

On the Lefschetz fixed point theorem for random multivalued mappings

Jan Andres and Lech Górniewicz

Abstract: The aim of this paper is to prove the Lefschetz fixed point theorem for random multivalued compact absorbing contractions on absolute neighbourhood multiretracts.

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Dedicated to the memory of Professor Francesco S. De Blasi

1 Introduction

The topological fixed point theory for random operators was systematically developed by ourselves in [3], where the appropriate topological degree was defined. The aim of this paper is to extend in an advanced way the Lefschetz fixed point theorem both to the deterministic as well as to the random case. For the classical Lefschetz fixed point theorem, see e.g. [4]. For its various multivalued generalizations, see e.g. [1, 5]. This will be done via transformation of a given random problem into the deterministic one. Such transformation is possible thanks to the existence of a measurable selection of a certain multivalued mapping determined by a given random operator (see Lemma 2.10 in [3]). Unfortunately, we can do it only for separable spaces. Our new Lefschetz type theorem, will be formulated for a large class of multivalued mappings, so called compact absorbing contractions (cf. [1, 5]), and for rather general spaces called absolute neighbourhood multiretracts (cf. [2, 7]).

2 Some auxiliary notions and definitions

In this paper all spaces are assumed to be metric and all single-valued mappings are assumed to be continuous. A compact nonempty space X is called *contractible* if

there exists a homotopy $h : X \times [0, 1] \rightarrow X$ such that: $h(x, 0) = x$ and $h(x, 1) = x_0$ for every $x \in X$; X is called an R_δ -set provided there exists a sequence of contractible spaces $\{X_n\}$ such that $X_{n+1} \subset X_n$ for every n and $X = \bigcap_n X_n$.

By $H = \{H_n\}$ we shall denote the Čech homology functor with compact carriers and coefficients in the field \mathbb{Q} of rational numbers (for details see [1, 5]).

A compact nonempty space X is called *acyclic* provided:

$$H_n(X) = \begin{cases} \mathbb{Q}, & n = 0, \\ 0, & n > 0 \end{cases}$$

Note that every R_δ space or in particular every contractible space is acyclic.

Definition 2.1. A mapping $p : Y \rightarrow X$ is called *Vietoris* provided:

- (2.1.1) p is proper, i.e., for every compact $K \subset X$ the set $p^{-1}(K)$ is compact,
- (2.1.2) $p^{-1}(x)$ is an acyclic space for every $x \in X$.

In what follows the symbol $p : Y \Longrightarrow X$ is reserved for Vietoris mappings.

Note that the composition of two Vietoris mappings is again a Vietoris map. We shall also consider multivalued mappings. We say that $\varphi : X \multimap Y$ is a *multivalued map* if for every $x \in X$ a nonempty closed subset $\varphi(x)$ of Y is given. For a multivalued map $\varphi : X \multimap Y$, we shall denote by

$$\Gamma_\varphi = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$$

the graph of φ . Then

$$X \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} Y,$$

where $p_\varphi(x, y) = x$, $q_\varphi(x, y) = y$ for every $(x, y) \in \Gamma_\varphi$ are called *natural projections* for φ .

A map $\varphi : X \multimap Y$ is called *upper semi-continuous* (u.s.c. for short) if for every open $U \subset Y$ the set $\varphi^{-1}(U) := \{x \in X \mid \varphi(x) \subset U\}$ is open in X ; φ is lower semi-continuous (l.s.c.) provided for every open $U \subset Y$ the set $\varphi_+^{-1}(U) := \{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$ is open in X .

A u.s.c. map $\varphi : X \multimap Y$ is called *acyclic* provided for every $x \in X$ the set $\varphi(x)$ is acyclic.

Observe that if $\varphi : X \multimap Y$ is acyclic, then $p_\varphi : \Gamma_\varphi \Longrightarrow X$ is a Vietoris map.

Definition 2.2. (cf. [5]) A multivalued map $\varphi : X \multimap Y$ is called *admissible* provided there exists a diagram

$$X \xleftarrow{p} \Gamma \xrightarrow{q} Y$$

in which p is a Vietoris map and q is a continuous map such that: $\varphi(x) = q(p^{-1}(x))$, for every $x \in X$.

For details concerning admissible maps see [5]. Below we recall only three important properties in the form of lemmas.

Lemma 2.3. *Any acyclic map is admissible.*

Lemma 2.4. *Any admissible map is u.s.c.*

Lemma 2.5. *Composition of two admissible maps is also admissible.*

Note that Lemma 2.5 is no longer true for two acyclic maps.

Definition 2.6. (cf. [1, 5]) An admissible map $\varphi: X \multimap X$ is called a *compact absorbing contraction* (CAC-map for short) provided there exists an open subset $U \subset X$ such that:

- (2.6.1) $\varphi(U) \subset U$,
- (2.6.2) $\tilde{\varphi}: U \multimap U$, where $\tilde{\varphi}(x) = \varphi(x)$ for every $x \in U$, is a compact map, i.e., $\tilde{\varphi}(U) \subset K \subset U$, where K is a compact set.
- (2.6.3) for every $x \in X$ there exists $n = n(x)$ such that $\varphi^n(x) \subset U$.

It is well known that the class of CAC-mappings is quite large. For example it contains (see [1, 5]): (2.7) compact admissible maps, (2.8) eventually compact admissible maps, (2.9) asymptotically compact admissible maps, (2.10) admissible mappings with compact attractor.

Finally, let us note that in (2.8)–(2.10) it is assumed that considered mappings are locally compact.

Let A be a subset of X and $\varphi: A \multimap X$ be a multivalued map. A point $x \in A$ is called a *fixed point* of φ provided $x \in \varphi(x)$.

We let:

$$\text{Fix } \varphi = \{x \in A \mid x \in \varphi(x)\}.$$

3 Multivalued random operators

By a *measurable space*, we shall mean as usually the pair (Ω, Σ) , where the set Ω is equipped with a σ -algebra Σ of subsets. We shall use $\mathbb{B}(X)$ to denote the Borel σ -algebra on X . The symbol $\Sigma \otimes \mathbb{B}(X)$ denotes the smallest σ -algebra on $\Omega \times X$ which contains all the sets $A \times B$, where $A \in \Sigma$ and $B \in \mathbb{B}(X)$.

Denoting, for $\varphi: X \multimap Y$, by

$$\varphi^{-1}(B) := \{x \in X \mid \varphi(x) \subset B\} \text{ and } \varphi_+^{-1}(B) := \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}$$

the small and large counter-images of $B \subset Y$, we can define (weakly) measurable multivalued maps as follows.

Definition 3.1. Let (Ω, Σ) be a measurable space and Y be a separable metric space. A map $\varphi: \Omega \multimap Y$ with closed values is called *measurable* if $\varphi^{-1}(B) \in \Sigma$, for each open $B \subset Y$, or equivalently, if $\varphi_+^{-1}(B) \in \Sigma$, for each closed $B \subset Y$. It is called *weakly measurable* if $\varphi_+^{-1}(B) \in \Sigma$, for each open $B \subset Y$, or equivalently, if $\varphi^{-1}(B) \in \Sigma$, for each closed $B \subset Y$.

It is well known (see e.g. [1, 6]) that, for compact-valued maps $\varphi: \Omega \multimap Y$, the notions of measurability and weak measurability coincide. Moreover, if φ and ψ are measurable, then so is their Cartesian product $\varphi \times \psi$. For more properties and details, see [1, 6].

As an important tool in our investigations, the classical Kuratowski–Ryll–Nardzewski selection theorem plays an important role, which we state here in the form of lemma (see e.g. [1, Theorem I.3.49]).

Lemma 3.2. *If $\varphi: \Omega \multimap Y$, where Ω is a complete measure space and Y is a complete separable metric space, is a measurable multivalued map, then φ possesses a measurable (single-valued) selection $f \subset \varphi$.*

The notions of a random operator and a random fixed point are essential in this paper. In the sequel, Ω will be always a complete measure space and X be always a complete separable metric space.

Definition 3.3. (see [3]) Let $A \subset X$ be a closed subset and $\varphi: \Omega \times A \multimap X$ be a multivalued map with closed values. We say that φ is a *random operator* if it is product-measurable (measurable in the whole), i.e. measurable w.r.t. minimal σ -algebra $\Sigma \otimes \mathcal{B}(X)$, generated by $\Sigma \times \mathcal{B}(X)$, where $\mathcal{B}(X)$ denotes the Borel sets of X . If $\varphi(\omega, \cdot): A \multimap X$ is still u.s.c. (or l.s.c.), then φ is called a *random u -operator* (or a *random l -operator*).

Remark 3.4. For the definition of a random operator, it is usually still required φ to be compact-valued (cf. [5]), and $\varphi(\omega, \cdot): A \multimap X$ to be u.s.c. (cf. again [5]) or h -continuous (cf. [6]), for almost all $\omega \in \Omega$. Since these restrictions are not necessary for us here, we omitted them in Definition 3.4.

Definition 3.5. Let $A \subset X$ be a closed subset and $\varphi: \Omega \times A \multimap X$ be a random operator. We say that φ has a *random fixed point* ξ if there exists a measurable mapping $\xi: \Omega \rightarrow A$ such that:

$$\xi(\omega) \in \varphi(\omega, \xi(\omega)), \quad \text{for every } \omega \in \Omega.$$

We let $\text{Fix}^{ra}(\varphi) = \{\xi: \Omega \rightarrow A \mid \xi \text{ is a random fixed point for } \varphi\}$.

The following lemma is crucial in our considerations.

Lemma 3.6. *Let X be a separable space, A a closed subset of X and $\varphi: \Omega \times X \multimap X$ a measurable map with nonempty closed values. We let $\varphi_\omega: A \multimap X$, $\varphi_\omega(x) := \varphi(\omega, x)$. Assume further that, for every $\omega \in \Omega$, the set $\text{Fix } \varphi_\omega := \{x \in X \mid x \in \varphi_\omega(x)\}$ of fixed points of φ_ω is nonempty and closed.*

Then the map $\Gamma: \Omega \multimap X$, given by $\Gamma(\omega) = \text{Fix } \varphi_\omega$, has a measurable selection.

For the proof see [3].

Note that if φ is a random l -operator, then it is sufficient to assume in Lemma 3.6 that only $\varphi(\cdot, x)$ is measurable, for every $x \in X$.

Remark 3.7. Applying a certain version of the Aumann selection theorem (see e.g. [6, Theorem 2.2.14]), instead of Lemma 3.2, the closedness of $\text{Fix } \varphi_\omega$ in Lemma 3.6 can be omitted (see [3, Corollary 2.12]).

Observe that if for almost all $\omega \in \Omega$ the set $\text{Fix } \varphi_\omega$ is nonempty, then $\text{Fix } \varphi = \{\xi: \Omega \rightarrow X \mid \xi(\omega) \in \varphi(\omega, \xi(\omega))\}$ is for almost all $\omega \in \Omega$ nonempty. Moreover, in view of Remark 3.7, $\text{Fix } \varphi_\omega$ need not be closed, for almost all $\omega \in \Omega$.

4 Multiretracts

In this section, following [7] (cf. also [2]), we recall the notion of absolute neighbourhood multiretracts (ANMR, for short).

Definition 4.1. (see [7]) A map $r: X \rightarrow Y$ of a space X onto a space Y is said to be a *multiretract* (*mr*-map, for short) if there is an admissible map $\varphi: Y \multimap X$ such that $r \circ \varphi = \text{id}_Y$.

An admissible map φ satisfying the condition $r \circ \varphi = \text{id}_Y$ is called an *admissible right inverse* of r . Now, we list some properties of *mr*-maps in the following lemmas.

Lemma 4.2. *The composition of two mr-maps is an mr-map.*

Lemma 4.3. *If $r: X \rightarrow Y$ is an mr-map, $Y_0 \subset Y$ and $X_0 = r^{-1}(Y_0)$, then the restriction $r_0: X_0 \rightarrow Y_0$ of r is an mr-map.*

Now, we shall still introduce the following definitions.

Definition 4.4. A space X is called an *absolute multiretract* (notation: $X \in \text{AMR}$) provided there exists a normed linear space E and an *mr*-map $r: E \rightarrow X$ from E onto X .

Definition 4.5. A space X is called an *absolute neighbourhood multiretract* (notation: $X \in \text{ANMR}$) provided there exists an open subset U of some normed linear space E and an *mr*-map $r: U \rightarrow X$ from U onto X .

Note that any absolute retract (AR-space) is an AMR-space and any absolute neighbourhood retract (ANR-space) is an ANMR-space. So we have the scheme:

$$\begin{array}{ccc} \text{AR} & \subset & \text{ANR} \\ & \cap & \cap \\ \text{AMR} & \subset & \text{ANMR} \end{array}$$

where all inclusions are proper.

In the next section we shall use the following property:

Lemma 4.6. (see [6]) *If $X \in \text{ANMR}$ and U is an open subset of X , then $U \in \text{ANMR}$.*

For more details concerning AMR-s and ANMR-s we refer to [6] (cf. also [2]).

5 Lefschetz fixed point theorem for random multivalued mappings

At first, we recall the celebrated Vietoris Mapping Theorem:

Theorem 5.1. (see [5]) *If $p: Y \rightrightarrows X$ is a Vietoris map, then $p_*: H_n(Y) \rightarrow H_n(X)$ is an isomorphism, i.e., for every $n \leq 0$ $p_{*n}: H_n(Y) \rightarrow H_n(X)$ is an isomorphism.*

Now, let $\varphi: X \multimap X$ be an admissible map. Then we have a diagram:

$$X \xleftarrow{p} \Gamma \xrightarrow{q} X$$

in which p is a Vietoris map and q is a continuous map such that:

$$\varphi(x) = q(p^{-1}(x)), \quad \text{for every } x \in X.$$

In what follows, having the above diagram, we say that (p, q) is a *selected pair* of φ and we use the following notation $(p, q) \subset \varphi$.

Applying Theorem 5.1 we define the *induced set* φ_* of an admissible map $\varphi: X \multimap X$ as follows:

$$\varphi_* = \{q_* \circ p_*^{-1} \mid (p, q) \subset \varphi\}.$$

Definition 5.2. An admissible map $\varphi: X \multimap X$ is called a *Lefschetz map* provided for every selected pair $(p, q) \subset \varphi$ the generalized Lefschetz number $\Lambda(p, q)$ of the pair (p, q) is well defined, where

$$\Lambda(p, q) = \Lambda(q_* \circ p_*^{-1}).$$

For more details see [5].

If $\varphi: X \multimap X$ is a Lefschetz map then we define the *Lefschetz set* $\Lambda(\varphi)$ of φ by letting (see [5]):

$$\Lambda(\varphi) = \{\Lambda(p, q) \mid (p, q) \subset \varphi\}.$$

Now, we can formulate the main results of this paper.

Theorem 5.3. *Let $\varphi: \Omega \times X \multimap X$ be a random u -operator. Assume furthermore that:*

- (5.3.1) X is a separable ANMR-space,
- (5.3.2) $\varphi_\omega: X \multimap X$, $\varphi_\omega(x) = \varphi(\omega, x)$ is a CAC-map for every $\omega \in \Omega$.

Then:

- “(i)” $\varphi_\omega: X \multimap X$ is a Lefschetz map and
- “(ii)” if $\Lambda(\varphi_\omega) \neq \{0\}$ for almost all $\omega \in \Omega$,

then $\text{Fix}^{ra}(\varphi) \neq \emptyset$.

Firstly, we shall prove the respective deterministic version of Theorem 5.3. Namely, we prove the following:

Theorem 5.4. *Let $X \in \text{ANMR}$ and let $\varphi: X \multimap X$ be a CAC-mapping. Then*

- “(5.4.1)” φ is a Lefschetz map and
- “(5.4.2)” if $\Lambda(\varphi) \neq \{0\}$, then $\text{Fix}(\varphi) \neq \emptyset$.

Proof. Let $(p, q) \subset \varphi$ be a selected pair of φ and let U be chosen according to Definition 2.6 (see (2.6.1)). So we have the diagram:

$$X \xleftarrow{p} \Gamma \xrightarrow{q} X.$$

We let:

$$\Gamma_0 = p^{-1}(U),$$

$p_0: \Gamma_0 \Rightarrow U$, $p_0(y) = p(y)$, for every $y \in \Gamma_0$ and $q_0: \Gamma_0 \rightarrow U$, $q_0(y) = q(y)$, for every $y \in \Gamma_0$. Observe that p_0 is a Vietoris map. Consequently we have the following diagram of mappings of pairs:

$$(X, U) \xleftarrow{\bar{p}} (\Gamma, \Gamma_0) \xrightarrow{\bar{q}} (X, U),$$

in which $\bar{p}(y) = p(y)$ and $\bar{q}(y) = q(y)$ for every $y \in \Gamma$. Since we consider the homology with compact carriers, in view of (2.6.3), we obtain that

$$\bar{q}_* \circ \bar{p}_*^{-1}: H_*(X, U) \rightarrow H_*(X, U)$$

is a weakly nilpotent linear map. Consequently the Lefschetz number $\Lambda(\bar{p}, \bar{q})$ of a pair (\bar{p}, \bar{q}) is well defined and

$$\Lambda(\bar{p}, \bar{q}) = 0. \quad (\text{i})$$

Now, it follows from Lemma 4.6 that $U \in \text{MNAR}$ and by means of (2.6.2) we obtain a compact admissible map $\varphi_0: U \multimap U$ for which (p_0, q_0) is a selected pair.

It is proved in [7] that φ_0 is a Lefschetz map so the Lefschetz number:

$$\Lambda(p_0, q_0) \text{ of } (p_0, q_0) \text{ is well defined.} \quad (\text{ii})$$

Since the Lefschetz numbers of (\bar{p}, \bar{q}) and (p_0, q_0) are well defined, by means of (11.5) in [5] we obtain that the Lefschetz number of (p, q) is also well defined and:

$$\Lambda(\bar{p}, \bar{q}) = \Lambda(p, q) - \Lambda(p_0, q_0). \quad (\text{iii})$$

Therefore from (i) and (iii) we deduce that:

$$\Lambda(p, q) = \Lambda(p_0, q_0). \quad (\text{iv})$$

Assuming that $\Lambda(p, q) \neq 0$, then $\Lambda(p_0, q_0) \neq 0$, and by the Lefschetz fixed point theorem proved in [7] we deduce that $\text{Fix}(\varphi_0) \neq \emptyset$. This already implies that $\text{Fix}(\varphi) \neq \emptyset$ which completes the proof of the deterministic part.

(Continued proof of Theorem 5.3) By means of Theorem 5.4 we infer that for every $\omega \in \Omega$ the map $\varphi_\omega: X \multimap X$ is a Lefschetz map and if $\Lambda(\varphi_\omega) \neq \{\emptyset\}$, then $\text{Fix}(\varphi_\omega) \neq \emptyset$.

Now, if we assume that, for almost every $\omega \in \Omega$, the Lefschetz set $\Lambda(\varphi_\omega)$ is different from $\{0\}$ and, in view of the assumption that X is a separable space, we can apply Lemma 3.6 (see also Remark 3.7). Thus, we get that

$$\text{Fix}^{ra}(\varphi) = \{\xi: \Omega \rightarrow X \mid \xi(\omega) \in \varphi(\omega, \xi(\omega)) \text{ for almost all } \omega \in \Omega\} \neq \emptyset.$$

This completes the proof of the random part. \square

As an immediate consequence of Theorem 5.4 we get:

Corollary 5.5. *If X is a separable MAR-space and $\varphi: \Omega \times X \multimap X$ is a random u -map such that for every $\omega \in \Omega$ the map $\varphi_\omega: X \multimap X$ is a CAC-map, then $\text{Fix}^{ra}(\varphi) \neq \emptyset$.*

Remark 5.6. Let us note that even the deterministic Theorem 5.3 is a new result. The same is in the random case true for Corollary 5.5 and all the better for Theorem 5.4.

Finally let us formulate two open problems:

Remark 5.7. (Open Problem) Is Theorem 5.3 true without the assumption that X is a separable space?

Remark 5.8. (Open Problem) Is it possible to formulate the problem of the existence of periodic points for random operators?

For some supporting arguments concerning Remark 5.8 and related references, see [3].

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J. Andres
Department of Mathematical Analysis,
Faculty of Science,
Palacký University,
17. listopadu 12,
771 46 Olomouc,
CZECH REPUBLIC
E-mail: jan.andres@upol.cz

L. Górniewicz
Institute of Mathematics,
University of Kazimierz Wielki,
Weysenhoffa 11,
85-072 Bydgoszcz,
POLAND
E-mail: gorn@mat.uni.torun.pl