \((T^n x)_{n \in \mathbb{N}}\) converges to a fixed point \( y^* \) of \( T \);

(3) \( y^* \) is the unique fixed point of \( T \) in the set \( Y = \{ y \in X \mid d(T^{n_0} x, y) < \infty \} \);

(4) \( d(y, y^*) \leq \frac{1}{1-L} d(y, Ty) \), for all \( y \in Y \).

One of the goals of the present note is to show that even for somehow very general nonlinear integral equations we may obtain conditions to ensure the Hyers-Ulam-Rassias stability without going to generalized metrics or using some alternative fixed point theorems. Instead of these techniques, we may simply use the classical Banach fixed point theorem if combined with an appropriate weighted metric framework. We should mention that this type of weighted spaces goes back to [12] and [4], where it was applied to other types of functional and integral equations. Here, it is also significant to recall the seminal paper by A. Bielecki [3], where specific exponential weighted metrics were introduced with the aim of obtaining global existence of solutions of certain functional equations. In the scope of Volterra integral equations, we also would like to refer [7, § 5] where general weighted spaces were introduced.
in view of obtaining consequent global solutions of corresponding vectorial Volterra integral equations.

Within this scope, in the present work we propose new conditions for obtaining the Hyers-Ulam-Rassias stability of Volterra integral equations with delay \[1, 8, 14, 19\] by using spaces of continuous functions endowed with a weighted metric. In particular, the present results generalize the main results of O. Baghani, M. Gachpazan and H. Baghani \[2, 11\] and present weaker conditions to obtain the Hyers-Ulam-Rassias stability of the class of integral equation under study when compared with the recent results of J.R. Morales and E.M. Rojas \[20\].

\section{Hyers-Ulam-Rassias stability of Volterra integral equations with delay}

Let us consider a fixed finite interval \(I = [a, b]\) \((a < b)\), and a non-decreasing continuous function \(\varphi : I \rightarrow (0, \infty)\). We will be using the space \(C(I)\) of continuous functions, \(g : I \rightarrow \mathbb{C}\), endowed with the metric

\[
d(u, v) = \sup_{x \in I} \frac{|u(x) - v(x)|}{\varphi(x)}.
\]  

We recall that \((C(I), d)\) is a complete metric space (cf., e.g., \[4\]).

The next theorem is the main result of this note and exhibits weaker conditions than those of \[20\], under which the Volterra integral equation introduced in \(1.2\) is Hyers-Ulam-Rassias stable. This result also generalizes several other previous known cases (like those of \[2, 11\]).

**Theorem 2.1.** Let us consider continuous given functions \(\mu : I \times I \rightarrow [0, \infty)\) and \(\eta : I \times I \rightarrow [0, \infty)\). Moreover, assume that \(g \in C(I)\), \(f : I \times I \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}\) is continuous, \(\alpha : I \rightarrow I\) is continuous and such that \(\alpha(x) \leq x\), for all \(x \in I\), and \(\Psi : C(I) \rightarrow C(I)\) is bounded in the sense that there exists \(K > 0\) such that:

\[
d(\Psi(h_1), \Psi(h_2)) \leq Kd(h_1, h_2).
\]  

In addition, suppose that there are \(\beta, \gamma \in [0, 1)\) such that

\[
\int_a^x \mu(x, t)\varphi(t)dt \leq \beta\varphi(x)
\]  

and

\[
\int_a^x \eta(x, t)\varphi(t)dt \leq \gamma\varphi(x),
\]  

where
and that

\[ |f(x, t, u(t), u(\alpha(t))) - f(x, t, v(t), v(\alpha(t)))| \leq \mu(x, t) |u(t) - v(t)| + \eta(x, t) |u(\alpha(t)) - v(\alpha(t))|, \]  

(2.5)

for all \( x, t \in I, u, v \in C(I) \).

If \( y \in C(I) \) is such that

\[ \left| y(x) - g(x) - \Psi \left( \int_a^x f(x, t, y(t), y(\alpha(t))) \, dt \right) \right| \leq \varphi(x), \quad x \in I, \]  

(2.6)

and \( K(\beta + \gamma) < 1 \), then there is a unique function \( y_0 \in C(I) \) such that

\[ y_0(x) = g(x) + \Psi \left( \int_a^x f(x, t, y_0(t), y_0(\alpha(t))) \, dt \right) \]  

(2.7)

and

\[ |y(x) - y_0(x)| \leq \frac{\varphi(x)}{1 - K(\beta + \gamma)}. \]  

(2.8)

This means that under the above conditions, the Volterra integral equation (1.2) has the Hyers-Ulam-Rassias stability.

**Proof.** We will consider the operator \( T : C(I) \to C(I) \), defined by

\[ (Tu)(x) = g(x) + \Psi \left( \int_a^x f(x, t, u(t), u(\alpha(t))) \, dt \right), \]  

(2.9)

for all \( x, t \in I \) and \( u \in C(I) \).

Under the present conditions, we will now deduce that the operator \( T \) is strictly contractive (with respect to the metric under consideration). Indeed, for all \( u, v \in C(I) \), we have:

\[ d(Tu, Tv) = \sup_{x \in I} \frac{|(Tu)(x) - (Tv)(x)|}{\varphi(x)} \]
\[
\begin{align*}
&= \sup_{x \in \mathcal{I}} \left| \Psi \left( \int_a^x f(x, t, u(t), u(\alpha(t))) dt \right) - \Psi \left( \int_a^x f(x, t, v(t), v(\alpha(t))) dt \right) \right| \\
&\leq K \sup_{x \in \mathcal{I}} \frac{\left| \int_a^x f(x, t, u(t), u(\alpha(t))) dt - \int_a^x f(x, t, v(t), v(\alpha(t))) dt \right|}{\varphi(x)} \\
&\leq K \sup_{x \in \mathcal{I}} \frac{\int_a^x \left| f(x, t, u(t), u(\alpha(t))) - f(x, t, v(t), v(\alpha(t))) \right| dt}{\varphi(x)} \\
&\leq K \sup_{x \in \mathcal{I}} \frac{\int_a^x (\mu(x, t) |u(t) - v(t)| + \eta(x, t) |u(\alpha(t)) - v(\alpha(t))|) dt}{\varphi(x)} \\
&= K \sup_{x \in \mathcal{I}} \frac{\int_a^x \mu(x, t) \varphi(t) \frac{|u(t) - v(t)|}{\varphi(t)} dt + \int_a^x \eta(x, t) \varphi(\alpha(t)) \frac{|u(\alpha(t)) - v(\alpha(t))|}{\varphi(\alpha(t))} dt}{\varphi(x)} \\
&\leq K \left[ \sup_{t \in \mathcal{I}} \frac{|u(t) - v(t)|}{\varphi(t)} \sup_{x \in \mathcal{I}} \int_a^x \mu(x, t) \varphi(t) dt \\
&\quad + \sup_{x \in \mathcal{I}} \frac{|u(\alpha(t)) - v(\alpha(t))|}{\varphi(\alpha(t))} \sup_{x \in \mathcal{I}} \int_a^x \eta(x, t) \varphi(\alpha(t)) dt \right] \\
&\leq K [d(u, v) \cdot \beta + d(u, v) \cdot \gamma] \\
&= K (\beta + \gamma) \cdot d(u, v)
\end{align*}
\]

Due to the fact that \(K(\beta + \gamma)<1\), it follows that \(T\) is strictly contractive. Thus, we can apply the Banach Fixed Point Theorem, which ensures by (2.6) that \(y_0\) is the unique function with the properties (2.7) and (2.8) – and so we have the Hyers-Ulam-Rassias stability for (1.2).

A direct application of the previous result for the particular case of \(\Psi g = \lambda g\), for some parameter \(\lambda\), yields the following corollary for corresponding linear Volterra integral equations.

**Corollary 2.2.** Let us consider continuous given functions \(\mu : I \times I \rightarrow [0, \infty)\) and \(\eta : I \times I \rightarrow [0, \infty)\). Assume additionally that \(g \in C(I)\), \(f : I \times I \times \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}\) is continuous and \(\alpha : I \rightarrow I\) is continuous and such that \(\alpha(x) \leq x\), for all \(x \in I\).

Suppose that there are \(\beta, \gamma \in [0, 1)\) such that

\[
\int_a^x \mu(x, t) \varphi(t) dt \leq \beta \varphi(x) \tag{2.10}
\]

and

\[
\int_a^x \eta(x, t) \varphi(t) dt \leq \gamma \varphi(x). \tag{2.11}
\]

Due to the fact that \(K(\beta + \gamma)<1\), it follows that \(T\) is strictly contractive. Thus, we can apply the Banach Fixed Point Theorem, which ensures by (2.6) that \(y_0\) is the unique function with the properties (2.7) and (2.8) – and so we have the Hyers-Ulam-Rassias stability for (1.2).
Finally, assume that
\[
|f(x, t, u(t), u(\alpha(t))) - f(x, t, v(t), v(\alpha(t)))| \\
\leq \mu(x, t) |u(t) - v(t)| + \eta(x, t) |u(\alpha(t)) - v(\alpha(t))|,
\]
(2.12)
for all \(x, t \in I, u, v \in C(I)\).

If \(y \in C(I)\) is such that
\[
\left| y(x) - g(x) - \lambda \int_a^x f(x, t, y(t), y(\alpha(t))) \, dt \right| \leq \varphi(x), \quad x \in I,
\]
(2.13)
and \(|\lambda| (\beta + \gamma) < 1\), then there is a unique function \(y_0 \in C(I)\) such that
\[
y_0(x) = g(x) + \lambda \int_a^x f(x, t, y_0(t), y_0(\alpha(t))) \, dt
\]
(2.14)
and
\[
|y(x) - y_0(x)| \leq \frac{\varphi(x)}{1 - |\lambda| (\beta + \gamma)}.
\]
(2.15)

This means that (under the indicated conditions)
\[
y(x) = g(x) + \lambda \int_a^x f(x, t, y(t), y(\alpha(t))) \, dt
\]
has the Hyers-Ulam-Rassias stability.

To conclude this section, we would like to consider a modification of the Volterra integral equation with delay (1.2) to the situation of infinite intervals instead of the compact case \(I = [a, b]\) (with \(b \in \mathbb{R}\)). Namely, within the context of bounded continuous functions, we will now consider
\[
y(x) = g(x) + \Psi \left( \int_a^x f(x, t, y(t), y(\alpha(t))) \, dt \right), \quad x \in [a, \infty),
\]
(2.16)
where \(a\) is a fixed real number, \(g : [a, \infty) \to \mathbb{C}\) and \(f : [a, \infty) \times [a, \infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}\), and \(\alpha : [a, \infty) \to [a, \infty)\) is a continuous delay function fulfilling \(\alpha(x) \leq x\), for all \(x \in [a, \infty)\).

The reasoning we apply to (2.16) is essentially the same as the one exhibited in the proof of Theorem 2.1, and so we will present the consequent result stated in the form of the next theorem (its proof being omitted here).
Theorem 2.3. Let us consider a fixed non-decreasing continuous function \( \varphi : [a, \infty) \to (\varepsilon, \omega) \), for some \( \varepsilon, \omega > 0 \), and the space \( C_b([a, \infty)) \) of bounded continuous functions endowed with the weighted metric

\[
d_\infty(u, v) = \sup_{x \in [a, \infty)} \frac{|u(x) - v(x)|}{\varphi(x)}. \tag{2.17}
\]

We assume to have continuous given functions \( \mu : [a, \infty) \times [a, \infty) \to [0, \infty) \) and \( \eta : [a, \infty) \times [a, \infty) \to [0, \infty) \), and assume also that \( g \in C_b([a, \infty)) \), \( \alpha : [a, \infty) \to [a, \infty) \) is continuous and such that \( \alpha(x) \leq x \), for all \( x \in [a, \infty) \), \( f : [a, \infty) \times [a, \infty) \times \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) is such that \( \int_a^x f(x, t, z(t), z(\alpha(t)))dt \in C_b([a, \infty)) \) for any \( z \in C_b([a, \infty)) \), and that \( \Psi : C_b([a, \infty)) \to C_b([a, \infty)) \) is bounded in the sense that there exists \( K > 0 \) so that

\[
d_\infty(\Psi(h_1), \Psi(h_2)) \leq K d_\infty(h_1, h_2) \tag{2.18}
\]

(for all \( h_1 \) and \( h_2 \)). In addition, suppose that there are \( \beta, \gamma \in [0, 1) \) such that

\[
\int_a^x \mu(x, t)\varphi(t)dt \leq \beta \varphi(x) \tag{2.19}
\]

and

\[
\int_a^x \eta(x, t)\varphi(t)dt \leq \gamma \varphi(x), \tag{2.20}
\]

and that

\[
\begin{align*}
|f(x, t, u(t), u(\alpha(t))) - f(x, t, v(t), v(\alpha(t)))| & \\
& \leq \mu(x, t)|u(t) - v(t)| + \eta(x, t)|u(\alpha(t)) - v(\alpha(t))|,
\end{align*} \tag{2.21}
\]

for all \( x, t \in [a, \infty) \), \( u, v \in C_b([a, \infty)) \).

If \( y \in C_b([a, \infty)) \) is such that

\[
\left| y(x) - g(x) - \Psi \left( \int_a^x f(x, t, y(t), y(\alpha(t)))dt \right) \right| \leq \varphi(x), \quad x \in [a, \infty), \tag{2.22}
\]

and \( K(\beta + \gamma) < 1 \), then there is a unique function \( y_0 \in C_b([a, \infty)) \) such that

\[
y_0(x) = g(x) + \Psi \left( \int_a^x f(x, t, y_0(t), y_0(\alpha(t)))dt \right) \tag{2.23}
\]

and

\[
|y(x) - y_0(x)| \leq \frac{\varphi(x)}{1 - K(\beta + \gamma)}. \tag{2.24}
\]

This means that under the above conditions, the Volterra integral equation (2.16) has the Hyers-Ulam-Rassias stability.

We point out that the reason why we are here considering the space of bounded continuous functions and \( \varphi \) taking values on \((\varepsilon, \omega)\) is to prevent \( d_\infty \) of being a generalized metric in the present case of \( I = [a, \infty) \).
3 Illustrative Examples

In this section, we will present some examples that illustrate the results and the importance of the conditions presented above. In the first one, we will exhibit an example of the application of Corollary 2.2. In the second and third examples, we will exemplify the possibility of having a variety of choices of weight functions \( \varphi \) when using Theorem 2.1. In a last example, we will consider an adaptation of the first example to an infinite interval case. This will help us to illustrate that it is not trivial to guarantee the conditions in Theorem 2.3.

Example 3.1. Consider the equation

\[
y(x) = \frac{1}{8}x + \frac{7}{8} + \frac{1}{2} \int_{1}^{x} \frac{1}{x} (y(t) - y(\alpha(t))) \, dt, \quad x \in [1, 10], \tag{3.1}
\]

where \( \alpha(t) = \frac{t}{2}, \ t \in [1, 10] \). Moreover, let us take \( \varphi : [1, 10] \to (0, \infty) \) such that \( \varphi(x) = x \).

Within the last section notation, if we perform the realization \( \Psi g = \lambda g \), for \( \lambda = \frac{1}{2} \), we may identify \( f(x, t, y(t), y(\alpha(t))) := \frac{1}{x}(y(t) - y(\alpha(t))) \). Moreover, we can see that all the remaining considered functions are under the conditions of Corollary 2.2. Namely:

- \( g : [1, 10] \to \mathbb{C} \) is continuous;

- \( \alpha : [1, 10] \to [1, 10] \) is continuous and such that \( \alpha(t) \leq t \), for all \( t \in [1, 10] \);

- \( f : [1, 10] \times [1, 10] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) is continuous and such that

\[
|f(x, t, u(t), u(\alpha(t))) - f(x, t, v(t), v(\alpha(t)))| = \left| \frac{1}{x} (u(t) - u(\alpha(t))) - \frac{1}{x} (v(t) - v(\alpha(t))) \right| \\
\leq \frac{1}{x} |u(t) - v(t)| + \frac{1}{x} |u(\alpha(t)) - v(\alpha(t))|;
\]

- from the previous item, we can see that the functions \( \mu : [1, 10] \to [0, \infty) \) and \( \eta : [1, 10] \to [0, \infty) \) are such that \( \mu(x, t) = \eta(x, t) = \frac{1}{x} \).
Now, we will compute $\beta$ and $\gamma$. Since $\mu(x,t) = \eta(x,t) = \frac{1}{x}$, we will have:

$$\int_{1}^{x} \mu(x,t)\varphi(t)dt = \int_{1}^{x} \eta(x,t)\varphi(t)dt$$

$$= \int_{1}^{x} \frac{1}{x} t dt$$

$$= \frac{1}{2} x \left(1 - \frac{1}{x^2}\right)$$

$$\leq \frac{1}{2} \varphi(x),$$

for all $x \in [1, 10]$. Therefore, we may take $\beta = \gamma = \frac{1}{2}$. Thus, $|\lambda|(\beta + \gamma) = \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2} < 1.$

If we simply choose $y(x) = 0$, it follows:

$$\left|y(x) - g(x) - \Psi \left(\int_{a}^{x} f(x,t,y(t),y(\alpha(t))) dt\right)\right|$$

$$= \left|0 - \frac{1}{8x} - \frac{7x}{8} - \frac{1}{2} \int_{1}^{x} \frac{1}{x} \cdot 0 dt\right|$$

$$= \left|\frac{1}{8x} + \frac{7x}{8}\right|$$

$$\leq x = \varphi(x),$$

for all $x \in [1, 10]$. Therefore, from Corollary 2.2 we conclude that the Volterra integral equation with delay (3.1) has the Hyers-Ulam-Rassias stability.

In addition, we notice that the exact solution of the equation (3.1) is $y_0(x) = x$. Indeed,

$$\frac{1}{8x} + \frac{7x}{8} + \frac{1}{2} \int_{1}^{x} \frac{1}{x} \cdot \left(t - \frac{t}{2}\right) dt = \frac{1}{8x} + \frac{7x}{8} + \frac{1}{2x} \int_{1}^{x} \frac{t}{2} dt$$

$$= x.$$

On the other hand, we obviously have

$$|y(x) - y_0(x)| = |0 - x| \leq \frac{x}{1 - \frac{1}{2}} = \frac{x}{\frac{1}{2}} = 2x = 2\varphi(x),$$

for all $x \in [1, 10]$.

**Example 3.2.** For this example, we will work with $\varphi : [1, 10] \rightarrow (0, \infty)$ such that $\varphi(x) = x^2$, and will consider the following equation very similar to (3.1) but different:

$$y(x) = \frac{1}{4x} + \frac{3x}{4} + \int_{1}^{x} \frac{1}{x} (y(t) - y(\alpha(t))) dt, \quad x \in [1, 10], \quad (3.2)$$
where $\alpha(t) = \frac{t}{2}$, $t \in [1, 10]$.

We are going to apply Theorem 2.1. In fact, we see that all the considered functions are under the conditions of Theorem 2.1:

- $g: [1, 10] \to \mathbb{C}$ is continuous;
- $\alpha: [1, 10] \to [1, 10]$ is continuous and such that $\alpha(t) \leq t$, for all $t \in [1, 10]$;
- $f: [1, 10] \times [1, 10] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is continuous and such that
  \[ |f(x, t, u(t), u(\alpha(t))) - f(x, t, v(t), v(\alpha(t)))| \]
  \[ = \left| \frac{1}{x}(u(t) - u(\alpha(t))) - \frac{1}{x}(v(t) - v(\alpha(t))) \right| \]
  \[ \leq \frac{1}{x} |u(t) - v(t)| + \frac{1}{x} |u(\alpha(t)) - v(\alpha(t))|; \]
- from the previous item, we can see that the functions $\mu$ and $\eta$ are such that
  \[ \mu: [1, 10] \to [0, \infty) \text{ and } \eta: [1, 10] \to [0, \infty) \text{ and } \mu(x, t) = \eta(x, t) = \frac{1}{x}; \]
- $d(\Psi(h_1), \Psi(h_2)) \leq 1 \cdot d(h_1, h_2)$, and so $K = 1$.

As about looking for $\beta$ and $\gamma$, we have:

\[ \int_1^x \mu(x, t)\varphi(t)dt = \int_1^x \eta(x, t)\varphi(t)dt \]
\[ = \int_1^x \frac{1}{x} dt \]
\[ = x^2 \left( \frac{1}{3} - \frac{1}{3x^3} \right) \leq \frac{1}{3} \varphi(x), \]
for all $x \in [1, 10]$.

In this way, it is enough to choose $\beta = \gamma = \frac{1}{3}$ since in this case $K(\beta + \gamma) = 1 \cdot \left( \frac{1}{3} + \frac{1}{3} \right) = \frac{2}{3} < 1$. If we choose $y(x) = 0$, it holds

\[ \left| y(x) - g(x) - \Psi \left( \int_a^x f(x, t, y(t), y(\alpha(t))) \, dt \right) \right| = \left| \frac{1}{4x} - \frac{3x}{4} - \int_1^x \frac{1}{x} \cdot 0 \, dt \right| \]
\[ = \left| \frac{1}{4x} + \frac{3x}{4} \right| \leq x^2 = \varphi(x), \]
for all $x \in [1, 10]$.

Therefore, in the present conditions, we conclude that the Volterra integral equation with delay (3.2) has the Hyers-Ulam-Rassias stability.
Example 3.3. In the present example we would like to illustrate the circumstance that –in general– there may exist several different weight functions $\varphi$ which lead to stable integral equations in the above sense. In view of this, we shall consider the same equation of the last example (equation (3.2)), but now, taking another function $\varphi : [1, 10] \to (0, \infty)$, defined simply by $\varphi(x) = x$.

In such a case, for the computation of the new $\beta$ and $\gamma$, we will have:

$$\int_1^x \mu(x,t)\varphi(t)dt = \int_1^x \eta(x,t)\varphi(t)dt$$

$$= \int_1^x \frac{1}{x} dt$$

$$= x \left( \frac{1}{2} - \frac{1}{2x^2} \right)$$

$$\leq \frac{99}{200} \varphi(x),$$

for all $x \in [1, 10]$. In this way, we have $\beta = \gamma = \frac{99}{200}$. Therefore, $K(\beta + \gamma) = 1 \cdot \left( \frac{99}{200} + \frac{99}{200} \right) = \frac{198}{200} < 1$ and so, also in this case, we can apply Theorem 2.1 and obtain the corresponding Hyers-Ulam-Rassias stability.

Example 3.4. We end up with an example which is outside the conditions of the results of the last section. We shall consider essentially the same conditions of Example 3.3 or Example 3.1 (and so with $K = 1$ or $K = 1/2$) but, instead of considering the interval $I = [1, 10]$, we will have the infinite interval $[1, \infty)$ (and can consider here any suitable bounded continuous function $g$, defined on $[1, \infty)$, instead of the previous continuous function $g$). Thus, within the framework of bounded continuous functions and taking $\varphi$ formally defined by $\varphi : [1, \infty) \to (1/4, 2)$, $\varphi(x) = x/2$ if $x \in [1, 2]$ and $\varphi(x) = 1$ if $x > 2$, by following Theorem 2.3 everything will occur in the same way as before up to the place where we need to compute $\beta$ and $\gamma$, and where we will have:

$$\int_1^x \mu(x,t)\varphi(t)dt = \int_1^x \eta(x,t)\varphi(t)dt$$

$$= \frac{x}{2} \left( \frac{1}{2} - \frac{1}{2x^2} \right)$$

$$\leq \frac{3}{8} \varphi(x), \quad \text{in case } x \in [1, 2]$$
and
\[ \int_1^x \mu(x,t)\phi(t)dt = \int_1^x \eta(x,t)\phi(t)dt = 1 - \frac{5}{4x} \leq \phi(x), \quad \text{in case } x > 2. \]

In this way, the best we have for the corresponding constants is \( \beta = \gamma = 1 \). Therefore, \( K(\beta + \gamma) \not< 1 \) and so, we cannot ensure the Hyers-Ulam-Rassias stability of the corresponding equation.

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