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Existence of solutions for quadratic integral inclusions

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Abstract: In this paper we study the existence of continuous solutions of a quadratic integral inclusion: we will look for solutions of that equation in a Banach space.

The theory of quadratic integral inclusions has many useful applications in mathematical physics, economics, biology, as well as in describing real world problems. The main tool used in our investigations is a fixed point result for the multivalued solution's map with acyclic values.

Keywords: Fixed point property; measure of noncompactness, multivalued mappings; acyclic valued; quadratic integral inclusions.

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Dedicated to the memory of Professor Francesco S. De Blasi

1 Introduction and Notations

In this paper, we are going to study the solvability of a nonlinear quadratic integral inclusion of the kind

$$x(t) \in (fx)(t) \int_0^t k(t,s)F(t,s,x(s))ds.$$

We will look for solutions of that equation in the Banach space of real functions being defined and continuous on a bounded and closed interval of the real line.

The main tool used in our investigations will be a special measure of noncompactness constructed in such a way that its use enables us to study the solvability of considered equations with a fixed point theorem for contraction mappings with compact acyclic values. Let us mention that the theory of integral equations has many useful applications in describing numerous events and problems of the real world. For example, integral equations are often applicable in engineering, mathematical physics, economics, and biology ([11], [16]). Let us pay attention to the fact that the so-called nonlinear quadratic integral equations are also often encountered in various applications; it is worthwhile mentioning the applications of those equations in the theory of radiative transfer, kinetic theory of gases, in the traffic theory and in the theory of neutron transport, for instance. Especially, the so-called quadratic integral equation of Chandrasekhar type can be very often encountered ([5], [13]).

In several areas of applied mathematics, like control theory, mathematical economics, mechanics, vehicular traffic theory, biology, queuing theory etc., we encounter problems that involve various types of ambiguity, indeterminacy or uncertainty turning sometimes on the impossibility of a comprehensive description of the dynamics of the system (see [25]). From many years it's well known that the evolution of such systems is then described by a multivalued equation (differential or integral). In recent years a number of papers have appeared concerning integral inclusions. For instance the following nonlinear functional-integral equation is studied ([12] and [20]):

$$x(t) = f(t, x(t)) \int_0^t F(t, s, x(s))ds; \quad t \in [0, T].$$

The study of integral equations has received much attention over the last fifty years or so (see [1], [2], [3], [4], [7], [8], [9], and their references). However very few results are available for integral inclusions. Our paper has the main objectives to show that the techniques applied in establishing existence results for integral equations transfer naturally to integral inclusions. Our ideas were motivated by results of Czarnowski and Pruszko (see [14]), and for integral inclusions by results in [10], [18], [23], [24], [26].

2 Preliminaries

In what follows $B_r(x_0)$ will denote an $r-$ ball (in a metric space $(\mathcal{N},d)$) i.e. the set $\{x \in \mathcal{N} : d(x, x_0) < r\}$, centered at x_0, where x_0 is any point in $\mathcal{N}$.

We will denote by $C = C(I, \mathbb{R})$, where $I = [0, T]$, the Banach space of all continuous functions (from I to $\mathbb{R}$) equipped with the sup norm, where I is a real interval.

If $\mathcal{E}$ is a Banach space we will denote by $\mathcal{BD}$ $(\mathcal{E})$ the family of all nonempty bounded closed subsets of $\mathcal{E}$, by $\mathcal{KC}$ $(\mathcal{E})$ the family of all nonempty convex closed subsets of $\mathcal{E}$ and by $\mathcal{KA}$ $(\mathcal{E})$ the family of all nonempty compact acyclic subsets of $\mathcal{E}$.

For $a \in \mathcal{E}$ and $A, B \in \mathcal{BD}(\mathcal{E})$ set $d_I(a, B) = \inf_{b \in B} d(a, b)$ and $d_S(A, B) = \sup_{a \in A} d_I(a, B)$. Denote by D the Hausdorff generalized metric on $\mathcal{BD}(\mathcal{E})$ defined by $D(A, B) = \max\{d_S(A, B), d_S(B, A)\}, \quad A, B \in \mathcal{BD}(\mathcal{E})$.

Let X be a nonempty set and let $F : X \longrightarrow \mathcal{BD}(\mathcal{E})$ be a set-valued map from X to $\mathcal{E}$. The range of F is the set $F(X) = \cup_{x\in X}F(x)$.

If F is a set-valued mapping from the metric space $\mathcal{N}$ into a metric space $\mathcal{M}$ (i.e. $F : \mathcal{N} \longrightarrow \mathcal{BD}(\mathcal{M})$), then F is called *upper semicontinuous* at the point $x \in \mathcal{N}$ if the following property holds: for any sequence $\{x_n\} \subset \mathcal{N}$ which converges in $\mathcal{N}$, $x_n \to x$ as $n \to +\infty$, we have $d_S(F(x_n), F(x)) \to 0$ as $n \to +\infty$.

Definition 1. Let X be a topological space and let $\breve{H}^p(X, Z)$ denote the reduced Čech cohomology group of X in dimension p with coefficients in Z. Then by an *acyclic* we mean a (non empty) topological space X such that $\breve{H}^p(X, Z) = 0$ for every $p \geq 0$ (see, e.g., [15]).

The function χ defined on $\mathcal{BD}$ $(\mathcal{E})$ by

$$\chi(X) = \inf\{\epsilon > 0 : X \text{ has a finite } \epsilon\text{-net in } E\ \}$$

is usually called the *Hausdorff measure of non compactness* (see [6]). In the Banach space $C(I, \mathbb{R})$ it is possible to show (see [7]) that $\chi(X) = \frac{1}{2}\omega_0(X)$ where $\omega_0(X) = \lim_{\epsilon\to 0} \sup\{\omega(x, \epsilon),\ x \in X\}$ and $\omega(x, \epsilon) = \sup\{|x(t)-x(s)|;\ t, s \in I,\ |t-s| < \epsilon\}$ (i.e. the so called modulus of continuity of the function $t \longrightarrow x(t)$).

The following definition will be useful in the sequel:

Definition 2. Let M be a nonempty set in a Banach space $\mathcal{E}$ and let $\mathbb{U} : M \longrightarrow \mathcal{BD}$ $(\mathcal{E})$ be an upper semicontinuous operator mapping bounded sets onto bounded ones. We will say that $\mathbb{U}$ satisfies the *Darbo condition* (with a constant $k \geq 0$) if for any bounded subset we have $\chi(\mathbb{U}(X)) \leq k\,\chi(X)$. It is well known that, if $k < 1$, the operator $\mathbb{U}$ is called a *contraction*.

Our main result is essentially based on the following fixed point result:

Fixed point Theorem: (see [17]) *Let M be a nonempty, closed and convex subset of a Banach space $\mathcal{E}$ and let $F : M \longrightarrow \mathcal{KA}$ $(\mathcal{E})$ be a contraction. Then F admits a fixed point.*

The well known *Gronwall Lemma*, from the standard theory of Ordinary Differential Equations, will also be used:

Gronwall Lemma *If, for $t_0 \leq t \leq t_1$, $\phi(t) \geq 0$ and $\psi(t) \geq 0$ are continuous functions such that the inequality $\phi(t) \leq c(t)+h \int_{t_0}^{t} \psi(s)\phi(s)ds$ holds on $t_0 \leq t \leq t_1$, with $c(t)$ and h positive function (or constant), then $\phi(t) \leq c(t) \exp\left(h \int_{t_0}^{t} \psi(s)ds\right)$ on $t_0 \leq t \leq t_1$* (see, e.g., [12]).

Proposition 1: (see [4], [19], [26] and related references). *Let* $F(t,x) : [0,T] \times B_r(0) \longrightarrow \mathcal{KC}(\mathbb{R}^n))$ *be an upper semicontinuous multis. Then there is a sequence* F_n *of multivalued functions defined from* $[0,T] \times B_r(0)$ *into* $\mathcal{KC}(\mathbb{R}^n)$ *such that*

1. $F_{n+1}(t,x) \subset F_n(t,x)$ *for every* $n \in \mathbb{N}$, *all* $t \in [0,T]$ *and any* $x \in B_r(0)$;
2. $F(t,x) = \cap_{n\in\mathbb{N}} F_n(t,x)$, *for every* $t \in [0,T]$ *and any* $x \in B_r(0)$;
3. *each* $F_n(t,x)$ *has a selection* $\phi_n : [0,T] \times B_r(0) \longrightarrow \mathbb{R}^n$ *such that*
    - *i) there is a positive* α_n *such that* $|\phi_n(t,x)| \leq \alpha_n$ *for every* $t \in [0,T]$ *and every* $x \in B_r(0)$;
    - *ii) the functions* $\phi_n : [0,T] \longrightarrow \mathbb{R}^n$ *are Lebesgue measurable for every* $x \in B_r(0)$;
    - *iii) for every* $t \in [0,T]$ *the functions* $\phi_n : B_r(0) \longrightarrow \mathbb{R}^n$ *are Lipschitz functions, i.e.* $|\phi_n(t,x) - \phi_n(t,y)| \leq \gamma_n|x-y|$.

Proposition 2:(see [21], [22] and related references). *For a compact metric space* X *the following are equivalent:*

- *i)* X *is the intersection of a decreasing sequence of compact contractible metric spaces; and*
- *ii)* X *is the intersection of a decreasing sequence of compact absolute retracts, i.e.* X *is an* R_δ*-set.*

Finally we observe that, from the continuity of the Čech cohomology functor, we can say that any R_δ set is also an acyclic space.

3 Main result

Here we want to deal with the quadratic integral equation of the type:

$$x(t) \in (fx)(t) \int_0^t k(t,s)F(s,x(s))ds, \quad t \in I = [0,T] \tag{3.1}$$

with $t \in I = [0,T] \subset \mathbb{R}$.

For such a kind of equation we assume the following conditions are satisfied:

i) $f : C = C(I,\mathbb{R}^n) \longrightarrow C(I,\mathbb{R}^n)$ is a continuous operator satisfying the Darbo property (with positive constant f_0), and such that $|(fx)(t)| \leq \beta_0|x|$, $\forall x \in C, \ \forall t \in I$;

ii) $k : I \times I \longrightarrow \mathbb{R}^n$ is a continuous function such that there is a bounded function $w : I \longrightarrow \mathbb{R}^+$, with $\lim_{t\to 0^+} w(t) = 0$ and $\int_0^T |k(t_2, s) - k(t_1, s)|ds \leq w(|t_2 - t_1|)$ for each couple $t_1, t_2 \in I$;

iii) $F : I \times \mathbb{R}^n \longrightarrow \mathcal{KC}(\mathbb{R}^n)$ is an upper semicontinuous multifunction such that $|F(t, x)| \leq \alpha(t) + \beta(t)|x|$, for every $x \in C$, for every $t \in I$ and where $t \longrightarrow \alpha(t)$ and $t \longrightarrow \beta(t)$ are continuous positive real functions defined on I; (here $|F(t, x)| = D(F(t, x), 0)$);

iv) the following inequality holds: $w_0 \leq min(\frac{1}{\beta_0\alpha_1}, \frac{1}{\alpha_1 f_0})$, where $\alpha_1 = \{\sup \alpha(t), t \in I\}$ and w_0 will be defined below.

Remark 1: Observe that from assumption ii) we have
$\int_0^T |k(t, s)|ds \leq \int_0^T |k(t, s) - k(0, s)|ds + \int_0^T |k(0, s)|ds \leq k_0 + w_1$
where
$k_0 = \int_0^T |k(0, s)|ds$ and $w_1 = \sup\{w(t),\ t \in [0, T]\}$.
This implies that $w_0 = \sup\{\int_0^T |k(t, s)|ds, t \in I\} < +\infty$.

Then we can formulate our main existence theorem:

Theorem 1. : Let the previous conditions i) - iv) be satisfied. Then the integral equation (1) has at least one solution in the space C .

Proof: Let M be a suitable ball $B_\rho(0)$ in the space $C(I, \mathbb{R})$ and let us consider, $\forall q \in M$, the map $\mathbb{U} : M \subset C(I, \mathbb{R}) \longrightarrow C(I, \mathbb{R})$ defined as follows: $x \in \mathbb{U}(q)$ if and only if $x(t) \in (fq)(t) \int_0^t k(t, s)F(s, x(s))ds,\ \ t \in [0, T]$.

Any possible fixed point of the (usually multivalued) function $\mathbb{U}$ will be a solution of our quadratic integral inclusion.

Let now $q \in B_\rho(0) = M \subset C$ be a given function: then $|q| \leq \rho$ and we have

$$|x(t)| \leq |(fq)(t)| \int_0^t |k(t, s)||\phi_x(s)|ds$$

$$\leq \beta_0\rho \int_0^t |k(t, s)||\alpha(s) + \beta(s)x(s)|ds \leq v(t) + \int_0^t h(s)|x(s)|\, ds$$

where $v(t) = \beta_0\rho \int_0^t |k(t, s)|\alpha(s)ds$ and $h(t) = \beta_0\rho|k(t, s)|\beta(s)$, if we denote by $\phi_x(t)$ a selector of the multis F and $\beta_1 = \sup\{\beta(t), t \in I\}$ (by iii)).

So, by applying the Gronwall lemma, we have:

$$|x(t)| \leq v(t) \exp(\int_0^t h(s) ds, \, t \in I.$$

So we can say that there is some constant, say $\rho_0 = \sup\{v(t) \exp(\int_0^t h(s) ds, \, t \in I\}$ such that $\mathbb{U}(q) \subset B_{\rho_0}(0)$, for every $q \in M$. For arbitrary $t_1, t_2 \in I$, we have

$$|x(t_2) - x(t_1)| \leq |(fq)(t_2) \int_0^{t_2} k(t_2, s) \phi_x(s) ds -$$

$$(fq)(t_1) \int_0^{t_1} k(t_1, s) \phi_x(s) ds| \leq$$

$$|(fq)(t_2) \int_0^{t_2} k(t_2, s) \phi_x(s) ds - (fq)(t_2) \int_0^{t_2} k(t_1, s) \ \phi_x(s) ds| +$$

$$|(fq)(t_2) \int_0^{t_2} k(t_1, s) \phi_x(s) ds - (fq)(t_2) \int_0^{t_1} k(t_1, s) \phi_x(s) ds| +$$

$$|(fq)(t_2) \int_0^{t_1} k(t_1, s) \phi_x(s) ds - (fq)(t_1) \int_0^{t_1} k(t_1, s) \phi_x(s) ds| \leq$$

$$|(fq)(t_2)| \int_0^{t_2} |k(t_2, s) - k(t_1, s)| \phi_x(s)| ds +$$

$$|(fq)(t_2)| \int_{t_1}^{t_2} |k(t_1, s) \phi_x(s)| ds + |(fq)(t_2) - (fq)(t_2)| \int_0^{t_1} |k(t_1, s) \phi_x(s)| ds \leq$$

$$\beta_0 \rho(\alpha_1 + \beta_1 \rho) w(|t_2 - t_1|) + \beta_0 \rho(\alpha_1 + \beta_1 \rho_0) w_0 |t_2 - t_1| + (\alpha_1 + \beta_1 \rho) w_0 \omega(fq, |t_2 - t_1|).$$

Hence, by using assumptions i) and ii), we have

$$\omega_0(\mathbb{U}(M)) \leq w_0(\alpha_1 + \beta_1 \rho) \omega_0(f(M))$$

and so

$$\chi(\mathbb{U}(M)) \leq w_0(\alpha_1 + \beta_1 \rho) \chi(f(M)) \leq w_0(\alpha_1 + \beta_1 \rho) f_0 \chi(M).$$

So, if $\rho < \dfrac{1 - f_0 w_0 \alpha_1}{w_0 \beta_1 f_0}$, the hypothesis iv) allows us to say that $\mathbb{U}$ is a contraction with respect to the measure (of non compactness) χ on the ball M: from the last consideration we trivially observe that $\mathbb{U}(q)$ is a compact set for each $q \in M$.

Now, in order to apply the fixed point theorem we need to show that $\mathbb{U}$ is an upper semicontinuous operator with compact and acyclic values mapping a ball in itself.

In order to show the upper semicontinuity of the operator $\mathbb{U}$ let us assume that in M $q_n \to q_0$ and let $x_n \in \mathbb{U}(q_n)$ be a sequence such that $x_n \to x_0$. We need to prove that $x_0 \in \mathbb{U}(q_0)$.

From $x_n(t) \in (fq_n)(t) \int_0^t k(t,s)F(s,x_n(s))ds$, by using i) and iii) we know that f is a continuous operator so it follows that $\lim_{n\to+\infty}(fq_n)(t) = (fq_0)(t)$; then if $y_n(t) \in F(t,x_n(t))$ and $x_n(t) = (fq_n)(t) \int_0^t k(t,s)y_n(s)ds$, we have:

$x_0(t) = \lim_{n\to+\infty} x_n(t) = \int_0^t k(t,s)y_0(s)ds$ where $y_0(t) \in F(t,x_0(t))$ (possibly by considering a subsequence, see for instance [12], [18] and related references).

Therefore $x_0 \in \mathbb{U}(q_0)$ and we are done.

We have to show that $\mathbb{U}(q)$ is an acyclic set for every $q \in M$; to this aim we want to show that the set of solutions is acyclic for every $q \in M$.

Since the solution set is a bounded set, we can confine ourselves with the set-valued mapping F defined in a suitable bounded set of $I \times \mathbb{R}^n$.

Let us first consider the equation:

$$x(t) \in (fq)(t) \int_0^t k(t,s)F_n(s,x(s))ds, \quad t \in I = [0,T], \tag{3.2}$$

where $\{F_n\}$ is the sequence of multivalued functions satisfying the conditions of Proposition 1.

The first step will be to show that the set of solutions of the equation (3.2) is a contractible set.

For every $n \in \mathbb{N}$ we have that the equation

$$x(t) = (fq)(t) \int_0^t k(t,s)\phi_n(s,x(s))ds, \quad t \in I = [0,T], \tag{3.3}$$

admits only one solution. As a matter of fact, if

$$y(t) = (fq)(t) \int_0^t k(t,s)\phi_n(s,y(s))ds,$$

is a different solution of the equation, we have, by using the Lipschitz condition (i.e. $|\phi_n(t,x) - \phi_n(t,y)| \leq \gamma_n|x-y|$):

$$|x(t) - y(t)| \leq |(fq)(t)| \int_0^t |k(t,s)|\gamma_n|x(s) - y(s)|ds \leq$$

$$A \int_0^t |x(s) - y(s)|ds,$$

whit $A = T\beta_0\rho K_0\gamma_n$, where β_0, ρ are as before, K_0 is the maximum of the function $|k(t,s)|$ in $I \times I$.

By using again the Gronwall lemma we have $|x(t) - y(t)| = 0,\ \ \forall t \in [0,T]$.

Now, let $x_n(t)$ denote the (unique) solution of the equation (3.3).

Then, for every $x \in S(F_n)$ (the set of solutions of the equation (3.2)), let us consider the homotopy

$$H(x,s)(t) = \begin{cases} x(t) & \text{if } 0 \le t \le sT,\ \ s < 1, \\ \sigma(sT,\phi_n)(t) & \text{if } sT \le t \le T,\ \ s < 1, \\ x(t) & \text{if } \ 0 \le t \le T,\ \ s = 1. \end{cases}$$

In the previous definition $\sigma(sT,\phi_n)(t)$ is the solution of the equation $x(t) = sT + (fq)(t)\int_{sT}^{t} k(t,s)\phi_n(s,x(s))ds, \quad t \in I = [0,T]$.

Let us remark that $H(x,1) = x,\ \ H(x,0) = x_n$.

Moreover we can see that H is defined from $S(F_n) \times [0,1]$ into $S(F_n)$; as a matter of fact, if $s < 1$ and by putting

$$\eta(t) = \begin{cases} x(t), & \text{if } 0 \le t \le sT, \\ \sigma(sT,\phi_n)(t), & \text{if } sT \le t \le T, \end{cases}$$

then $\eta(t) \in \int_0^t k(t,s)F_n(s,\eta(s))ds$: to get this result it will be enough to consider (separately) the cases when $t \in [0,sT]$ and when $t \in [sT,T]$.

Thus $H(s,t)$ is a continuous homotopy transforming $S(F_n)$ into the point x_n.

But we have $F(t,x) = \cap_{n\in\mathbb{N}} F_n(t,x)$: from that we easily see that $S(F) = \cap_{n\in\mathbb{N}} S(F_n)$.

Finally $S(F)$ is an R_δ set since it is the (countable) intersection of contractible and compact sets.

In order to end the proof we have to show that there is a ball $B_R(0)$ such that $\mathbb{U}(B_R(0)) \subset B_R(0)$.

To that aim, if $|q| \le R$ we have

$|x| \le |(fq)(t)| \int_0^t |k(t,s)||\phi_x(s)|ds \le \beta_0 R w_0 \alpha_1 + \beta_0 R w_0 \beta_1 |x|$;

Since $1 - \beta_0 w_0 \alpha_1 > 0$ (from iv)), we can choose $R < \dfrac{1-\beta_0 w_0 \alpha_1}{\beta_0 w_0 \beta_1}$. This inequality implies $R < \dfrac{1}{\beta_0 w_0 \beta_1}$. Hence we have:

$$|x| \le \frac{\beta_0 R w_0 \alpha_1}{1 - \beta_0 R w_0 \beta_1} \le R.$$

Moreover, if we choose $r = min\{R, \rho\}$, the inequality in assumption iv) allows us to say the Darbo condition is also satisfied in the ball $B_r(0)$ and that will be enough to end the proof.

Example Consider now the integral equation written as

$$x(t) \in \frac{\sin x(t)}{x^2(t)+1} \int_0^t \frac{t}{2(t+s)} F(t, x(s))ds$$

where

$$F(t, x(t)) = \begin{cases} [0, \frac{t}{2}] & \text{if } (t, x) \in [0, \frac{1}{2}] \times \mathbb{R} \\ \exp(-t \sin x(t)) & \text{if } (t, x) \in [\frac{1}{2}, 1] \times \mathbb{R} \end{cases}$$

We easily see that $w_0 = \sup_{t\in[0,1]} \int_0^1 \frac{t}{2(t+s)} ds = \frac{\ln 2}{2}$; $\beta_0 = 1,\ f_0 = 1,\ \alpha_1 = e$ and $\beta_1 = 0$.

So we can claim the existence of at least one solution of our integral equation.

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