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Global solvability of some double-diffusive convection system coupled with Brinkman-Forchheimer equations

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Abstract: In this paper, the global solvability of the initial boundary value problem and the periodic problem are discussed for a double-diffusive convection system under the homogeneous Neumann boundary condition in a bounded domain. This system is coupled with the so-called Brinkman-Forchheimer equations, which is similar to the Stokes equations and contains some convection terms similar to that in the Navier-Stokes equations. However, in contrast to the Navier-Stokes equations, it is shown that the global solvability in L^2-spaces holds true for the 3-dimensional problems.

Keywords: Global solvability, double-diffusive convection, Brinkman-Forchheimer equations, Neumann boundary condition, Soret's coefficient.

MSC2010: 34A60, 47J25

Dedicated to the memory of Professor Francesco S. De Blasi

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and consider the following double-diffusive convection system based upon Brinkman-Forchheimer Equations.

$$\text{(BF)}\begin{cases} \partial_t \boldsymbol{u} = \nu\Delta\boldsymbol{u} - a\boldsymbol{u} - \nabla p + \boldsymbol{g}T + \boldsymbol{h}C + \boldsymbol{f}_1 & (x,t)\in\Omega\times[0,S], \\ \partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + f_2 & (x,t)\in\Omega\times[0,S], \\ \partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho\Delta T + f_3 & (x,t)\in\Omega\times[0,S], \\ \nabla\cdot\boldsymbol{u} = 0 & (x,t)\in\Omega\times[0,S], \\ \boldsymbol{u} = 0,\ \dfrac{\partial T}{\partial n} = 0,\ \dfrac{\partial C}{\partial n} = 0, & (x,t)\in\partial\Omega\times[0,S], \end{cases}$$

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where n denotes the unit outward normal vector on $\partial\Omega$ and $\boldsymbol{u}, T, C, p$ are unknown functions and represent the solenoidal velocity of the fluid, the temperature of the fluid, the concentration of a solute, the pressure of the fluid respectively. Given constant vectors $\boldsymbol{g}, \boldsymbol{h}$ are derived from the gravity. The positive constants ν, ρ, a are called the viscosity coefficient, Soret's coefficient and Darcy's coefficient respectively and $\boldsymbol{f}_1, f_2, f_3$ are the given external forces. Throughout this paper, $\partial_t \boldsymbol{u}$ or $\boldsymbol{u}_t$ designates the time derivative of $\boldsymbol{u}$, i.e., $\dfrac{\partial \boldsymbol{u}}{\partial t}$.

The first equation of (BF) is called the Brinkman-Forchheimer equations, which describes the behavior of the fluid velocity in some porous medium. Originally, the Brinkman-Forchheimer equations has a convection term and another nonlinear term, and in each term of the equations, there appears another space-dependent function which stands for the rate of the void space in the porous medium (which is called the porosity).

We assume that the medium is homogeneous, whence it follows that the porosity is constant. Moreover we presume that the flow is relatively so calm that we can neglect nonlinear terms which are very small, which is plausible when we are concerned with the porous medium, which disturbs the flow.

It is also known that the nonlinear terms in the Brinkman-Forchheimer equations become negligibly small when we deal with the convection of the temperature and the concentration together. We also assume that the porosity of the porous medium is sufficiently large, which makes the diffusion term more effective than the nonlinear terms. Under these assumption, we derive the linearized Brinkman-Forchheimer equations given in (BF). Here $\boldsymbol{g}\,T, \boldsymbol{h}\,C$ are the effects from the gravity.

The second equation and the third equation of (BF) originate from the result of the irreversible thermodynamics. The term $\rho\,\Delta T$, which is called Soret's effect, describes the certain interaction between the temperature of the fluid and the concentration of a solute. Naturally, the second equation also contains a interaction term $\rho' \Delta C$, which is called Dufour's effect. However, Dufour's effect is generally much smaller than Soret's effect, especially for the case where the fluid is a liquid. Therefore we here consider only Soret's effect.

There are many studies for (BF), for example, about the continuous dependence of the solutions on the Soret's coefficient ρ and so on. However, to the best of our knowledge, it seems that there are very few studies for the solvability of (BF). The first attempt in this direction is made in [12], where the initial boundary value problem for (BF) with the homogeneous Dirichlet boundary condition is studied and it is shown that this problem admits a unique global solution even for the 3-dimensional case.

In [11], the global solvability of the time periodic problem is shown for (BF) with the homogeneous Dirichlet boundary condition both for 2 and 3-dimensional cases.

Since (BF) contains the convection terms $\boldsymbol{u}\cdot\nabla T, \boldsymbol{u}\cdot\nabla C$, which are quite similar to that appearing in the Navier-Stokes equations, it apparently seems that it would be very difficult to obtain " the global solvability " of (BF) in 3-dimensional case, i.e., the existence of the unique global solution of the initial boundary-value problem for arbitrarily large initial data or the existence of time-periodic solutions for arbitrarily large external forces. However, it is revealed that the global solvability holds true for these problems even for the 3-dimensional case in [12] and [11].

The main purpose of this paper is to show the similar global solvability results still hold true for (BF) with the homogeneous Neumann boundary condition for T and C. In order to carry out this purpose, we follow the basic strategy adopted in [12] and [11], i.e., we reduce our problem to some abstract equation in an appropriate Hilbert space and we rely on the abstract theory developed in [9] and [10]. However, the lack of the coercivity of the Laplacian with the Neumann boundary condition causes some difficulty in this procedure. Especially for the periodic problem, we need to introduce some approximate system involving some dissipation terms and cut-off functions as in [11]. Unfortunately this hinders establishing desirable a priori estimates under the Neumann boundary condition. In order to cope with this difficulty, we introduce another step of approximations for the original system.

In section 2, our main results are stated and some preliminary results are fixed for later use. In section 3 and 4, we give proofs of the main results for the initial boundary value problem and the periodic problem respectively.

2 Preliminaries and Main Results

2.1 Notation and Main Results

In this paper, in order to formulate our results, we use following notations.

$$\Omega: \text{ bounded domain in } \mathbb{R}^N, \quad Q = \Omega\times(0,S),$$

$$\mathbb{C}_\sigma^\infty(\Omega) = \{\,\boldsymbol{u} = (u^1,u^2,\cdots,u^N)^t\,;\, u^j\in C_0^\infty(\Omega)\ {}^\forall j = 1,2,\cdots,N,\ \nabla\cdot\boldsymbol{u} = 0\,\},$$

$$\mathbb{L}^2(\Omega) = (L^2(\Omega))^N, \quad \mathbb{H}^1(\Omega) = (H^1(\Omega))^N = (W^{1,2}(\Omega))^N,$$

$$\mathbb{L}^2_\sigma(\Omega) = \text{The closure of } \mathbb{C}_\sigma^\infty(\Omega) \text{ under the } \mathbb{L}^2(\Omega)\text{-norm},$$

$$\mathbb{H}^1_\sigma(\Omega) = \text{The closure of } \mathbb{C}_\sigma^\infty(\Omega) \text{ under the } \mathbb{H}^1(\Omega)\text{-norm},$$

$$H = \mathbb{L}^2_\sigma(\Omega)\times L^2(\Omega)\times L^2(\Omega), \quad H^0 = \mathbb{L}^2(\Omega)\times L^2(\Omega)\times L^2(\Omega),$$

$$C_\pi([0,S];H) = \{\,U\in C([0,S]\,;\,H)\,;\,U(0) = U(S)\,\},$$

$$\mathcal{P}_\Omega = \text{ The orthogonal projection from } \mathbb{L}^2(\Omega) \text{ onto } \mathbb{L}^2_\sigma(\Omega),$$

$\mathcal{A} = -\mathcal{P}_\Omega \Delta$: The Stokes operator with domain $D(\mathcal{A}) = \mathbb{H}^2(\Omega) \cap \mathbb{H}^1_\sigma(\Omega)$,

$A_N = -\Delta$ with domain $D(A_N) = \{\, u \in H^2(\Omega)\,;\, \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \,\}$,

$A_D = -\Delta$ with domain $D(A_D) = H^2(\Omega) \cap H^1_0(\Omega)$.

$\mathcal{A}^\alpha$, A^α_D and A^α_N denote the fractional powers of $\mathcal{A}$, A_D and A_N of order α.

Then our main results are stated as follows.

Theorem 2.1 (Initial Boundary Value Problem) *Let N be 2 or 3 and let $(\boldsymbol{f}_1, f_2, f_3)^t \in L^2(0,S;H^0)$. Then for each initial data $U_0 = (\boldsymbol{u}_0, T_0, C_0)^t \in D(\mathcal{A}^\alpha) \times D(A^\alpha_N) \times D(A^\alpha_N)$ with $\alpha \in [1/4, 1/2]$,* (BF) *admits a unique solution $U = (\boldsymbol{u},\ T,\ C)^t \in C([0,S];H)$ satisfying $U(0) = U_0$ and*

$$(\#)_\alpha \quad \begin{cases} t^{1/2-\alpha}\, \partial_t \boldsymbol{u},\ t^{1/2-\alpha} \mathcal{A}\, \boldsymbol{u} \in L^2(0,S\,;\,\mathbb{L}^2_\sigma(\Omega)), \\ t^{1/2-\alpha}\, ||\nabla \boldsymbol{u}||_{\mathbb{L}^2(\Omega)} \in L^p_*(0,S) \quad \textit{for all } p \in [2,\infty], \\ t^{1/2-\alpha}\, \partial_t\, T,\ t^{1/2-\alpha}\, \partial_t\, C,\ t^{1/2-\alpha} \Delta\, T,\ t^{1/2-\alpha} \Delta\, C \in L^2(0,S;L^2(\Omega)), \\ t^{1/2-\alpha}\, ||\nabla T||_{L^2(\Omega)},\ t^{1/2-\alpha}\, ||\nabla C||_{L^2(\Omega)} \in L^p_*(0,S) \quad \textit{for all } p \in [2,\infty], \end{cases}$$

where $L^p_* = L^p(dt/t)$, i.e., $||f||_{L^p_*(0,S)} = \big(\int_0^S |f(t)|^p\, t^{-1}\, dt\big)^{1/p}$ for $1 \leqslant p < \infty$ and $L^\infty_* = L^\infty$.

Theorem 2.2 (Periodic Problem) *Let N be 2 or 3 and let $(\boldsymbol{f}_1,\ f_2,\ f_3)^t \in L^2(0,S;H^0)$ such that*

$$f_2,\ f_3 \in \Big\{\, f\,;\ \int_Q f(x,t)\, dx\, dt = 0 \,\Big\}. \tag{2.1}$$

Then (BF) *admits a solution $U = (\boldsymbol{u},\ T,\ C)^t \in C_\pi([0,S]\,;\,H)$ satisfying*

$$(\#)_{1/2} \quad \begin{cases} \partial_t \boldsymbol{u},\ \mathcal{A}\, \boldsymbol{u} \in L^2(0,S;\mathbb{L}^2_\sigma(\Omega)), \\ \boldsymbol{u} \in C([0,S];\mathbb{H}^1_\sigma(\Omega)), \\ \partial_t\, T,\ \partial_t\, C,\ \Delta\, T,\ \Delta\, C \in L^2(0,S;L^2(\Omega)), \\ T,\ C \in C([0,S];H^1(\Omega)). \end{cases}$$

Remark 1 (1) If U_0 belongs to $D(\mathcal{A}^{1/2}) \times D(A^{1/2}_N) \times D(A^{1/2}_N) = \mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times$

$H^1(\Omega)$ in Theorem 2.1, then the solution U satisfies property $(\#)_{1/2}$ given in Theorem 2.2.

(2) In [12], the same result as in Theorem 2.1 is given for (BF) with the homogeneous Neumann boundary condition replaced by the homogeneous Dirichlet boundary condition only for the case $\alpha = 1/2$. However, with obvious modifications, we can show that if $U_0 \in D(\mathcal{A}^\alpha) \times D(A_D^\alpha) \times D(A_D^\alpha)$ with $\alpha \in [1/4, 1/2]$, then the Dirichlet problem for (BF) admits a unique solution U satisfying $(\#)_\alpha$.

(3) The characterizations for the domains of the fractional powers of A_N, A_D and $\mathcal{A}$ can be found in [7] and [6].

(4) Condition (2.1) is the necessary condition for the existence of the periodic solution of (BF) satisfying the homogeneous Neumann boundary condition. In fact, integrating (4.19) (which will be given later) with $\lambda = 0$, we can derive (2.1).

2.2 Reduction to an Abstract Problem

Let φ be a proper lower semi-continuous convex function from H into $(-\infty, +\infty]$. Define the effective domain of φ by $D(\varphi) = \{\, U \in H \,;\, \varphi(U) < +\infty \,\}$ and the subdifferential of φ by

$$\partial\varphi(U) = \{\, f \in H \,;\, \varphi(V) - \varphi(U) \geqslant (f, V-U)_H \quad \text{for all } V \in H \,\}$$

with domain $D(\partial\varphi) = \{\, U \in H \,;\, \partial\varphi(U) \neq \varnothing \,\}$. Then $\partial\varphi$ becomes a maximal monotone operator.

It is well known that for any maximal monotone operator A in H, $J_\lambda = (I + \lambda\, A)^{-1}$, $\lambda > 0$, the resolvent of A, is well defined on H and $J_\lambda U \to U$ as $\lambda \to 0$ for all $U \in \overline{D(A)}$. Then for $\alpha \in (0,1)$, $p \in [1, \infty]$, by measuring how fast $J_\lambda U$ converges to U, we can define a nonlinear interpolation class $\mathcal{B}_{\alpha,p}(A)$ associated with A by

$$\mathcal{B}_{\alpha,p}(A) = \{\, U \in \overline{D(A)}\,;\, t^{-\alpha}\,|\,U - J_t\,U\,|_H \in L^p_*(0,1)\,\}.$$

We often use the notation

$$|U|_{\mathcal{B}_{\alpha,p}(A)} = \left|\, t^{-\alpha}|U - J_t\,U|_H \,\right|_{L^p_*(0,1)}.$$

This nonlinear interpolation class $\mathcal{B}_{\alpha,p}(A)$ covers a very wide class of already known interpolation spaces such as Besov spaces, in particular, if A is non-negative self-adjoint operator, then the domain of the fractional power of A of order α is given by $D(A^\alpha) = \mathcal{B}_{\alpha,2}(A)$ (See [2], [3] and [4]). In what follows, we use this nonlinear interpolation theory for the special case where $A = \partial\varphi$.

In the later arguments, it will be shown that the leading terms $(\mathcal{A}, A_N, A_N)^t$ can be given as the subdifferential of a suitable lower semi-continuous convex function on H. Generally, subdifferential operators are multivalued maximal monotone operators. However, since the subdifferential operators used in this paper are always single-valued, we restrict ourselves to the single-valued subdifferential operators in the following setting.

In order to reduce our problem to an abstract problem in the Hilbert space H, we first operate the projection $\mathcal{P}_\Omega$ to the first equation of (BF) to erase the pressure term ∇p. Then we obtain the following equations.

$$\begin{cases} \partial_t \boldsymbol{u} + \nu\, \mathcal{A}\, \boldsymbol{u} = -a\, \boldsymbol{u} + \mathcal{P}_\Omega\, \boldsymbol{g}\, T + \mathcal{P}_\Omega\, \boldsymbol{h}\, C + \mathcal{P}_\Omega\, \boldsymbol{f}_1, \\ \partial_t T + A_N\, T + \boldsymbol{u}\cdot\nabla T = f_2, \\ \partial_t C + A_N\, C + \boldsymbol{u}\cdot\nabla C = -\rho\, A_N\, T + f_3. \end{cases} \tag{2.2}$$

Here, for each parameter $\eta \in (0,1]$, H_η designates the Hilbert space H endowed with the following inner product:

$$\begin{aligned} (U_1,\ U_2)_H = (\boldsymbol{u}_1, \boldsymbol{u}_2)_{\mathbb{L}^2_\sigma} + (T_1, T_2)_{L^2} + \frac{\eta^2}{9\rho^2}\,(C_1, C_2)_{L^2} \\ \text{for}\ \ U_i = (\boldsymbol{u}_i, T_i, C_i)^t,\ \ (i = 1, 2). \end{aligned} \tag{2.3}$$

Here we put the weight depending on η and ρ in front of the inner product for C so that the term $\rho\,\Delta T = -\rho\, A_N\, T$ can be treated as a small perturbation.

Next define φ by

$$\varphi(U) = \begin{cases} \dfrac{\nu}{2}\, \|\,|\nabla \boldsymbol{u}|\,\|^2_{L^2} + \dfrac{1}{2}\,\|\,\nabla T\,\|^2_{L^2} + \dfrac{\eta^2}{18\rho^2}\,\|\,\nabla C\,\|^2_{L^2} & \text{if}\ \ U \in D(\varphi), \\ +\infty & \text{if}\ \ U \in H \backslash D(\varphi). \end{cases} \tag{2.4}$$

where $D(\varphi) = \mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$ is the effective domain of φ. Then it is easy to see that φ becomes a lower semi-continuous convex function from H_η into $[0, +\infty]$. Moreover the subdifferential $\partial\varphi$ is given by

$$\partial\varphi(U) = \begin{pmatrix} -\nu\, \mathcal{P}_\Omega\, \Delta \boldsymbol{u} \\ -\Delta\, T \\ -\Delta\, C \end{pmatrix} \text{ with domain } D(\partial\varphi) = (\mathbb{H}^2 \cap \mathbb{H}^1_\sigma) \times D(A_N) \times D(A_N). \tag{2.5}$$

Furthermore, we put

$$U(t)=\begin{pmatrix}\boldsymbol{u}(t)\\ T(t)\\ C(t)\end{pmatrix},\quad \frac{dU}{dt}(t)=\begin{pmatrix}\partial_t\boldsymbol{u}(t)\\ \partial_t T(t)\\ \partial_t C(t)\end{pmatrix},$$

$$B(U(t))=\begin{pmatrix}a\boldsymbol{u}(t)-\mathcal{P}_\Omega\,\boldsymbol{g}\,T(t)-\mathcal{P}_\Omega\,\boldsymbol{h}\,C(t)\\ \boldsymbol{u}\cdot\nabla T(t)\\ \boldsymbol{u}\cdot\nabla C(t)-\rho\,\Delta\,T(t)\end{pmatrix},\quad F(t)=\begin{pmatrix}\mathcal{P}_\Omega\,\boldsymbol{f}_1(t)\\ f_2(t)\\ f_3(t)\end{pmatrix}. \tag{2.6}$$

Then the initial boundary value problem for (2.2) is reduced to the following abstract Cauchy problem in H:

$$\text{(CP)}\begin{cases}\dfrac{dU}{dt}(t)+\partial\varphi\,(U(t))+B(U(t))=F(t)\quad t\in[0,S],\\ U(0)=U_0,\end{cases} \tag{2.7}$$

and the periodic problem for (2.2) is reduced to the following abstract periodic problem in H:

$$\text{(AP)}\begin{cases}\dfrac{dU}{dt}(t)+\partial\varphi\,(U(t))+B(U(t))=F(t)\quad t\in[0,S],\\ U(0)=U(S).\end{cases} \tag{2.8}$$

2.3 Abstract Results

In order to prove Theorems 2.1 and 2.2, we rely on abstract results given in [9] and [10]. To formulate these results, we introduce the following conditions.

(A1) For any $L\in(0,+\infty)$, the set $\{\,U\in H\,;\,\varphi(U)+\|\,U\,\|_H^2\leqslant L\,\}$ is compact in H.

(A2) $B(\cdot)$ is φ-demiclosed in the following sense:
$U_n\to U$ strongly in $C([0,S];H)$, $\partial\varphi(U_n)\rightharpoonup\partial\varphi(U)$ weakly in $L^2(0,S;H)$, $B(U_n)\rightharpoonup b$ weakly in $L^2(0,S;H)$, then $b(t)=B(U(t))$ holds for a.e. $t\in[0,S]$.

(A3)$^0_\alpha$ For a given exponent $\alpha\in(0,1/2)$, there exists a monotone increasing function $\ell(\cdot)$ such that

$$\|B(U)\|_H\leqslant\ell(\|U\|_H)\,\{\,\varepsilon\,\|\partial\varphi(U)\|_H+\frac{1}{\varepsilon}\,|\varphi(U)|^{\frac{1-\alpha}{1-2\alpha}}+1\,\}\quad {}^\forall U\in D(\partial\varphi),$$

where ε is a positive constant determined by the initial data U_0 and the external force $F(t)$, more precisely, ε is a monotone decreasing function of $|U_0|_H + |U_0|_{\mathcal{B}_{\alpha,p}(\partial\varphi)} + |F|_{L^2(0,S;H)}$.

(A4) There exists a monotone increasing function $\ell(\cdot)$ and $k \in (0,1)$ such that
$$\|B(U)\|_H^2 \le k\,\|\partial\varphi(U)\|_H^2 + \ell\,(\varphi(U) + \|U\|_H^2) \quad {}^\forall U \in D(\partial\varphi).$$

(A5) There exists a monotone increasing function $\ell(\cdot)$ and a constant $k \in [0,1)$ such that
$$\|\,B(U)\,\|_H^2 \leqslant k\,\|\,\partial\varphi(U)\,\|_H^2 + \ell\,(\|\,U\,\|_H)(\,\varphi(U) + 1\,)^2 \quad {}^\forall U \in D(\partial\varphi).$$

(A6) There exist positive constants $\alpha,\ K$ such that
$$(\,-\,\partial\varphi(U) - B(U),\ U\,)_H + \alpha\,\varphi(U) \leqslant K \quad {}^\forall U \in D(\partial\varphi).$$

Then the following results hold (see [9] and [10]).

Theorem 2.3 *Let $U_0 \in \mathcal{B}_{\alpha,p}(\partial\varphi)$ with $p \in [1,2]$ and $F \in L^2(0,S;H)$, and let* (A1), (A2) *and* $(\mathrm{A3})_\alpha^0$ *be satisfied. Then there exists $S_0 \in (0,S]$ depending on $|U_0|_H$ and $|U_0|_{\mathcal{B}_{\alpha,p}(\partial\varphi)}$ such that* (CP) *has a solution $U(t)$ in $[0,S_0]$ satisfying*
$$t^{1/2-\alpha}\,dU/dt,\ t^{1/2-\alpha}\,\partial\varphi(U(t)),\ t^{1/2-\alpha}\,B(U(t)) \in L^2(0,S_0\,;H),$$
$$t^{-\alpha}\,\|U(t)-U_0\|_H,\ t^{1/2-\alpha}\,|\varphi(U(t))|^{1/2} \in L^q_*(0,S_0) \quad {}^\forall q \in [2,\infty].$$

Theorem 2.4 *Let* (A1), (A2) *and* (A4) *be satisfied and let $U_0 \in D(\varphi)$ and $F \in L^2(0,S;H)$. Then there exists $S_0 \in (0,S]$ depending on $|U_0|_H$ and $\varphi(U_0)$ such that* (CP) *has a solution $U(t)$ in $[0,S_0]$ satisfying*
$$dU/dt,\ \partial\varphi(U(t)),\ B(U(t)) \in L^2(0,S_0\,;H),$$
$$\varphi(U(t)) \ \ \textit{is absolutely continuous on } [0,S_0].$$

Theorem 2.5 *Let* (A1), (A2), (A5) *and* (A6) *be satisfied. Then for every $F \in L^2(0,S;H)$,* (AP) *has a strong solution $U \in C_\pi([0,S];H)$ such that*
$$dU/dt,\ \partial\varphi(U),\ B(U) \in L^2(0,S;H),$$
$$\varphi(U) \ \ \textit{is absolutely continuous on } \ [0,S] \textit{ and } \varphi(U(0)) = \varphi(U(S)).$$

Remark 2 In [9], Theorem 2.3 is actually proved under a different assumption $(\mathrm{A3})_\alpha$ which is slightly stronger than $(\mathrm{A3})_\alpha^0$. However it is easy to see that the proof of Theorem 2.3 holds true with $(\mathrm{A3})_\alpha$ replaced by $(\mathrm{A3})_\alpha^0$ (see the proof of Theorem I in [9]).

3 Initial Boundary Value Problems

In this section, we give a proof of Theorem 2.1, which is divided into three parts, i.e., the local existence, the global existence and the uniqueness. Theorems 2.3 and 2.4 are applied to assure the existence of local solutions and some appropriate a priori estimates are established to show that local solutions can be extended as global solutions.

3.1 Local Existence

In order to prove the local existence result, we are going to check conditions assumed in Theorems 2.3 and 2.4. For this purpose, we choose $\eta = \varepsilon$ and we denote H_η simply by H, where ε is an exponent appearing in $(\mathrm{A3})^0_\alpha$.

Check of (A1) For any $L \in (0, +\infty)$, the set $\{\, U \in H\,;\, \varphi(U) + \|\, U\,\|_H^2 \leqslant L \,\}$ is compact in H.

Proof. The level set $\{\, U \in H\,;\, \varphi(U) + \|\, U\,\|_H^2 \leqslant L \,\}$ is bounded in the function space $\mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$. Therefore it is clear that the level set is compact in H by virtue of Rellich's compactness theorem. □

Check of (A2) $B(\cdot)$ is φ-demiclosed.

Proof. Assume

$$\begin{cases} \boldsymbol{u}_k \to \boldsymbol{u} \quad \text{strongly in } \; C([0,S]; \mathbb{L}^2_\sigma(\Omega)), \\ T_k \to T \quad \text{strongly in } \; C([0,S]; L^2(\Omega)), \\ C_k \to C \quad \text{strongly in } \; C([0,S]; L^2(\Omega)), \end{cases} \tag{3.1}$$

$$\begin{cases} -\nu\, \mathcal{P}_\Omega\, \Delta \boldsymbol{u}_k \rightharpoonup -\nu\, \mathcal{P}_\Omega\, \Delta \boldsymbol{u} & \text{weakly in } \; L^2(0,S; \mathbb{L}^2_\sigma(\Omega)), \\ -\Delta\, T_k \rightharpoonup -\Delta\, T & \text{weakly in } \; L^2(0,S; L^2(\Omega)), \\ -\Delta\, C_k \rightharpoonup -\Delta\, C & \text{weakly in } \; L^2(0,S; L^2(\Omega)), \end{cases} \tag{3.2}$$

and let

$$\begin{cases} a\, \boldsymbol{u}_k - \mathcal{P}_\Omega\, \boldsymbol{g}\, T_k - \mathcal{P}_\Omega\, \boldsymbol{h}\, C_k \rightharpoonup h_1 & \text{weakly in } \; L^2(0,S; \mathbb{L}^2_\sigma(\Omega)), \\ \boldsymbol{u}_k \cdot \nabla T_k \rightharpoonup h_2 & \text{weakly in } \; L^2(0,S; L^2(\Omega)), \\ \boldsymbol{u}_k \cdot \nabla C_k - \rho\, \Delta\, T_k \rightharpoonup h_3 & \text{weakly in } \; L^2(0,S; L^2(\Omega)). \end{cases} \tag{3.3}$$

From the strong convergences of (3.1), we easily get

$$h_1 = a\, \boldsymbol{u} - \mathcal{P}_\Omega\, \boldsymbol{g}\, T - \mathcal{P}_\Omega\, \boldsymbol{h}\, C.$$

Using the fact that $\boldsymbol{u}_k$ is a solenoidal and applying the integration by parts, we obtain

$$\langle\, \boldsymbol{u}_k \cdot \nabla T_k,\ \phi\,\rangle = -\,\langle\, \boldsymbol{u}_k\, T_k,\ \nabla\phi\,\rangle \to -\,\langle\, \boldsymbol{u}\,T, \nabla\phi\,\rangle = \langle\, \boldsymbol{u} \cdot \nabla T,\ \phi\,\rangle \tag{3.4}$$

for all $\phi \in C_0^\infty(\Omega \times (0,S))$. Consequently we find $h_2 = \boldsymbol{u}\cdot\nabla T$. Similarly, by (3.2), we find $h_3 = \boldsymbol{u}\cdot\nabla C - \rho\,\Delta\, T$. □

Check of $(\mathbf{A3})_\alpha^0$ For $\alpha \in (0,1/2)$, $^\exists\ell(\cdot)$ such that

$$\|B(U)\|_H \leqslant \ell(\|U\|_H)\{\varepsilon\,\|\partial\varphi(U)\|_H + \frac{1}{\varepsilon}\,|\varphi(U)|^{\frac{1-\alpha}{1-2\alpha}} + 1\} \quad {}^\forall U \in D(\partial\varphi).$$

Proof. In what follows, let the space dimension N be 3. For the case where $N = 2$, the proof can be done by the same or much easier arguments. By the definition of $B(U)$ and the inner product of H, we get

$$\begin{aligned} \|B(U)\|_H \leqslant & a\,\|\boldsymbol{u}\|_{\mathbb{L}^2_\sigma} + |\boldsymbol{g}|\,\|T\|_{L^2} + |\boldsymbol{h}|\,\|C\|_{L^2} \\ & + \|\,|\boldsymbol{u}|\,\nabla T\|_{L^2} + \frac{\varepsilon}{3\,\rho}\,(\,\|\,|\boldsymbol{u}|\,\nabla C\|_{L^2} + \rho\,\|\Delta T\|_{L^2}\,). \end{aligned} \tag{3.5}$$

Here using Hölder's inequality and the fact that $|w|_{L^3}^2 \leqslant |w|_{L^2}|w|_{L^6}$, we get

$$\begin{aligned} \Big(\int_\Omega |\boldsymbol{u}|^2|\nabla T|^2 dx\,\Big)^{1/2} & \leqslant \|\,|\boldsymbol{u}|\,\|_{L^6}\,\|\nabla T\|_{L^3} \leqslant \|\,|\boldsymbol{u}|\,\|_{L^6}\,\|\nabla T\|_{L^2}^{1/2}\,\|\nabla T\|_{L^6}^{1/2} \\ & \leqslant \gamma_0\,\|\,|\nabla\boldsymbol{u}|\,\|_{L^2}\,\|\nabla T\|_{L^2}^{1/2}(\|\nabla T\|_{L^2}^{1/2} + \|\Delta T\|_{L^2}^{1/2}) \\ & \leqslant \frac{\varepsilon}{4}\,\|\Delta T\|_{L^2} + \frac{\gamma_0}{\varepsilon}\,(\,\|\,|\nabla\boldsymbol{u}|\,\|_{L^2} + 1\,)^2\,\|\nabla T\|_{L^2}, \end{aligned} \tag{3.6}$$

where γ_0 is a constant depending on some Sobolev's embedding constants. Similarly we have

$$\Big(\int_\Omega |\boldsymbol{u}|^2|\nabla C|^2 dx\,\Big)^{1/2} \leqslant \frac{\varepsilon}{4}\,\|\Delta C\|_{L^2} + \frac{\gamma_0}{\varepsilon}\,(\,\|\,|\nabla\boldsymbol{u}|\,\|_{L^2} + 1\,)^2\,\|\nabla C\|_{L^2}. \tag{3.7}$$

Hence, by virtue of (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned} \|B(U)\|_H & \leqslant (\,\frac{\varepsilon}{4} + \frac{\varepsilon}{3}\,)\,\|\Delta T\|_{L^2} + \frac{\varepsilon}{4}\,\frac{\varepsilon}{3\rho}\,\|\Delta C\|_{L^2} \\ & \quad + \frac{\gamma_1}{\varepsilon}\,(\,\|\,|\nabla\boldsymbol{u}|\,\|_{L^2}^3 + \|\nabla T\|_{L^2}^3 + \frac{\varepsilon}{3\rho}\,\|\nabla C\|_{L^2}^3\,) + \gamma_2\,(\,\|U\|_H + 1\,) \\ & \leqslant \varepsilon\,\|\partial\varphi(U)\|_H + \frac{\gamma_3}{\varepsilon}\,(\,\varphi^{3/2}(U) + \|U\|_H + 1\,), \end{aligned} \tag{3.8}$$

where γ_1, γ_2 and γ_3 are constants depending on γ_0.
Then this ensures $(\mathrm{A3})^0_\alpha$ with $\alpha \in [1/4, 1/2)$ and (A4). Thus, Theorems 2.3 and 2.4 assures the existence of local solution $U(t)$ on $[0, S_0]$ satisfying $(\#)_\alpha$ with S replaced by S_0 when $U_0 = (\boldsymbol{u}_0, T_0, C_0)^t \in D(\mathcal{A}^\alpha) \times D(A_N^\alpha) \times D(A_N^\alpha)$ for $\alpha \in [1/4, 1/2)$ and satisfying $(\#)_{1/2}$ with S replaced by S_0 when $U_0 = (\boldsymbol{u}_0, T_0, C_0)^t \in D(\mathcal{A}^{1/2}) \times D(A_N^{1/2}) \times D(A_N^{1/2}) = \mathbb{H}^1_\sigma(\Omega) \times H^1(\Omega) \times H^1(\Omega)$. □

3.2 Global Existence

In this subsection, we show that every local solutions can be continued globally to $[0, S]$ by establishing some a priori estimates.

First Energy Estimates of T

Multiplying the second equation of (BF) by T and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{dt}\|T(t)\|^2_{L^2} + \|\nabla T(t)\|^2_{L^2} \leqslant \|f_2(t)\|_{L^2}\,\|T(t)\|_{L^2}. \tag{3.9}$$

Here we used the fact that

$$\int_\Omega (\boldsymbol{u}\cdot\nabla T)\,T\,dx = \int_\Omega \boldsymbol{u}\cdot\nabla(\,\frac{1}{2}T^2\,)\,dx = \int_\Omega \operatorname{div}\boldsymbol{u}\,\frac{1}{2}T^2\,dx = 0. \tag{3.10}$$

Then we get

$$\frac{d}{dt}\|T(t)\|_{L^2} \leqslant \|f_2(t)\|_{L^2}.$$

Hence we can derive the a priori bound for $\|T(t)\|_{L^2}$ and substituting this estimate in (3.9), we obtain

$$\sup_{0\le t\le S}\|T(t)\|^2_{L^2} + \int_0^S \|\nabla T(t)\|^2_{L^2}dt \leqslant \gamma\,(\,\|T_0\|_{L^2(\Omega)},\quad \|f_2\|_{L^2(Q)}\,). \tag{3.11}$$

Here and henceforth, γ denotes general constants which may vary from place to place. In order to make clear that γ depends on $a, b, \cdots$, we use the notation $\gamma\,(\,a, b, \cdots)$ instead of γ.

First Energy Estimates of C

Multiplying the third equation of (BF) by C and using the property (3.10) with T replaced by C, we get

$$\frac{1}{2}\frac{d}{dt}\|C(t)\|^2_{L^2} + \|\nabla C(t)\|^2_{L^2} = -\int_\Omega(\boldsymbol{u}\cdot\nabla C)\,C\,dx + \int_\Omega \rho\,\nabla T\cdot\nabla C\,dx + \int_\Omega f_3\,C\,dx$$

$$\leqslant \rho\,\|\nabla T\|_{L^2}\,\|\nabla C\|_{L^2} + \|f_3\|_{L^2}\,\|C\|_{L^2} \leqslant \frac{1}{2}\|\nabla C\|^2_{L^2} + \frac{\rho^2}{2}\|\nabla T\|^2_{L^2} + \|f_3\|_{L^2}\,\|C\|_{L^2}.$$

Then, by the same argument as above and (3.11), we obtain

$$\sup_{0\le t\le S} \|C(t)\|_{L^2}^2 + \int_0^S \|\nabla C(t)\|_{L^2}^2\,dt \leqslant \gamma\,(\|T_0\|_{L^2}, \|C_0\|_{L^2}, \|f_2\|_{L^2(Q)}, \|f_3\|_{L^2(Q)}). \quad (3.12)$$

Second Energy Estimates of $\boldsymbol{u}$

Multiplying the first equation of (BF) by $\boldsymbol{u}_t$, we get

$$\begin{aligned}\|\,\boldsymbol{u}_t\,\|_{\mathbb{L}^2_\sigma}^2 + \frac{\nu}{2}\frac{d}{dt}\int_\Omega |\nabla \boldsymbol{u}|^2\,dx + \frac{a}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}|^2 dx &\\ \leqslant \|\boldsymbol{u}_t\|_{\mathbb{L}^2_\sigma}\,(\,|\boldsymbol{g}|\,\|T\|_{L^2} + |\boldsymbol{h}|\,\|C\|_{L^2} + \|\boldsymbol{f}_1\|_{\mathbb{L}^2}\,) &\qquad (3.13)\\ \leqslant \frac{1}{2}\,\|\boldsymbol{u}_t\|_{\mathbb{L}^2_\sigma}^2 + (\,|\boldsymbol{g}|\,\|T\|_{L^2} + |\boldsymbol{h}|\,\|C\|_{L^2}\,)^2 + \|\boldsymbol{f}_1\|_{\mathbb{L}^2}^2.&\end{aligned}$$

Then integrating (3.13) over $[0,t]$ with $t \in (0,S]$ and using (3.11) and (3.12), we obtain

$$\begin{aligned}\sup_{0\le t\le S}\,\big(\,\nu\,\|\,|\nabla \boldsymbol{u}(t)|\,\|_{\mathbb{L}^2}^2 + a\,\|\boldsymbol{u}(t)\|_{\mathbb{L}^2_\sigma}^2\,\big) + \int_0^S \|\boldsymbol{u}_t\|_{\mathbb{L}^2_\sigma}^2\,dt &\qquad (3.14)\\ \leqslant \gamma\,(\,\|\,|\nabla \boldsymbol{u}_0|\,\|_{L^2},\ \|U_0\|_H,\ \|F\|_{L^2(0,S;H)}\,),&\end{aligned}$$

which also implies

$$\int_0^S \|\mathcal{A}\boldsymbol{u}\|_{\mathbb{L}^2_\sigma}^2\,dt \leqslant \gamma\,(\,\|\,|\nabla \boldsymbol{u}_0|\,\|_{L^2},\ \|U_0\|_H,\ \|F\|_{L^2(0,S;H)}\,). \quad (3.15)$$

Second Energy Estimates of T

Multiplying the second equation of (BF) by $-\Delta T$, we obtain, by the same argument as for (3.6) with $\varepsilon = 1$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,\|\nabla T\|_{L^2}^2 + \|\Delta\,T\|_{L^2}^2 &\leqslant \frac{1}{2}\int_\Omega |\boldsymbol{u}|^2\,|\nabla T|^2\,dx + \frac{1}{2}\,\|\Delta\,T\|_{L^2}^2 + \|f_2\|_{L^2}\,\|\Delta\,T\|_{L^2}\\ &\leqslant (\,\frac{1}{8}+\frac{1}{2}+\frac{1}{4}\,)\,|\Delta T|_{L^2}^2 + 2\,\gamma_0^2\,(\,\|\,|\nabla \boldsymbol{u}|\,\|_{L^2} + \|\boldsymbol{u}\|_{\mathbb{L}^2_\sigma}\,)^4\,\|\nabla T\|_{L^2}^2 + \|f_2\|_{L^2}^2. \quad (3.16)\end{aligned}$$

Then Gronwall's inequality with (3.14) yields

$$\begin{aligned}\sup_{0\le t\le S}\,\|\nabla T(t)\|_{L^2}^2 + \int_0^S \|\Delta T(t)\|_{L^2}^2\,dt &\qquad (3.17)\\ \leqslant \gamma\,(\,\|\,|\nabla \boldsymbol{u}_0|\,\|_{L^2},\ \|\nabla T_0\|_{L^2},\ \|C_0\|_{L^2},\ \|F\|_{L^2(0,S;H)}).&\end{aligned}$$

Second Energy Estimates of C

Multiplying the third equation of (BF) by $-\Delta C$ and applying the same argument as for (3.17), we now have

$$\frac{1}{2}\frac{d}{dt}\|\nabla C\|_{L^2}^2 + \frac{1}{2}\|\Delta C\|_{L^2}^2 \leqslant \int_\Omega |\boldsymbol{u}|^2 |\nabla C|^2\,dx + 2\|\Delta T\|_{L^2}^2 + 2\|f_3\|_{L^2}^2$$

$$\leqslant \frac{1}{4}\|\Delta C\|_{L^2}^2 + 4\gamma^2 (\,\|\,|\nabla \boldsymbol{u}|\,\|_{L^2} + \|\boldsymbol{u}\|_{\mathbb{L}^2_\sigma}\,)^4 \|\nabla C\|_{L^2}^2 + 2\|\Delta T\|_{L^2}^2 + 2\|f_3\|_{L^2}^2. \quad (3.18)$$

Then integrating (3.18) over $[0,S]$ and applying Gronwall's inequality with estimates (3.14) and (3.17), we obtain

$$\begin{aligned}\sup_{0\le t\le S}\|\nabla C(t)\|_{L^2}^2 + \int_0^S \|\Delta C(t)\|_{L^2}^2\,dt \\ \leqslant \gamma\,(\,\|\,|\nabla \boldsymbol{u}_0|\,\|_{L^2},\ \|T_0\|_{H^1},\ \|C_0\|_{H^1},\ \|F\|_{L^2(0,S;H)}\,).\end{aligned} \quad (3.19)$$

Thus, a priori estimates (3.14), (3.17) and (3.19) assure that every local solutions can be continued globally up to $[0,S]$, provided that $U_0 \in D(\varphi) = D(\mathcal{A}^{1/2}) \times D(A_N^{1/2}) \times D(A_N^{1/2})$. As for the general case where $U_0 \in D(\mathcal{A}^\alpha) \times D(A_N^\alpha) \times D(A_N^\alpha)$ with $\alpha \in [1/4, 1/2)$, since $U(t)$ enjoys $(\#)_\alpha$, there exists a $t_0 \in (0, S_0)$ such that $U(t_0) \in D(\varphi)$. Hence, regarding $U(t_0)$ as an initial data and applying the global existence result for the case where $U_0 \in D(\varphi)$, we can derive the global existence result for the general case.

3.3 Uniqueness

In this subsection, we are going to prove the uniqueness of the solution of the initial boundary value problem for (BF).

Proof. Let U^1 and U^2 be solutions of (BF) for the same initial data:

$$U^i = \begin{pmatrix} \boldsymbol{u}^{\boldsymbol{i}} \\ T^i \\ C^i \end{pmatrix} \quad (i = 1, 2)$$

and let W be the difference of these two solutions

$$W = U^1 - U^2 = \begin{pmatrix} \boldsymbol{w} \\ \tau \\ \theta \end{pmatrix}.$$

Then $W = (\boldsymbol{w}, \tau, \theta)^t$ satisfies

$$
\text{(D)} \begin{cases} \partial_t \boldsymbol{w} = \nu \mathcal{P}_\Omega \Delta \boldsymbol{w} - a \boldsymbol{w} + \mathcal{P}_\Omega \boldsymbol{g} \tau + \mathcal{P}_\Omega \boldsymbol{h} \theta, \\ \partial_t \tau = \Delta \tau - \boldsymbol{u}^1 \cdot \nabla \tau + \boldsymbol{w} \cdot \nabla T^2, \\ \partial_t \theta = \Delta \theta + \rho \Delta \tau - \boldsymbol{u}^1 \cdot \nabla \theta + \boldsymbol{w} \cdot \nabla C^2. \end{cases}
$$

Multiplying the first equation of (D) by $\boldsymbol{w}$, we get

$$
\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\boldsymbol{w}\|^2_{\mathbb{L}^2_\sigma} + \nu \| \, |\nabla \boldsymbol{w}| \, \|^2_{L^2} + a \|\boldsymbol{w}\|^2_{\mathbb{L}^2_\sigma} &\leqslant |\boldsymbol{g}| \, \|\tau\|_{L^2} \, \|\boldsymbol{w}\|_{\mathbb{L}^2_\sigma} + |\boldsymbol{h}| \, \|\theta\|_{L^2} \, \|\boldsymbol{w}\|_{\mathbb{L}^2_\sigma} \\ &\leqslant \frac{1}{2} \left(|\boldsymbol{g}| + |\boldsymbol{h}| \right) \|\boldsymbol{w}\|^2_{\mathbb{L}^2_\sigma} + \frac{1}{2} |\boldsymbol{g}| \, \|\tau\|^2_{L^2} + \frac{1}{2} |\boldsymbol{h}| \, \|\theta\|^2_{L^2}. \end{aligned} \tag{3.20}
$$

Multiplying the second equation of (D) by τ and using the property (3.10) with $\boldsymbol{u}$ and T replaced by $\boldsymbol{u}^1$ and τ respectively and $|v|^2_{L^3} \leqslant |v|_{L^2} |v|_{L^6}$, we obtain

$$
\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tau\|^2_{L^2} + \|\nabla \tau\|^2_{L^2} &\leqslant \int_\Omega |\boldsymbol{w}| \, |\nabla T^2| \, |\tau| \, dx \\ &\leqslant \|\tau\|_{L^6} \, \|\nabla T^2\|_{L^2} \, \| \, |\boldsymbol{w}| \, \|_{L^3} \\ &\leqslant \frac{1}{4} \left(\|\nabla \tau\|^2_{L^2} + \|\tau\|^2_{L^2} \right) + \gamma \, \|\nabla T^2\|^2_{L^2} \, \|\boldsymbol{w}\|_{\mathbb{L}^2_\sigma} \, \| \, |\nabla \boldsymbol{w}| \, \|_{L^2} \\ &\leqslant \frac{1}{4} \|\nabla \tau\|^2_{L^2} + \frac{\nu}{4} \| |\nabla \boldsymbol{w}| \|^2_{L^2} + \frac{\gamma^2}{\nu} \|\nabla T^2\|^4_{L^2} \|\boldsymbol{w}\|^2_{\mathbb{L}^2_\sigma} + \frac{1}{4} \|\tau\|^2_{L^2}. \end{aligned} \tag{3.21}
$$

By the argument similar to that for (3.21), we obtain

$$
\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\theta\|^2_{L^2} + \|\nabla \theta\|^2_{L^2} \leqslant \int_\Omega |\boldsymbol{w}| \, |\nabla C^2| \, |\theta| \, dx + \rho \int_\Omega |\nabla \tau| \, |\nabla \theta| \, dx \\ &\leqslant \|\theta\|_{L^6} \, \|\nabla C^2\|_{L^2} \, \| \, |\boldsymbol{w}| \, \|_{L^3} + \rho \, \|\nabla \tau\|_{L^2} \, \|\nabla \theta\|_{L^2} \\ &\leqslant \frac{1}{4} \left(\|\nabla \theta\|^2_{L^2} + \|\theta\|^2_{L^2} \right) + \gamma \, \|\nabla C^2\|^2_{L^2} \, \|\boldsymbol{w}\|_{\mathbb{L}^2_\sigma} \, \| \, |\nabla \boldsymbol{w}| \, \|_{L^2} + \frac{1}{4} \, \|\nabla \theta\|^2_{L^2} + \rho^2 \, \|\nabla \tau\|^2_{L^2} \\ &\leqslant \frac{1}{2} \|\nabla \theta\|^2_{L^2} + \rho^2 \|\nabla \tau\|^2_{L^2} + \frac{\rho^2 \nu}{4} \| \, |\nabla \boldsymbol{w}| \, \|^2_{L^2} + \frac{\gamma^2}{\rho^2 \nu} \|\nabla C^2\|^4_{L^2} \, \|\boldsymbol{w}\|^2_{\mathbb{L}^2_\sigma} + \frac{1}{4} \|\theta\|^2_{L^2}. \end{aligned} \tag{3.22}
$$

Put $y(t) = \|\boldsymbol{w}(t)\|^2_{\mathbb{L}^2_\sigma} + \|\tau(t)\|^2_{L^2} + \frac{1}{2\rho^2} \, \|\theta(t)\|^2_{L^2}$ and sum up (3.20), (3.21) and (3.22)$\times \frac{1}{2\rho^2}$, then we get

$$
\begin{aligned} \frac{1}{2} \frac{d}{dt} y(t) &\leqslant \gamma \, y(t) + \frac{\gamma^2}{\nu} \, \|\nabla T^2(t)\|^4_{L^2} \, \|\boldsymbol{w}(t)\|^2_{\mathbb{L}^2_\sigma} + \frac{\gamma^2}{2\rho^4 \nu^2} \, \|\nabla C^2|^4_{L^2} \, \|\boldsymbol{w}(t)\|^2_{\mathbb{L}^2_\sigma} \\ &\leqslant \gamma \left(\|\nabla T^2\|^4_{L^2} + \|\nabla C^2\|^4_{L^2} + 1 \right) y(t). \end{aligned}
$$

Here we note that $(\#)_\alpha$ with $\alpha \in [1/4, 1/2]$ implies that

$$t^{1/2-\alpha}\,\|\nabla T^2\|_{L^2},\ t^{1/2-\alpha}\,\|\nabla C^2\|_{L^2} \in L^4_*(0,S) \quad \Rightarrow \quad \|\nabla T^2\|_{L^2},\ \|\nabla C^2\|_{L^2} \in L^4(0,S).$$

Hence, the uniqueness follows from Gronwall's inequality. $\square$

4 Periodic Problem

In this section, we give a proof of Theorem 2.2, which is divided into three parts, i.e., the existence of periodic solutions of approximate systems, some a priori estimates for solutions of approximate equations and convergence of solutions. Theorem 2.5 is applied for proving the existence of solutions of the approximate systems.

4.1 Approximate Equations

When one tries to apply Theorem 2.3 to (AP), one faces some difficulties. The most serious one arises in checking (A5). In fact, we recall that estimate (3.8) gives

$$\|\, B(U) \,\|_H^2 \leqslant \varepsilon^2 \,\|\, \partial\varphi(U) \,\|_H^2 + \frac{\gamma}{\varepsilon^2}\,(\,\varphi(U)^3 + \|\, U \,\|_H^2\,),$$

whose growth order for $\varphi(U)$ is cubic which does not satisfy the required growth order in (A5).

Moreover, when the constant vectors $\boldsymbol{g}$, $\boldsymbol{h}$ are very large, it is difficult to examine whether condition (A6) is satisfied. From these reasons, we are led to introduce the same type of relaxed approximate problems as in [11].

However, the approximate problems introduced in [11] prevents establishing the desirable a priori estimates under the homogeneous Neumann boundary condition. In order to manage with this difficulty, we introduce another approximation procedure.

More precisely, we replace T, C by their cut-off function, $[T]_\varepsilon$, $[C]_\varepsilon$, and we add some nonlinear dissipation terms and linear coercive terms to the second and the third equation. Indeed, we consider the following approximate equations.

$$(\mathrm{BF})_{\varepsilon,\lambda} \quad \begin{cases} \partial_t\, \boldsymbol{u} = \nu\, \mathcal{P}_\Omega\, \Delta\, \boldsymbol{u} - a\, \boldsymbol{u} + \mathcal{P}_\Omega\, \boldsymbol{g}\, [T]_\varepsilon + \mathcal{P}_\Omega\, \boldsymbol{h}\, [C]_\varepsilon + \mathcal{P}_\Omega\, \boldsymbol{f}_1, \\ \partial_t\, T + \boldsymbol{u}\cdot\nabla T = \Delta\, T - \varepsilon\, |T|^{p-2} T - \lambda\, T + f_2, \\ \partial_t\, C + \boldsymbol{u}\cdot\nabla C = \Delta\, C + \rho\, \Delta\, T - \varepsilon\, |C|^{p-2} C - \lambda\, C + f_3, \end{cases} \tag{4.1}$$

where ε, $\lambda \in (0,1)$ are approximation parameters and the cut-off function $[T]_\varepsilon$ is defined by

$$[T]_\varepsilon = \begin{cases} T & \text{if} \ \ |T| \leqslant 1/\varepsilon, \\ (\mathrm{Sgn}\ T)\, 1/\varepsilon & \text{if} \ \ |T| \geqslant 1/\varepsilon, \quad \varepsilon \in (0,1), \end{cases} \tag{4.2}$$

and p is a large exponent to be fixed later on.

Here we are going to reduce these approximate equations (4.1) to an abstract problem similar to (AP). For the perturbation term, we replace it by

$$B_\varepsilon(U) = \begin{pmatrix} a\boldsymbol{u} - \mathcal{P}_\Omega\, \boldsymbol{g}\, [T]_\varepsilon - \mathcal{P}_\Omega\, \boldsymbol{h}\, [C]_\varepsilon \\ \boldsymbol{u}\cdot\nabla T \\ \boldsymbol{u}\cdot\nabla C - \rho\,\Delta\, T \end{pmatrix}. \tag{4.3}$$

We also need to replace the lower semi-continuous convex function φ by $\varphi_{\varepsilon,\lambda}$ which is given by

$$\varphi_{\varepsilon,\lambda}(U) = \varphi(U) + \psi_{\varepsilon,\lambda}(U),$$

$$\psi_{\varepsilon,\lambda}(U) = \begin{cases} \dfrac{\varepsilon}{p}\|T\|_{L^p}^p + \dfrac{\varepsilon}{9\rho^2 p}\ \|C\|_{L^p}^p + \dfrac{\lambda}{2}\|T\|_{L^2}^2 + \dfrac{\lambda}{18\rho^2}\|C\|_{L^2}^2 & \text{if } U \in D(\psi_{\varepsilon,\lambda}) \\ +\infty & \text{if } U \in H\backslash D(\psi_{\varepsilon,\lambda}), \end{cases}$$

where $D(\psi_{\varepsilon,\lambda}) = \mathbb{L}^2_\sigma(\Omega) \times L^p(\Omega) \times L^p(\Omega)$.

Here and henceforth, we choose $\eta = 1$ in (2.3), definition of the inner product of H. Then it is clear that $\psi_{\varepsilon,\lambda}$ is a lower semi-continuous convex function on H and Fréchet differentiable on $D(\psi_{\varepsilon,\lambda})$ and that the subdifferential of $\psi_{\varepsilon,\lambda}$ coincides with the sum of the dissipation term and the coercive term, i.e.,

$$\partial\psi_{\varepsilon,\lambda}(U) = (\,0,\ \varepsilon\,|T|^{p-2}T + \lambda\,T,\ \varepsilon\,|C|^{p-2}C + \lambda\,C\,)^t.$$

In general, the sum of two subdifferentials is not always maximal monotone. But for this case, we have the following good property:

$$\begin{aligned} (\,\partial\varphi(U),\ \partial\psi_{\varepsilon,\lambda}(U))_H &= (-\Delta\,T,\ \varepsilon\,|T|^{p-2}T + \lambda\,T\,)_{L^2} + (\,-\,\Delta\,C,\ \varepsilon\,|C|^{p-2}C\,+\lambda\,C)_{L^2} \\ &= \lambda\,\|\nabla T\|_{L^2}^2 + \varepsilon\,(p-1)\int_\Omega |T|^{p-2}|\nabla T|^2 dx \\ &\quad + \lambda\,\|\nabla C\|_{L^2}^2 + \varepsilon\,(p-1)\int_\Omega |C|^{p-2}|\nabla C|^2 dx \geqslant 0. \end{aligned} \tag{4.4}$$

By virtue of (4.4), together with Proposition 2.17, Theorem 4.4 and Proposition 4.6 in Brézis [5], we can deduce that $\partial\varphi + \partial\psi_{\varepsilon,\lambda}$ becomes maximal monotone, and hence we get $\partial\,(\varphi + \psi_{\varepsilon,\lambda}) = \partial\varphi + \partial\psi_{\varepsilon,\lambda}$ with $D(\partial\,(\varphi + \psi_{\varepsilon,\lambda})) = D(\partial\varphi) \cap D(\partial\psi_{\varepsilon,\lambda})$.

Thus, we have another abstract problem associated with approximate problems:

$$(\mathrm{AP})_{\varepsilon,\lambda} \begin{cases} \dfrac{dU(t)}{dt} + \partial\varphi_{\varepsilon,\lambda}(U(t)) + B_\varepsilon(U(t)) = F(t) \quad t \in [0, S], \\ U(0) = U(S). \end{cases} \tag{4.5}$$

4.2 Solvability of the Approximate Systems

In this subsection, we are going to verify that Theorem 2.5 can be applied to $(\mathrm{AP})_{\varepsilon,\lambda}$, that is to say, we are going to show that (A1), (A2), (A5) and (A6) are satisfied with φ and B replaced by $\varphi_{\varepsilon,\lambda}$ and B_ε respectively. With obvious modifications, (A1) and (A2) can be verified as in §3.1.

Check of (A5) $\quad {}^\exists\ell(\cdot),\ {}^\exists k \in [0,1)$ such that

$$\| B(U) \|_H^2 \leqslant k \, \| \partial\varphi(U) \|_H^2 + \ell\,(\| U \|_H)(\,\varphi(U) + 1\,)^2 \quad {}^\forall U \in D(\partial\varphi).$$

Proof. By the definition of $B_\varepsilon(U)$ and the inner product of H, we get (see (3.5))

$$\| B_\varepsilon(U) \|_H^2 \leqslant \beta \, \| U \|_H^2 + 2\int_\Omega |\boldsymbol{u}\cdot\nabla T|^2 dx + \frac{2}{9\rho^2}\int_\Omega |\boldsymbol{u}\cdot\nabla C|^2 dx + \frac{2}{9}\,\| \Delta T \|_{L^2}^2 \quad (4.6)$$

for some constant β.

We begin with the estimate for the convection terms. Since $\nabla\cdot\boldsymbol{u} = 0$, the integration by parts gives

$$\int_\Omega |\boldsymbol{u}\cdot\nabla T|^2 dx = \int_\Omega \nabla T\cdot\boldsymbol{u}(\boldsymbol{u}\cdot\nabla T)\,dx = -\int_\Omega T\boldsymbol{u}\nabla(\boldsymbol{u}\cdot\nabla T)\,dx \leqslant \int_\Omega |T|\,|\boldsymbol{u}|\,|\nabla(\boldsymbol{u}\cdot\nabla T)|\,dx.$$

Then by the elliptic estimate and Hölder's inequality, we have

$$\begin{aligned} 2\int_\Omega |\boldsymbol{u}\cdot\nabla T|^2 dx &\leqslant \beta\int_\Omega |T|\,|\boldsymbol{u}|\,|\boldsymbol{u}|\,|\Delta T|\,dx + \beta\int_\Omega |T|\,|\boldsymbol{u}|\,|\nabla\boldsymbol{u}|\,|\nabla T|\,dx \\ &\leqslant \beta\| T \|_{L^{12}} \, \| |\boldsymbol{u}| \|_{L^6} \, \| |\boldsymbol{u}| \|_{L^4} \, \| \Delta T \|_{L^2} \\ &\qquad + \beta\| T \|_{L^{12}} \, \| |\boldsymbol{u}| \|_{L^6} \, \| |\nabla\boldsymbol{u}| \|_{L^4} \, \| \nabla T \|_{L^2}. \end{aligned}$$

Hence, by using the inequality $\|w\|_{L^4}^4 \leqslant \|w\|_{L^2}\,\|w\|_{L^6}^3$, Sobolev's inequality and Young's inequality, we get

$$\begin{aligned} 2\int_\Omega |\boldsymbol{u}\cdot\nabla T|^2 dx \leqslant{}& \frac{1}{9}\,\| \Delta T \|_{L^2}^2 + \beta\,(\,\| |\nabla\boldsymbol{u}| \|_{L^2}^4 + \| T \|_{L^{12}}^{16} \, \| \boldsymbol{u} \|_{\mathbb{L}^2_\sigma}^4) \\ &+ \frac{1}{6}\,\| \mathcal{A}\boldsymbol{u} \|_{\mathbb{L}^2_\sigma}^2 + \beta\,(\,\| |\nabla\boldsymbol{u}| \|_{L^2}^4 + \| \nabla T \|_{L^2}^4 + \| T \|_{L^{12}}^{16}\,), \end{aligned}$$

for some constant β. The convection term for C can be estimated in the same way.

Consequently, by taking $p \geq 12$, we obtain

$$\| B_\varepsilon(U) \|_H^2 \leqslant \frac{1}{3}\,\| \partial\varphi_{\varepsilon,\lambda}(U) \|_H^2 + \ell\,(\,\| U \|_H\,)\,(\,\varphi_{\varepsilon,\lambda}(U) + 1\,)^2,$$

whence follows (A5) with $k = 1/3$, provided that $p \geqslant 12$. $\square$

Check of (A6) $\quad {}^{\exists}\alpha,\ {}^{\exists}K$ such that

$$(-\partial\varphi(U) - B(U),\ U\,)_H + \alpha\,\varphi(U) \leqslant K \quad {}^{\forall}U \in D(\partial\varphi).$$

Proof. The definition of the inner product of H gives

$$\begin{aligned}(\partial\varphi_{\varepsilon,\lambda}(U),\ U\,)_H &= \nu\|\,|\nabla \boldsymbol{u}|\,\|_{L^2}^2 + \|\nabla T\|_{L^2}^2 + \frac{1}{9\rho^2}\|\nabla C\|_{L^2}^2 \\ &\qquad + \varepsilon\|T\|_{L^p}^p + \lambda\|T\|_{L^2}^2 + \frac{\varepsilon}{9\rho^2}\,\|C\|_{L^p}^p + \frac{\lambda}{9\rho^2}\,\|C\|_{L^2}^2 \\ &\geqslant 2\,\varphi_{\varepsilon,\lambda}(U).\end{aligned}$$

Moreover, noting that $(\,\boldsymbol{u}\cdot\nabla T,\ T\,)_{L^2} = (\,\boldsymbol{u}\cdot\nabla C,\ C\,)_{L^2} = 0$ and the cut-off function is bounded by $1/\varepsilon$, we get

$$\begin{aligned}(B_\varepsilon(U),\ U\,)_H &\geqslant a\,\|\,\boldsymbol{u}\,\|_{\mathbb{L}^2_\sigma}^2 - |\boldsymbol{g}|\,\|\,\boldsymbol{u}\,\|_{\mathbb{L}^2_\sigma}\,\|\,[T]_\varepsilon\,\|_{L^2} - |\boldsymbol{h}|\,\|\,\boldsymbol{u}\,\|_{\mathbb{L}^2_\sigma}\,\|\,[C]_\varepsilon\,\|_{L^2} \\ &\qquad - \frac{1}{9\rho}\,\|\,\nabla T\,\|_{L^2}\,\|\,\nabla C\,\|_{L^2} \\ &\geqslant a\,\|\,\boldsymbol{u}\,\|_{\mathbb{L}^2_\sigma}^2 - 2\cdot\frac{a}{2}\,\|\,\boldsymbol{u}\,\|_{\mathbb{L}^2_\sigma}^2 - \frac{|\boldsymbol{g}|^2 + |\boldsymbol{h}|^2}{2a}\{\,\|\,[T]_\varepsilon\,\|_{L^2}^2 + \|\,[C]_\varepsilon\,\|_{L^2}^2\} \\ &\qquad - \frac{1}{2}\,\|\,\nabla T\,\|_{L^2}^2 - \frac{1}{18\rho^2}\,\|\,\nabla C\,\|_{L^2}^2 \\ &\geqslant -\frac{\beta}{\varepsilon^2} - \varphi_{\varepsilon,\lambda}(U),\end{aligned}$$

where we used Cauchy's inequality and β is a suitable constant.

Hence we get

$$(-\partial\varphi_{\varepsilon,\lambda}(U) - B_\varepsilon(U),\ U\,)_H + \varphi_{\varepsilon,\lambda}(U) \leqslant \frac{\beta}{\varepsilon^2},$$

whence follows (A6) with $K = \frac{\beta}{\varepsilon^2}$ and $\alpha = 1$. □

Thus the existence of solutions $(\mathrm{AP})_{\varepsilon,\lambda}$ is assured by Theorem 2.2.

4.3 A Priori Estimates 1

In this subsection, we are going to establish some a priori estimates independent of the approximation parameter ε for fixed λ. In what follows, let $U_\varepsilon = (\boldsymbol{u}_\varepsilon,\ T_\varepsilon,\ C_\varepsilon)^t$ be the periodic solutions of approximate equations $(\mathrm{BF})_{\varepsilon,\lambda}$ and let $\varphi_{\varepsilon,\lambda}$ be denoted by φ_ε. We also introduce the general constant γ depending on $\|\,\boldsymbol{f}_1\,\|_{\mathbb{L}^2(0,S;\mathbb{L}^2_\sigma(\Omega))}$, $\|\,f_2\,\|_{L^2(0,S;L^2(\Omega))}$, $\|\,f_3\,\|_{L^2(0,S;L^2(\Omega))}$, $|\,\boldsymbol{g}\,|$ and $|\,\boldsymbol{h}\,|$ but not on ε.

The First Energy Estimates

(i) Multiply the second equation of (4.1) by T_ε, then recalling that $(\boldsymbol{u}_\varepsilon \cdot \nabla T_\varepsilon,\ T_\varepsilon)_{L^2} = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\,\|\,T_\varepsilon\,\|_{L^2}^2 + \|\,\nabla T_\varepsilon\,\|_{L^2}^2 + \varepsilon\,\|\,T_\varepsilon\,\|_{L^p}^p + \lambda\,\|T_\varepsilon\|_{L^2}^2 = \int_\Omega f_2\,T_\varepsilon\,dx. \tag{4.7}$$

Since the periodic condition gives

$$\int_0^S \frac{d}{d\tau}\|\,T_\varepsilon(\tau)\,\|_{L^2}^2\,d\tau = \|\,T_\varepsilon(S)\,\|_{L^2}^2 - \|\,T_\varepsilon(0)\,\|_{L^2}^2 = 0,$$

integrating (4.7) over $[0,S]$, we obtain

$$\|\nabla T_\varepsilon\|_{L^2(Q)}^2 + \lambda\,\|T_\varepsilon\|_{L^2(Q)}^2 \leqslant \|f_2\|_{L^2(Q)}\,\|T_\varepsilon\|_{L^2(Q)}. \tag{4.8}$$

Since $T_\varepsilon \in C([0,S];L^2(\Omega))$, there exists $t_0 \in [0,S]$ where $\|T_\varepsilon(t)\|_{L^2}$ attains its minimum, i.e.,

$$\|T_\varepsilon(t_0)\|_{L^2} = \min_{0\leqslant t\leqslant S}\|T_\varepsilon(t)\|_{L^2}.$$

Hence (4.8) gives

$$\|T_\varepsilon(t_0)\|_{L^2(\Omega)} \leqslant \frac{1}{\lambda S}\,\|f_2\|_{L^2(Q)} \leqslant \gamma.$$

Then, integrating (4.7) over $[t_0,t]$ $(t_0 \leqslant t \leqslant t_0+S)$ and over $[t_0,t_0+S]$, we obtain

$$\|\,T_\varepsilon\,\|_{C([0,S];L^2(\Omega))},\ \|\,\nabla T_\varepsilon\,\|_{L^2(Q)},\ \varepsilon\,\|\,T_\varepsilon\,\|_{L^p(Q)}^p \leqslant \gamma. \tag{4.9}$$

(ii) Multiplying the third equation of (4.1) by C_ε, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\,C_\varepsilon\,\|_{L^2}^2 + \|\,\nabla C_\varepsilon\,\|_{L^2}^2 + \varepsilon\,\|\,C_\varepsilon\,\|_{L^p}^p + \lambda\,\|C_\varepsilon\|_{L^2}^2 &= \rho\int_\Omega C_\varepsilon\,\Delta\,T_\varepsilon\,dx + \int_\Omega f_3\,C_\varepsilon\,dx\\ &\leqslant \frac{1}{2}\,\|\,\nabla C_\varepsilon\,\|_{L^2}^2 + \frac{\rho^2}{2}\,\|\,\nabla T_\varepsilon\,\|_{L^2}^2 + \|\,f_3\,\|_{L^2}\,\|\,C_\varepsilon\,\|_{L^2}\end{aligned}$$

Since we already know the boundedness of $\|\,\nabla T_\varepsilon\,\|_{L^2(Q)}^2$, repeating the same arguments as above, we obtain

$$\|\,C_\varepsilon\,\|_{C([0,S];L^2(\Omega))},\ \|\,\nabla C_\varepsilon\,\|_{L^2(Q)},\ \varepsilon\,\|\,C_\varepsilon\,\|_{L^p(Q)}^p \leqslant \gamma. \tag{4.10}$$

(iii) Multiplying the first equation of (4.1) by $\boldsymbol{u}_\varepsilon$ and integrating over Ω, by (4.9) and (4.10), we see that for any $\mu > 0$ there exists a constant C_μ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\,\boldsymbol{u}_\varepsilon\,\|_{\mathbb{L}^2_\sigma}^2 + \nu\,\|\,|\nabla \boldsymbol{u}_\varepsilon|\,\|_{L^2}^2 + a\,\|\,\boldsymbol{u}_\varepsilon\,\|_{\mathbb{L}^2_\sigma}^2 \\
&\qquad = \int_\Omega [T_\varepsilon]_\varepsilon\,\boldsymbol{g}\cdot\boldsymbol{u}_\varepsilon\,dx + \int_\Omega [C_\varepsilon]_\varepsilon\,\boldsymbol{h}\cdot\boldsymbol{u}_\varepsilon\,dx + \int_\Omega \boldsymbol{f}_1\cdot\boldsymbol{u}_\varepsilon\,dx \\
&\qquad \leqslant |\,\boldsymbol{g}\,|\,\|\,T_\varepsilon\,\|_{L^2}\,\|\,\boldsymbol{u}_\varepsilon\,\|_{\mathbb{L}^2_\sigma} + |\,\boldsymbol{h}\,|\,\|\,C_\varepsilon\,\|_{L^2}\,\|\,\boldsymbol{u}_\varepsilon\,\|_{\mathbb{L}^2_\sigma} + \|\,\boldsymbol{f}_1\,\|_{\mathbb{L}^2}\,\|\,\boldsymbol{u}_\varepsilon\,\|_{\mathbb{L}^2_\sigma} \\
&\qquad \leqslant \mu\,\|\,\boldsymbol{u}_\varepsilon\,\|_{\mathbb{L}^2_\sigma}^2 + C_\mu\,(\,\gamma + \|\,\boldsymbol{f}_1\,\|_{\mathbb{L}^2}^2\,).
\end{aligned}$$

Then, as above, we get

$$\|\,\boldsymbol{u}_\varepsilon\,\|_{C([0,S];\mathbb{L}^2_\sigma(\Omega))},\ \ \|\,|\nabla \boldsymbol{u}_\varepsilon|\,\|_{L^2(Q)} \leqslant \gamma. \tag{4.11}$$

The Second Energy Estimates

(i) Multiplying the first equation of (4.1) by $\partial_t \boldsymbol{u}_\varepsilon$, we have

$$\begin{aligned}
&\|\,\partial_t\,\boldsymbol{u}_\varepsilon\|_{\mathbb{L}^2_\sigma}^2 + \frac{\nu}{2}\frac{d}{dt}\|\,|\nabla \boldsymbol{u}_\varepsilon|\,\|_{\mathbb{L}^2}^2 + \frac{a}{2}\frac{d}{dt}\|\,\boldsymbol{u}_\varepsilon\,\|_{\mathbb{L}^2_\sigma}^2 \\
&\qquad = \int_\Omega ([T_\varepsilon]_\varepsilon\,\boldsymbol{g} + [C_\varepsilon]_\varepsilon \boldsymbol{h} + \boldsymbol{f}_1)\cdot\partial_t \boldsymbol{u}_\varepsilon\,dx \\
&\qquad \leqslant (\,|\,\boldsymbol{g}\,|\,\|\,T_\varepsilon\,\|_{L^2} + |\,\boldsymbol{h}\,|\,\|\,C_\varepsilon\,\|_{L^2} + \|\,\boldsymbol{f}_1\,\|_{\mathbb{L}^2}\,)\,\|\,\partial_t \boldsymbol{u}_\varepsilon\,\|_{\mathbb{L}^2_\sigma} \\
&\qquad \leqslant (\,\gamma + \|\,\boldsymbol{f}_1\,\|_{\mathbb{L}^2}^2\,) + \frac{1}{2}\,\|\,\partial_t \boldsymbol{u}_\varepsilon\,\|_{\mathbb{L}^2_\sigma}^2.
\end{aligned} \tag{4.12}$$

On the other hand, in view of (4.9), (4.10) and (4.11), we find that $\|\,\varphi_\varepsilon(U_\varepsilon(t))\|_{L^1(0,S)} \leqslant \gamma$. Hence, since $\varphi_\varepsilon(U_\varepsilon(t))$ is absolutely continuous on $[0,S]$, there exists $t_1 \in [0,S]$, where $\varphi_\varepsilon(U_\varepsilon(t))$ attains its minimum at $t = t_1$, i.e.,

$$\varphi_\varepsilon(U_\varepsilon(t_1)) = \min_{0\leqslant t\leqslant S}\varphi_\varepsilon(U_\varepsilon(t)) \leqslant \frac{1}{S}\int_0^S \varphi_\varepsilon(U_\varepsilon(\tau))d\tau \leqslant \frac{\gamma}{S}$$

whence follows

$$\|\,\nabla \boldsymbol{u}_\varepsilon(t_1)\,\|_{\mathbb{L}^2}^2,\ \|\,\nabla T_\varepsilon(t_1)\,\|_{L^2}^2,\ \|\,\nabla C_\varepsilon(t_1)\,\|_{L^2}^2,\ \varepsilon\,\|\,T_\varepsilon(t_1)\,\|_{L^p}^p,\ \varepsilon\,\|\,C_\varepsilon(t_1)\,\|_{L^p}^p \leqslant \gamma. \tag{4.13}$$

Then integrating (4.12) over $[t_1,t]$ and $[t_1,t_1+S]$, we derive

$$\|\,\nabla \boldsymbol{u}_\varepsilon\,\|_{C(0,S;\mathbb{L}^2(\Omega))},\ \|\,\partial_t \boldsymbol{u}_\varepsilon\,\|_{L^2(0,S;\mathbb{L}^2_\sigma(\Omega))} \leqslant \gamma. \tag{4.14}$$

Furthermore, by using the first equation of (4.1), we also obtain

$$\| \mathcal{A}\boldsymbol{u}_\varepsilon \|_{L^2(0,S;\mathbb{L}^2_\sigma(\Omega))} \leqslant \gamma. \tag{4.15}$$

(ii) Multiplying the second equation of (4.1) by $-\Delta T_\varepsilon$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\| \nabla T_\varepsilon \|^2_{L^2} + \| \Delta T_\varepsilon \|^2_{L^2} \\
&\qquad \leqslant -\varepsilon\,(p-1)\int_\Omega |T_\varepsilon|^{p-2}|\nabla T_\varepsilon|^2 dx - \int_\Omega \Delta T_\varepsilon\, \boldsymbol{u}_\varepsilon \cdot \nabla T_\varepsilon\, dx - \int_\Omega f_2\, \Delta T_\varepsilon\, dx. \\
&\qquad \leqslant \int_\Omega |\Delta T_\varepsilon|\, |\boldsymbol{u}_\varepsilon|\, |\nabla T_\varepsilon|\, dx + \| f_2 \|_{L^2} \| \Delta T_\varepsilon \|_{L^2}.
\end{aligned}$$

Here, using again $\| T_\varepsilon \|^2_{L^3} \leqslant \| T_\varepsilon \|_{L^2} \| T_\varepsilon \|_{L^6}$, we get

$$\begin{aligned}
&\int_\Omega |\boldsymbol{u}_\varepsilon|\, |\nabla T_\varepsilon|\, |\Delta T_\varepsilon|\, dx \\
&\qquad \leqslant \| \,|\boldsymbol{u}_\varepsilon|\, \|_{L^6} \| \nabla T_\varepsilon \|_{L^3} \| \Delta T_\varepsilon \|_{L^2} \\
&\qquad \leqslant \kappa \| \,|\nabla \boldsymbol{u}_\varepsilon|\, \|_{L^2} \| \nabla T_\varepsilon \|^{1/2}_{L^2} \left(\| \Delta T_\varepsilon \|^{1/2}_{L^2} + \|\nabla T_\varepsilon\|^{1/2}_{L^2} \right) \| \Delta T_\varepsilon \|_{L^2} \\
&\qquad \leqslant \frac{1}{2} \| \Delta T_\varepsilon \|^2_{L^2} + \frac{27}{4}\kappa^4 \| \,|\nabla \boldsymbol{u}_\varepsilon|\, \|^4_{L^2} \| \nabla T_\varepsilon \|^2_{L^2} + \kappa^2 \| \,|\nabla \boldsymbol{u}_\varepsilon|\, \|^2_{L^2} \| \nabla T_\varepsilon \|^2_{L^2},
\end{aligned}$$

where κ is the constant which depends on Sobolev's embedding constant. Therefore, using previous estimates, we obtain

$$\frac{1}{2}\frac{d}{dt}\| \nabla T_\varepsilon \|^2_{L^2} + \frac{1}{4} \| \Delta T_\varepsilon \|^2_{L^2} \leqslant \gamma \| \nabla T_\varepsilon \|^2_{L^2} + \| f_2 \|^2_{L^2}.$$

Hence, by Gronwall's inequality, we obtain

$$\| \nabla T_\varepsilon \|_{C(0,S;L^2(\Omega))},\ \| \Delta T_\varepsilon \|_{L^2(Q)} \leqslant \gamma. \tag{4.16}$$

Next multiplying the second equation of (4.1) by $\partial_t T_\varepsilon$ and integrating over Ω, we get

$$\|\partial_t T_\varepsilon\|^2_{L^2} + \frac{d}{dt}\left(\frac{1}{2}\|\nabla T_\varepsilon\|^2_{L^2} + \frac{\varepsilon}{p}\|T_\varepsilon\|^p_{L^p} + \frac{\lambda}{2}\|T_\varepsilon\|^2_{L^2}\right) = \int_\Omega \partial_t T_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla T_\varepsilon\, dx + \int_\Omega f_2 \partial_t T_\varepsilon\, dx.$$

The above argument with ΔT_ε replaced by $\partial_t T_\varepsilon$ gives

$$\frac{1}{4} \| \partial_t T_\varepsilon \|^2_{L^2} + \frac{d}{dt}\left(\frac{1}{2} \| \nabla T_\varepsilon \|^2_{L^2} + \frac{\varepsilon}{p} \| T_\varepsilon \|^p_{L^p} + \frac{\lambda}{2} \| T_\varepsilon \|^2_{L^2} \right) \leqslant \gamma \| \nabla T_\varepsilon \|^2_{L^2} + \| f_2 \|^2_{L^2},$$

whence follows

$$\| \partial_t T_\varepsilon \|_{L^2(Q)}, \ \sup_{0 \leqslant t \leqslant S} \varepsilon \| T_\varepsilon \|^p_{L^p(\Omega)} \leqslant \gamma. \tag{4.17}$$

(iii) We multiply the third equation of (4.1) by $-\Delta\, C_\varepsilon$ or $\partial_t\, C_\varepsilon$. Since we already obtain the a priori bounds for T_ε, by the same arguments as above, we obtain

$$\| \nabla C_\varepsilon \|_{C(0,S;L^2(\Omega))}, \ \| \Delta\, C_\varepsilon \|_{L^2(Q)}, \ \| \partial_t\, C_\varepsilon \|_{L^2(Q)}, \ \sup_{0 \leqslant t \leqslant S} \varepsilon \| C_\varepsilon \|^p_{L^p(\Omega)} \leqslant \gamma. \tag{4.18}$$

Convergence

Since we already have the same a priori estimates for U_ε as in [11], we can repeat the same arguments for the convergence of U_ε as $\varepsilon \to 0$ as in [11] to find that U_ε converges to $U = (\boldsymbol{u}, T, C)^t$ which satisfies the following equations.

$$(\mathrm{BF})_\lambda \ \begin{cases} \partial_t\, \boldsymbol{u} = \nu\, \mathcal{P}_\Omega\, \Delta\, \boldsymbol{u} - a\, \boldsymbol{u} + \mathcal{P}_\Omega\, \boldsymbol{g}\, T + \mathcal{P}_\Omega\, \boldsymbol{h}\, C + \mathcal{P}_\Omega\, \boldsymbol{f}_1, \\ \partial_t\, T + \boldsymbol{u} \cdot \nabla T = \Delta\, T - \lambda\, T + f_2, \\ \partial_t\, C + \boldsymbol{u} \cdot \nabla C = \Delta\, C + \rho\, \Delta\, T - \lambda\, C + f_3, \end{cases}$$

with periodic conditions.

4.4 A Priori Estimates 2

Let $U_\lambda = (\boldsymbol{u}_\lambda, T_\lambda, C_\lambda)^t$ be the periodic solutions of $(\mathrm{BF})_\lambda$ and assume that $f_2,\ f_3 \in L^2(0, S; L^2(\Omega))$ satisfy (2.1).

In this subsection, we are going to establish some a priori estimates for U_λ independent of the second approximation parameter λ. We here denote by γ the general constant depending on $\| \boldsymbol{f}_1 \|_{\mathbb{L}^2(0,S;\mathbb{L}^2_\sigma(\Omega))}$, $\| f_2 \|_{L^2(0,S;L^2(\Omega))}$, $\| f_3 \|_{L^2(0,S;L^2(\Omega))}$, $|\, \boldsymbol{g}\, |$ and $|\, \boldsymbol{h}\, |$ but not on λ.

The First Energy Estimates

(i) Integrate the second equation of $(\mathrm{BF})_\lambda$ over Ω, then by the boundary condition, we get

$$\frac{d}{dt} \int_\Omega T_\lambda(x,t)\, dx + \lambda \int_\Omega T_\lambda(x,t)\, dx = \int_\Omega f_2(x,t)\, dx \quad {}^\forall t \in [0, S]. \tag{4.19}$$

Here we used the following facts.

$$\int_\Omega \Delta T_\lambda dx = \int_{\partial\Omega} \frac{\partial T_\lambda}{\partial n} dS = 0, \ \int_\Omega \boldsymbol{u}_\lambda \cdot \nabla T_\lambda dx = \int_\Omega \mathrm{div}(\boldsymbol{u}_\lambda T_\lambda) dx = \int_{\partial\Omega} \boldsymbol{u}_\lambda T_\lambda dS = 0.$$

Integrating (4.19) with respect to t on $(0,S)$ and using the periodic condition and (2.1), we find that

$$\lambda \int_0^S\!\!\int_\Omega T_\lambda(x,t)\,dx\,dt = 0 \quad \text{so} \quad {}^\exists t_0 \in [0,S] \;\text{ such that }\; \int_\Omega T_\lambda(x,t_0)\,dx = 0.$$

Hence by (4.19), we obtain

$$\int_\Omega T_\lambda(x,t)\,dx = \int_{t_0}^t e^{-\lambda(t-s)}\!\int_\Omega f_2(x,t)\,dx\,dt \quad {}^\forall t \in [t_0, t_0+S]. \tag{4.20}$$

Then applying Poincaré-Wirtinger's inequality

$$\|v-\overline{v}\|_{L^2} \leqslant C_W\,\|\nabla v\|_{L^2} \quad {}^\forall v \in H^1(\Omega), \quad \overline{v} = \frac{1}{|\Omega|}\int_\Omega v(x)\,dx,$$

we obtain

$$\|T_\lambda\|_{L^2(Q)} \leqslant C_W\,\|\nabla T_\lambda\|_{L^2(Q)} + S\,\|f_2\|_{L^2(Q)}. \tag{4.21}$$

(ii) Multiply the second equation of $(\mathrm{BF})_\lambda$ by T_λ, then recalling that $(\boldsymbol{u}_\lambda\cdot\nabla T_\lambda,\ T_\lambda)_{L^2} = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\,\|\,T_\lambda\,\|_{L^2}^2 + \|\,\nabla T_\lambda\,\|_{L^2}^2 + \lambda\,\|T_\lambda\|_{L^2}^2 = \int_\Omega f_2\,T_\lambda\,dx. \tag{4.22}$$

Integrating (4.22) over $[0,S]$ and using the periodic condition, we get

$$\|\nabla T_\lambda\|_{L^2(Q)}^2 + \lambda\,\|T_\lambda\|_{L^2(Q)}^2 \leqslant \|f_2\|_{L^2(Q)}\,\|T_\lambda\|_{L^2(Q)}. \tag{4.23}$$

Then substituting (4.21) into (4.23), we get

$$\|\nabla T_\lambda\|_{L^2(Q)}^2 \leqslant (\,C_W^2 + 2\,S\,)\,\|f_2\|_{L^2(Q)}^2,$$

which together with (4.21) gives

$$\|T_\lambda\|_{L^2(Q)}^2 \leqslant \{\,2\,C_W^2(C_W^2 + 2\,S) + 2\,S^2\,\}\,\|f_2\|_{L^2(Q)}^2. \tag{4.24}$$

Since $T_\lambda \in C([0,S];L^2(\Omega))$, there exists $t_0 \in [0,S]$ where $\|T_\lambda(t)\|_{L^2}$ attains its minimum, i.e.,

$$\|T_\lambda(t_0)\|_{L^2} = \min_{0\leqslant t\leqslant S}\|T_\lambda(t)\|_{L^2}.$$

Hence by (4.24), we have

$$\|T_\lambda(t_0)\|_{L^2}^2 \leqslant \frac{1}{S}\,\{\,2\,C_W^2(C_W^2 + 2\,S) + 2\,S^2\,\}\,\|f_2\|_{L^2(Q)}^2 \leqslant \gamma.$$

Then, integrating (4.22) over $[t_0, t]$ $(t_0 \leqslant t \leqslant t_0 + S)$ and over $[t_0, t_0 + S]$, we obtain

$$\| T_\lambda \|_{C([0,S];L^2(\Omega))}, \ \| \nabla T_\lambda \|_{L^2(Q)}, \ \lambda \| T_\lambda \|^2_{L^2(Q)} \leqslant \gamma. \tag{4.25}$$

(iii) Multiplying the third equation of $(\mathrm{BF})_\lambda$ by C_λ, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\| C_\lambda \|^2_{L^2} + \| \nabla C_\lambda \|^2_{L^2} + \lambda \|C\|^2_{L^2} &= \rho \int_\Omega C_\lambda \, \Delta \, T_\lambda \, dx + \int_\Omega f_3 \, C_\lambda \, dx \\ &\leqslant \frac{1}{2} \| \nabla C_\lambda \|^2_{L^2} + \frac{\rho^2}{2} \| \nabla T_\lambda \|^2_{L^2} + \| f_3 \|_{L^2} \| C_\lambda \|_{L^2}.\end{aligned}$$

Since we already know the boundedness of $\| \nabla T_\lambda \|^2_{L^2(Q)}$, repeating the same arguments as above, we obtain

$$\| C_\lambda \|_{C([0,S];L^2(\Omega))}, \ \| \nabla C_\lambda \|_{L^2(Q)}, \ \lambda \| C_\lambda \|^2_{L^2(Q)} \leqslant \gamma. \tag{4.26}$$

(iv) Multiplying the first equation of $(\mathrm{BF})_\lambda$ by $\boldsymbol{u}_\lambda$, by (4.25) and (4.26), we see that for any $\mu > 0$ there exists a constant C_μ such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt} \| \boldsymbol{u}_\lambda \|^2_{\mathbb{L}^2_\sigma} + \nu \| \, |\nabla \boldsymbol{u}_\lambda| \, \|^2_{L^2} + a \| \boldsymbol{u}_\lambda \|^2_{\mathbb{L}^2_\sigma} \\ &\quad = \int_\Omega T_\lambda \, \boldsymbol{g} \cdot \boldsymbol{u}_\lambda \, dx + \int_\Omega C_\lambda \, \boldsymbol{h} \cdot \boldsymbol{u}_\lambda \, dx + \int_\Omega \boldsymbol{f}_1 \cdot \boldsymbol{u}_\lambda \, dx \\ &\quad \leqslant |\, \boldsymbol{g} \,| \, \| T_\lambda \|_{L^2} \| \boldsymbol{u}_\lambda \|_{\mathbb{L}^2_\sigma} + |\, \boldsymbol{h} \,| \, \| C_\lambda \|_{L^2} \| \boldsymbol{u}_\lambda \|_{\mathbb{L}^2_\sigma} + \| \boldsymbol{f}_1 \|_{\mathbb{L}^2} \| \boldsymbol{u}_\lambda \|_{\mathbb{L}^2_\sigma} \\ &\quad \leqslant \mu \| \boldsymbol{u}_\lambda \|^2_{\mathbb{L}^2_\sigma} + C_\mu \, (\, \gamma + \| \boldsymbol{f}_1 \|^2_{\mathbb{L}^2} \,).\end{aligned}$$

Then, as above, we get

$$\| \boldsymbol{u}_\lambda \|_{C([0,S];\mathbb{L}^2_\sigma(\Omega))}, \ \| \, |\nabla \boldsymbol{u}_\lambda| \, \|_{L^2(Q)} \leqslant \gamma. \tag{4.27}$$

The Second Energy Estimates

(i) Multiplying the first equation of $(\mathrm{BF})_\lambda$ by $\partial_t \boldsymbol{u}_\lambda$, we have

$$\| \partial_t \, \boldsymbol{u}_\lambda \|^2_{\mathbb{L}^2_\sigma} + \frac{\nu}{2}\frac{d}{dt} \| \, |\nabla \boldsymbol{u}_\lambda| \, \|^2_{\mathbb{L}^2} + \frac{a}{2}\frac{d}{dt} \| \boldsymbol{u}_\lambda \|^2_{\mathbb{L}^2_\sigma} \tag{4.28}$$

$$\begin{aligned}&= \int_\Omega (\, T_\lambda \, \boldsymbol{g} + C_\lambda \, \boldsymbol{h} + \boldsymbol{f}_1 \,) \cdot \partial_t \boldsymbol{u}_\lambda \, dx \\ &\leqslant (\, |\, \boldsymbol{g} \,| \, \| T_\lambda \|_{L^2} + |\, \boldsymbol{h} \,| \, \| C_\lambda \|_{L^2} + \| \boldsymbol{f}_1 \|_{\mathbb{L}^2} \,) \, \| \partial_t \boldsymbol{u}_\lambda \|_{\mathbb{L}^2_\sigma} \\ &\leqslant (\, \gamma + \| \boldsymbol{f}_1 \|^2_{\mathbb{L}^2} \,) + \frac{1}{2} \| \partial_t \boldsymbol{u}_\lambda \|^2_{\mathbb{L}^2_\sigma}.\end{aligned} \tag{4.29}$$

On the other hand, in view of (4.25), (4.26) and (4.27), we find that

$$\| \varphi_\lambda(U_\lambda(t)) \|_{L^1(0,S)} \leqslant \gamma, \quad \varphi_\lambda(U) = \varphi(U) + \frac{\lambda}{2} (\|T\|_{L^2}^2 + \|C\|_{L^2}^2).$$

Hence, since $\varphi_\lambda(U_\lambda(t))$ is absolutely continuous on $[0, S]$, there exists $t_1 \in [0, S]$, where $\varphi_\lambda(U_\lambda(t))$ attains its minimum at $t = t_1$, i.e.,

$$\varphi_\lambda(U_\lambda(t_1)) = \min_{0 \leqslant t \leqslant S} \varphi_\lambda(U_\lambda(t)) \leqslant \frac{1}{S} \int_0^S \varphi_\lambda(U_\lambda(\tau)) d\tau \leqslant \frac{\gamma}{S}$$

whence follows

$$\| \nabla \boldsymbol{u}_\lambda(t_1) \|_{\mathbb{L}^2}^2, \ \| \nabla T_\lambda(t_1) \|_{L^2}^2, \ \| \nabla C_\lambda(t_1) \|_{L^2}^2, \ \lambda \| T_\lambda(t_1) \|_{L^2}^2, \ \lambda \| C_\lambda(t_1) \|_{L^2}^2 \leqslant \gamma. \tag{4.30}$$

Then integrating (4.29) over $[t_1, t]$ and $[t_1, t_1 + S]$, we derive

$$\| \nabla \boldsymbol{u}_\lambda \|_{C(0,S;\mathbb{L}^2(\Omega))}, \ \| \partial_t \boldsymbol{u}_\lambda \|_{L^2(0,S;\mathbb{L}^2_\sigma(\Omega))} \leqslant \gamma. \tag{4.31}$$

Furthermore, by using the first equation of $(\mathrm{BF})_\lambda$, we also obtain

$$\| \mathcal{A} \boldsymbol{u}_\lambda \|_{L^2(0,S;\mathbb{L}^2_\sigma(\Omega))} \leqslant \gamma. \tag{4.32}$$

(ii) Multiplying the second equation of $(\mathrm{BF})_\lambda$ by $-\Delta T_\lambda$, we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \| \nabla T_\lambda \|_{L^2}^2 + \| \Delta T_\lambda \|_{L^2}^2 \\
&\qquad = -\lambda \int_\Omega |\nabla T_\lambda|^2 dx - \int_\Omega \Delta T_\lambda \, \boldsymbol{u}_\lambda \cdot \nabla T_\lambda \, dx - \int_\Omega f_2 \, \Delta T_\lambda \, dx. \\
&\qquad \leqslant \int_\Omega |\Delta T_\lambda| \, |\boldsymbol{u}_\lambda| \, |\nabla T_\lambda| \, dx + \| f_2 \|_{L^2} \| \Delta T_\lambda \|_{L^2}.
\end{aligned}$$

Here, using again $\| T_\lambda \|_{L^3}^2 \leqslant \| T_\lambda \|_{L^2} \| T_\lambda \|_{L^6}$, we get

$$\begin{aligned}
&\int_\Omega |\boldsymbol{u}_\lambda| \, |\nabla T_\lambda| \, |\Delta T_\lambda| \, dx \\
&\quad \leqslant \| \, |\boldsymbol{u}_\lambda| \, \|_{L^6} \| \nabla T_\lambda \|_{L^3} \| \Delta T_\lambda \|_{L^2} \\
&\quad \leqslant \kappa \| \, |\nabla \boldsymbol{u}_\lambda| \, \|_{L^2} \| \nabla T_\lambda \|_{L^2}^{1/2} (\| \Delta T_\lambda \|_{L^2}^{1/2} + \| \nabla T_\lambda \|_{L^2}^{1/2}) \| \Delta T_\lambda \|_{L^2} \\
&\quad \leqslant \frac{1}{2} \| \Delta T_\lambda \|_{L^2}^2 + \frac{27}{4} \kappa^4 \| \, |\nabla \boldsymbol{u}_\lambda| \, \|_{L^2}^4 \| \nabla T_\lambda \|_{L^2}^2 + \kappa^2 \| \, |\nabla \boldsymbol{u}_\lambda| \, \|_{L^2}^2 \| \nabla T_\lambda \|_{L^2}^2
\end{aligned}$$

where κ is the constant which depends on Sobolev's embedding constant. Therefore, using previous estimates, we obtain

$$\frac{1}{2}\frac{d}{dt}\| \nabla T_\lambda \|_{L^2}^2 + \frac{1}{4}\| \Delta T_\lambda \|_{L^2}^2 \leqslant \gamma \| \nabla T_\lambda \|_{L^2}^2 + \| f_2 \|_{L^2}^2.$$

Hence, by Gronwall's inequality, we obtain

$$\| \nabla T_\lambda \|_{C(0,S;L^2(\Omega))},\ \| \Delta T_\lambda \|_{L^2(Q)} \leqslant \gamma. \tag{4.33}$$

Next multiplying the second equation of $(\mathrm{BF})_\lambda$ by $\partial_t T_\lambda$, we get

$$\| \partial_t T_\lambda \|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2}\| \nabla T_\lambda \|_{L^2}^2 + \frac{\lambda}{2}\| T_\lambda \|_{L^2}^2 \right) = \int_\Omega \partial_t T_\lambda \, \boldsymbol{u}_\lambda \cdot \nabla T_\lambda \, dx + \int_\Omega f_2 \, \partial_t T_\lambda \, dx.$$

The above argument with ΔT_λ replaced by $\partial_t T_\lambda$ gives

$$\frac{1}{4}\| \partial_t T_\lambda \|_{L^2}^2 + \frac{d}{dt}\left(\frac{1}{2}\| \nabla T_\lambda \|_{L^2}^2 + \frac{\lambda}{2}\| T_\lambda \|_{L^2}^2 \right) \leqslant \gamma \| \nabla T_\lambda \|_{L^2}^2 + \| f_2 \|_{L^2}^2,$$

whence follows

$$\| \partial_t T_\lambda \|_{L^2(Q)},\ \sup_{0\leqslant t\leqslant S} \lambda \| T_\lambda \|_{L^2(\Omega)}^2 \leqslant \gamma. \tag{4.34}$$

(iii) We multiply the third equation of $(\mathrm{BF})_\lambda$ by $-\Delta C_\lambda$ or $\partial_t C_\lambda$. Since we already obtain the a priori bounds for T_λ, by the same arguments as above, we obtain

$$\| \nabla C_\lambda \|_{C(0,S;L^2(\Omega))},\ \| \Delta C_\lambda \|_{L^2(Q)},\ \| \partial_t C_\lambda \|_{L^2(Q)},\ \sup_{0\leqslant t\leqslant S} \lambda \| C_\lambda \|_{L^2(\Omega)}^2 \leqslant \gamma. \tag{4.35}$$

4.5 Convergence

In this subsection, making use of a priori estimates given in the previous subsection, we shall discuss the convergence of solutions of the approximate equations.

We first recall that (4.31) and (4.33)-(4.35) assure

$$\sup_{0\leqslant t\leqslant S} \varphi_\lambda(U_\lambda(t)) \leqslant \gamma. \tag{4.36}$$

Therefore by virtue of Rellich's compactness theorem, the sequence of the solution $\{U_\lambda(t)\}_{\lambda>0}$ is pre-compact in H for all $t \in [0, S]$. Moreover, noting that $\partial_t U_\lambda$ is bounded in $L^2(0, S; H)$ by (4.31), (4.34) and (4.35), we find

$$\| U_\lambda(t) - U_\lambda(s) \|_H \leqslant \int_s^t \| \partial_\tau U_\lambda(\tau) \|_H \, d\tau \leqslant \| \partial_t U_\lambda \|_{L^2(0,T;H)}^{1/2} \, | t - s |^{1/2},$$

which implies that $\{U_\lambda(t)\}_{\lambda>0}$ forms an equi-continuous subset in $C_\pi([0,S];H)$. Hence, by virtue of Ascoli's theorem, there exists a sequence U_{λ_n}, denoted by U_n, with $\lambda_n \to 0$ as $n \to \infty$ such that

$$U_n \to U \quad \text{strongly in } C_\pi([0,S];H) \text{ as } n \to \infty. \tag{4.37}$$

Furthermore, in view of (4.32), (4.33), (4.35) and (4.36), we have

$$\frac{dU_n}{dt} \rightharpoonup \frac{dU}{dt} = (\, \partial_t \boldsymbol{u},\ \partial_t T,\ \partial_t C\,)^t \text{ weakly in } L^2(0,S;H) \text{ as } n \to \infty, \tag{4.38}$$

$$\nabla U_n \rightharpoonup \nabla U = (\,\nabla \boldsymbol{u},\ \nabla T,\ \nabla C\,)^t \text{ weakly star in } L^\infty(0,S;H) \text{ as } n \to \infty, \tag{4.39}$$

$$\partial\varphi(U_n) \rightharpoonup \partial\varphi(U) = (\mathcal{A}\boldsymbol{u}, -\Delta T, -\Delta C)^t \text{ weakly in } L^2(0,S;H) \text{ as } n \to \infty. \tag{4.40}$$

Since the absolute continuity of $\|\nabla U(t)\|_H$ follows from the fact that U_t and $\partial\varphi(U)$ belong to $L^2(0,S;H)$ (see [5]), using the uniformly convexity of H, we can easily deduce

$$\nabla U = (\,\nabla \boldsymbol{u},\ \nabla T,\ \nabla C\,)^t \in C_\pi([0,S];H).$$

Now to complete the proof, it remains to show that the limit function $(\,\boldsymbol{u},\ T,\ C\,)^t$ becomes a solution of (2.2). We first note that (4.37) implies

$$\lambda_n T_n \to 0 \quad \lambda_n C_n \to 0 \quad \text{ strongly in } C_\pi([0,S];L^2(\Omega)).$$

Hence, by (4.40), we find

$$\partial\varphi_{\lambda_n}(U_{\lambda_n}) \rightharpoonup \partial\varphi(U) \quad \text{ weakly in } L^2(0,S;H) \text{ as } n \to \infty. \tag{4.41}$$

As for the convection terms, since $\boldsymbol{u}_{\lambda_n}\cdot\nabla T_{\lambda_n}$ and $\boldsymbol{u}_{\lambda_n}\cdot\nabla C_{\lambda_n}$ are bounded in $L^2(Q)$ (see (3.6), (4.32) and (4.36)), $\boldsymbol{u}_{\lambda_n}\cdot\nabla T_{\lambda_n}$ and $\boldsymbol{u}_{\lambda_n}\cdot\nabla C_{\lambda_n}$ are weakly convergent to some functions in $L^2(Q)$.

Moreover, we note by (4.37) that $u^j_{\lambda_n} T_{\lambda_n}$ and $u^j_{\lambda_n} C_{\lambda_n}$ converge to $u^j\, T$ and $u^j\, C$ in $C(0,S;L^1(\Omega))$ as $n \to \infty$ for all $j = 1, \cdots, N$. Then, by the same argument as for (3.4), we get

$$\boldsymbol{u}_{\lambda_n}\cdot\nabla T_{\lambda_n} \rightharpoonup \boldsymbol{u}\cdot\nabla T,\ \ \boldsymbol{u}_{\lambda_n}\cdot\nabla C_{\lambda_n} \rightharpoonup \boldsymbol{u}\cdot\nabla C \quad \text{weakly in } L^2(0,S;L^2(\Omega)) \text{ as } n \to \infty.$$

Hence, in view of (4.40), we find

$$B_{\lambda_n}(U_{\lambda_n}) \rightharpoonup B(U) \quad \text{weakly in } L^2(0,S;L^2(\Omega)) \text{ as } n \to \infty. \tag{4.42}$$

Thus, it follows from (4.37), (4.38), (4.41) and (4.42) that $U = (\,\boldsymbol{u},\ T,\ C\,)^t$ is a periodic solution of (BF). $\square$

Remark 3 If we impose the assumption

$$f_2(t),\ f_3(t) \in L^2_N(\Omega) = \{\, f \in L^2(\Omega)\,;\ \int_\Omega f(x)\,dx = 0\,\} \quad \text{for a.e.} \quad t \in (0, S),$$

which is slightly stronger than (2.1), then the periodic solution given in Theorem 2.2 satisfies $T(t),\ C(t) \in L^2_N(\Omega)$ for a.e. $t \in (0, S)$, which is easily derived from (4.20).

Remark 4 Repeating exactly the same arguments as those in [11] with Poincaré's inequality replaced by Poincaré-Wirtinger's inequality, we can prove the following uniqueness result.

Theorem 4.1 *Let* $\boldsymbol{f}_1 \in \mathbb{L}^2(0, S; \mathbb{L}^2_\sigma(\Omega))$, *then there exists a* (*sufficiently small*) *constant* γ_0 *depending on* $|\,\boldsymbol{g}\,|$, $|\,\boldsymbol{h}\,|$, ν, ρ *and* $\|\,\boldsymbol{f}_1\,\|_{\mathbb{L}^2(0,S;\mathbb{L}^2_\sigma(\Omega))}$ *such that if* $\|\,f_2\,\|_{L^2(0,S;L^2(\Omega))} \leqslant \gamma_0$ *and* $\|\,f_3\,\|_{L^2(0,S;L^2(\Omega))} \leqslant \gamma_0$, *then the periodic solution of* (BF) *satisfying* $(\#)_{1/2}$ *in* Theorem 2.2 *is unique.*

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