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Lower semi-continuous differential inclusions with p-Laplacian

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Abstract: The existence of solutions to lower semi-continuous, closed-valued differential inclusions with p-Laplacian is investigated under various growth conditions. Proofs exploit the Bressan-Colombo-Fryszkowski Continuous Selection Theorem and fixed point arguments.

Keywords: p-Laplacian, differential inclusion, lower semi-continuous multifunction.

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Dedicated to the memory of Professor Francesco S. De Blasi

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$, let $1 < p < +\infty$, and let F be a multifunction from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$ with nonempty closed values. Consider the Dirichlet problem

$$\begin{cases} -\Delta_p u \in F(x, u, \nabla u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Δ_p denotes the p-Laplace operator, namely

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \forall\, u \in W_0^{1,p}(\Omega).$$

A function $u \in W_0^{1,p}(\Omega)$ is called a (weak) solution to (1.1) provided that there exists $v \in L^{p'}(\Omega)$, being p' the conjugate exponent of p, satisfying $v(x) \in F(x, u(x), \nabla u(x))$ for almost every $x \in \Omega$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w \, dx = \int_\Omega vw \, dx \quad \forall\, w \in W_0^{1,p}(\Omega).$$

This paper treats the existence of solutions to (1.1) by chiefly assuming that the multifunction $(x, z, w) \mapsto F(x, z, w)$ is measurable in $x \in \Omega$, lower semi-continuous with respect to $(z,w) \in \mathbb{R} \times \mathbb{R}^N$, and, moreover, fulfills a growth condition of the type

$$\inf_{y \in F(x,z,w)} |y| < a(x) + \psi(|z|,|w|), \quad (x,z,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

where, roughly speaking, $a \in L^{p'}(\Omega)$ while $\psi : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is nondecreasing in each variable separately. Various choices of ψ are made and worked out in detail; see Section 3. The technical approach we adopt basically combines Bressan-Colombo-Fryszkowski's Continuous Selection Theorem with Leray-Schauder's or Schauder's Fixed Point Theorem.

Problem (1.1) exhibits at least two interesting features:

(*i*) Contrary to the most part of elliptic differential inclusions previously investigated, here, the right-hand side is neither convex nor upper semi-continuous and depends on the gradient of the solution. This does not allow the use of variational methods for possibly non-smooth functionals, which instead have been the main tools in numerous papers; see, for instance, [13, 10] and the references therein.

(*ii*) Despite (*i*), the involved differential operator, namely the p-Laplacian, turns out to be fully variational.

As far as we know, there are not many existence results for lower semi-continuous elliptic differential inclusions. Actually, we can only mention [2, Section 3], [3, Theorem 2], as well as [12, Theorem 3.1]. All of them deal with elliptic operators in non-divergence form and require growth conditions more restrictive than those adopted here.

Let us finally point out that, reasoning exactly as in [12], the results below might be exploited to establish existence theorems for implicit elliptic equations of the type $f(x,u,\nabla u,\Delta_p u) = 0$.

2 Preliminaries

Let X be a topological space and let V be a subset of X. We denote by $\overline{V}$ the closure of V while $\mathcal{B}(X)$ indicates the Borel σ-algebra of X. If (X,d) is a metric space, $x \in X$, $r > 0$, and $V \neq \emptyset$ then

$$B(x,r) := \{z \in X : d(z,x) < r\}, \quad d(x,V) := \inf_{z \in V} d(x,z).$$

If X is a normed space then $\overline{\mathrm{co}}(V)$ denotes the closed convex hull of V. Let X and Z be two nonempty sets. A multifunction Φ from X into Z (briefly, $\Phi : X \to 2^Z$) is a function which associates to every $x \in X$ a nonempty subset $\Phi(x)$ of Z. A function

$\varphi : X \to Z$ is called a *selection* of Φ provided $\varphi(x) \in \Phi(x)$ for all $x \in X$. Given $W \subseteq Z$, we define

$$\Phi^-(W) := \{x \in X : \Phi(x) \cap W \neq \emptyset\}.$$

When $(X, \mathcal{F})$ is a measure space, Z is a topological space, and for every open set $W \subseteq Z$ one has $\Phi^-(W) \in \mathcal{F}$, we say that Φ is *measurable.* If X and Z are two topological spaces and $\Phi^-(W)$ is open in X for any open subset W of Z, then the multifunction Φ is called *lower semi-continuous.* When (Z, δ) is a metric space, Φ turns out to be lower semi-continuous iff for every $z \in Z$ the real-valued function $x \mapsto \delta(z, \Phi(x))$, $x \in X$, is upper semi-continuous. When, moreover, X is first countable, Φ turns out to be lower semi-continuous iff for each $x \in X$, $\{x_n\} \subseteq X$ fulfilling $x_n \to x$, and $z \in \Phi(x)$ there exists $\{z_n\} \subseteq Z$ such that $z_n \to z$ and $z_n \in \Phi(x_n)$, $n \in \mathbb{N}$. The monographs [1, 7] are general references on these subjects.

Let T be a nonempty, bounded, open set in $(\mathbb{R}^N, |\cdot|)$, the real Euclidean N-space, let $(Z, \|\cdot\|)$ be a real Banach space, and let $s \in [1, +\infty)$. We denote by $L^s(T, Z)$ the space of all (equivalence classes of) strongly Lebesgue measurable functions $w : T \to Z$ such that $t \mapsto \|w(t)\|^s$ is Lebesgue integrable. The norm in this space is given by

$$\|w\|_{L^s(T,Z)} := \left(\int_T \|w(t)\|^s dt \right)^{1/s}, \quad w \in L^s(T, Z).$$

When $Z = \mathbb{R}$ and there is no ambiguity, we simply write $L^s(T)$ in place of $L^s(T, \mathbb{R})$. A nonempty set $K \subseteq L^s(T, Z)$ is said to be *decomposable* provided for every $w_1, w_2 \in K$ and every measurable set $A \subseteq T$ one has

$$\chi_A w_1 + (1 - \chi_A) w_2 \in K,$$

being χ_A the characteristic function of A.

A basic relationship between decomposable-valued lower semi-continuous multifunctions and continuous selections is established by the next result, usually called Bressan-Colombo-Fryszkowski's Continuous Selection Theorem; see [3, Proposition 2] and [12, Theorem 2.1].

Theorem 2.1. *Let X be a separable metric space and let $\Phi : X \to 2^{L^s(T,Z)}$ be a lower semi-continuous multifunction with decomposable closed values. Then Φ admits a continuous selection.*

From now on, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a smooth boundary $\partial\Omega$, the symbol $\mathcal{L}(\Omega)$ (respectively, $m(\Omega)$) denotes the Lebesgue σ-algebra (respectively, measure) of Ω, while $W_0^{1,s}(\Omega)$ indicates the closure of $C_0^\infty(\Omega)$ in $W^{1,s}(\Omega)$. On $W_0^{1,s}(\Omega)$ we introduce the norm

$$\|u\|_s := \left(\int_\Omega |\nabla u(x)|^s dx \right)^{1/s}, \quad u \in W_0^{1,s}(\Omega).$$

Let s^* be the critical exponent for the Sobolev embedding $W_0^{1,s}(\Omega) \subseteq L^r(\Omega)$. Recall that

$$s^* = \begin{cases} Ns/(N-s) & \text{if } s < N, \\ +\infty & \text{otherwise.} \end{cases}$$

If $s \neq N$ then to each $r \in [1, s^*]$ there corresponds a constant $c_{rs} > 0$ satisfying

$$\|u\|_{L^r(\Omega)} \leq c_{rs} \|u\|_s \quad \forall\, u \in W_0^{1,s}(\Omega)$$

whereas, when $s = N$, for every $r \in [1, +\infty)$ one has

$$\|u\|_{L^r(\Omega)} \leq c_{rN} \|u\|_N, \quad u \in W_0^{1,N}(\Omega).$$

Finally, the embedding $W_0^{1,s}(\Omega) \hookrightarrow L^r(\Omega)$ is compact provided $1 \leq r < s^*$.

Lemma 2.2. *Let $G : \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ be a closed-valued multifunction. Assume that*

- (a_1) *G is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N)$-measurable,*
- (a_2) *for almost every $x \in \Omega$ the multifunction $G(x, \cdot, \cdot)$ turns out to be lower semi-continuous, and*
- (a_3) *there exist $q, s \geq 1$, $a \in L^q(\Omega, \mathbb{R}_0^+)$, $b, c \geq 0$ complying with*

$$\sup_{y \in G(x,z,w)} |y| \leq a(x) + b|z|^{s/q} + c|w|^{s/q} \quad \text{in} \quad \Omega \times \mathbb{R} \times \mathbb{R}^N. \tag{2.1}$$

Then the associated Nemytskii operator $\mathbb{G} : W_0^{1,s}(\Omega) \to 2^{L^q(\Omega)}$ defined by

$$\mathbb{G}(u) := \{v \in L^q(\Omega) : \, v(x) \in G(x, u(x), \nabla u(x)) \text{ a.e. in } \Omega\} \tag{2.2}$$

for all $u \in W_0^{1,s}(\Omega)$ takes decomposable closed values and is lower semi-continuous.

Proof. Pick any $u \in W_0^{1,s}(\Omega)$. In view of (a_1) and [7, Theorem III.23] the multifunction $x \mapsto G(x, u(x), \nabla u(x))$ is measurable, because for any open set $B \subseteq \mathbb{R}$ one has

$$\begin{aligned} &\{x \in \Omega : \, G(x, u(x), \nabla u(x)) \cap B \neq \emptyset\} \\ &= \operatorname{proj}_\Omega \left(G^-(B) \cap \{(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N : \, z = u(x), \, w = \nabla u(x)\} \right), \end{aligned}$$

being $\operatorname{proj}_\Omega$ the projection map from $\Omega \times \mathbb{R} \times \mathbb{R}^N$ onto Ω. Hence, the classical Kuratowski and Ryll-Nardzewski Selection Theorem [7, Theorem III.6] gives a measurable function $v : \Omega \to \mathbb{R}$ such that $v(x) \in G(x, u(x), \nabla u(x))$ for almost every $x \in \Omega$. Thanks to (a_1) we obtain

$$\|v\|_{L^q(\Omega)} \leq \|a\|_{L^q(\Omega)} + b\|u\|_{L^s(\Omega)}^{s/q} + c\|\nabla u\|_{L^s(\Omega, \mathbb{R}^N)}^{s/q} < +\infty, \tag{2.3}$$

namely $v \in \mathbb{G}(u)$ and in particular $\mathbb{G}(u) \neq \emptyset$. A simple argument then shows that $\mathbb{G}(u)$ turns out to be a decomposable closed subset of $L^q(\Omega)$.

Let us next verify that the multifunction $\mathbb{G}$ is lower semi-continuous. With this aim, fix $u_0 \in W_0^{1,s}(\Omega)$, $v_0 \in \mathbb{G}(u_0)$, as well as $\{u_n\} \subseteq W_0^{1,s}(\Omega)$ such that $u_n \to u_0$. Taking a subsequence if necessary, this entails

$$u_n(x) \to u_0(x), \quad |u_n(x)| \le l(x) \quad \text{a.e. in} \quad \Omega, \tag{2.4}$$

$$\nabla u_n(x) \to \nabla u_0(x), \quad |\nabla u_n(x)| \le m(x) \quad \text{a.e. in} \quad \Omega \tag{2.5}$$

for appropriate $l, m \in L^s(\Omega)$; cf. [4, Theorem 4.9]. Now, given $n \in \mathbb{N}$, the Kuratowski and Ryll-Nardzewski Selection Theorem and [7, Theorem III.41] yield a measurable function $v_n : \Omega \to \mathbb{R}$ fulfilling

$$v_n(x) \in G(x, u_n(x), \nabla u_n(x)), \tag{2.6}$$

$$|v_n(x) - v_0(x)| = d(v_0(x), G(x, u_n(x), \nabla u_n(x))).$$

As before, (2.3) forces $v_n \in \mathbb{G}(u_n)$. Moreover, $v_n \to v_0$ almost everywhere in Ω. Indeed, by (a$_2$), the function $(z, w) \mapsto d(v_0(x), G(x, z, w))$ turns out to be upper semi-continuous for almost all $x \in \Omega$. Thus, on account of (2.4)–(2.5),

$$\begin{aligned}\limsup_{n\to+\infty} |v_n(x) - v_0(x)| &= \limsup_{n\to+\infty} d(v_0(x), G(x, u_n(x), \nabla u_n(x))) \\ &\le d(v_0(x), G(x, u_0(x), \nabla u_0(x))) = 0\end{aligned}$$

because $v_0 \in \mathbb{G}(u_0)$, and the assertion follows. Combining (2.4)–(2.6) with (2.1) we next achieve

$$|v_n(x)| \le a(x) + bl(x)^{s/q} + cm(x)^{s/q} \quad \text{a.e. in} \quad \Omega.$$

Consequently, by the Lebesgue Dominated Convergence Theorem, $v_n \to v_0$ in $L^q(\Omega)$. A standard sub-subsequence reasoning finally leads to the lower semi-continuity of $\mathbb{G}$. □

Given $p \in]1, +\infty[$, the symbol p' will denote the conjugate exponent of p while $W^{-1,p'}(\Omega)$ indicates the dual space of $W_0^{1,p}(\Omega)$. Through [4, Theorem 6.4] we see that $L^{p'}(\Omega)$ compactly embeds in $W^{-1,p'}(\Omega)$. So, there exists $C > 0$ satisfying

$$\|v\|_{W^{-1,p'}(\Omega)} \le C\|v\|_{L^{p'}(\Omega)} \quad \forall\, v \in L^{p'}(\Omega). \tag{2.7}$$

Let $A_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ be the nonlinear operator stemming from the negative p-Laplacian, i.e.,

$$\langle A_p(u), v\rangle := \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)\, dx, \quad u, v \in W_0^{1,p}(\Omega), \tag{2.8}$$

and let $\lambda_{1,p}$ be its first eigenvalue in $W_0^{1,p}(\Omega)$. The following facts are known; see, e.g., [14, Appendix A].

(p$_1$) A_p is bijective and uniformly continuous on bounded sets.

(p$_2$) The inverse operator A_p^{-1} turns out to be continuous.

(p$_3$) $\|A_p(u)\|_{W^{-1,p'}(\Omega)} = \|u\|_p^{p-1}$ in $W_0^{1,p}(\Omega)$.

(p$_4$) $\|u\|_{L^p(\Omega)}^p \leq \frac{1}{\lambda_{1,p}} \|u\|_p^p$ for all $u \in W_0^{1,p}(\Omega)$.

The constant C that appears in (2.7) can easily be evaluated through (p$_4$). Indeed, for any $v \in L^{p'}(\Omega)$,

$$\begin{aligned} \|v\|_{W^{-1,p'}(\Omega)} &:= \sup_{\|u\|_p \leq 1} \left| \int_\Omega u(x)v(x)\,dx \right| \\ &\leq \sup_{\|u\|_p \leq 1} \|u\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)} \leq \lambda_{1,p}^{-1/p} \|v\|_{p'}, \end{aligned}$$

whence $C \leq \lambda_{1,p}^{-1/p}$.

3 Differential inclusions with p-Laplacian

From now on, $p \in]1, +\infty[$ while $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \to 2^{\mathbb{R}}$ denotes a closed-valued multifunction. The following hypotheses will be posited.

(h$_1$) F is $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R}^N)$-measurable.

(h$_2$) For almost every $x \in \Omega$ the multifunction $(z, w) \to F(x, z, w)$ turns out to be lower semi-continuous.

(h$_3$) There exist $a \in L^{p'}(\Omega, \mathbb{R}_0^+)$, $b, c \geq 0$ complying with

$$\inf_{y \in F(x,z,w)} |y| < a(x) + b|z|^{p-1} + c|w|^{p-1} \quad \text{in} \quad \Omega \times \mathbb{R} \times \mathbb{R}^N.$$

Our first result basically comes out from Lemma 2.2 and the Leray-Schauder method.

Theorem 3.1. *Let* (h$_1$)–(h$_3$) *be satisfied. Assume further that*

$$\frac{b}{\lambda_{1,p}} + \frac{c}{\lambda_{1,p}^{1/p}} < 1. \tag{3.1}$$

Then Problem (1.1) *possesses at least one solution.*

Proof. Define, provided $(x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$,

$$\varphi(x, z, w) := a(x) + b|z|^{p-1} + c|w|^{p-1} \tag{3.2}$$

as well as

$$G(x, z, w) := \overline{F(x, z, w) \cap B(0, \varphi(x, z, w))}. \tag{3.3}$$

By (h_3) the set $G(x, z, w)$ is nonempty and compact. Theorems 0.45 and 0.48 of [16] ensure that the multifunction $(z, w) \mapsto G(x, z, w)$ turns out to be lower semi-continuous for almost every $x \in \Omega$. Since one evidently has

$$\sup_{y \in G(x,z,w)} |y| \le a(x) + b|z|^{p-1} + c|w|^{p-1}, \tag{3.4}$$

Lemma 2.2 can be applied, and the multifunction $\mathbb{G} : W_0^{1,p}(\Omega) \to 2^{L^{p'}(\Omega)}$ defined in (2.2), with $s := p$ and $q := p'$, takes decomposable closed values and is lower semi-continuous. Thus, Theorem 2.1 yields a continuous selection $g : W_0^{1,p}(\Omega) \to L^{p'}(\Omega)$ of $\mathbb{G}$, which, by (3.4), is bounded on bounded sets. Through (p_1)–(p_3) we know that A_p^{-1} is a continuous bounded bijection from $W^{-1,p'}(\Omega)$ into $W_0^{1,p}(\Omega)$. Since the natural embedding $j_{p'} : L^{p'}(\Omega) \to W^{-1,p'}(\Omega)$ is compact, $A_p^{-1} \circ j_{p'}$ enjoys the same property. Consequently, the nonlinear operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ defined by

$$T(u) := A_p^{-1} \circ j_{p'} \circ g(u) \quad \forall\, u \in W_0^{1,p}(\Omega)$$

turns out to be compact as well and any fixed point $u \in W_0^{1,p}(\Omega)$ of T is a weak solution to the equation

$$A_p(u) = g(u) \quad \text{in} \quad W^{-1,p'}(\Omega). \tag{3.5}$$

On the other hand, if $u \in W_0^{1,p}(\Omega)$ complies with (3.5) then it solves Problem (1.1), because

$$g(u) \in \mathbb{G}(u) \subseteq L^{p'}(\Omega).$$

To get a fixed point of T we shall employ the Leray-Shauder alternative. Suppose $u = \sigma T(u)$ for some $\sigma \in [0, 1]$. The choice of T forces

$$\langle A_p(u), v\rangle = \sigma^{p-1} \int_\Omega g(u)(x) v(x)\, dx, \quad \forall v \in W_0^{1,p}(\Omega). \tag{3.6}$$

From $g(u)(x) \in G(x, u(x), \nabla u(x))$, $\sigma \in [0, 1]$ and (3.2), it evidently follows

$$|\sigma^{p-1} g(u)(x)| \le \varphi(x, u(x), \nabla u(x)).$$

Letting $v := u$ in (3.6) and exploiting (2.8), (3.2), and (p_4), we thus obtain

$$\begin{aligned}
\|u\|_p^p &\leq \int_\Omega \big(a(x)|u(x)| + b|u(x)|^p + c|\nabla u(x)|^{p-1}|u(x)|\big)\, dx \\
&\leq \|a\|_{L^{p'}(\Omega)}\|u\|_{L^p(\Omega)} + b\|u\|^p_{L^p(\Omega)} + c\|\nabla u\|^{p/p'}_{L^p(\Omega,\mathbb{R}^N)}\|u\|_{L^p(\Omega)} \\
&\leq \frac{1}{\lambda_{1,p}^{1/p}}\|a\|_{L^{p'}(\Omega)}\|u\|_p + \left(\frac{b}{\lambda_{1,p}} + \frac{c}{\lambda_{1,p}^{1/p}}\right)\|u\|_p^p.
\end{aligned}$$

Therefore, under (3.1), any fixed point u of σT is bounded by a constant which does not depend on u and σ. Now, the Leray-Shauder Fixed Point Theorem [5, Theorem 4.6] leads to the conclusion. □

The above technique furnishes a solution of (1.1) every time the equation $A_p u = g(u)$ allows an estimate of $\|u\|_{L^p(\Omega)}$ through terms coming only from data, e.g., when some kind of a-priori estimate holds. Due to (h_3) we always have

$$|g(u)(x)| \leq a(x) + b|u(x)|^{p-1} + c|\nabla u(x)|^{p-1} \quad \forall\, u \in W_0^{1,p}(\Omega).$$

Hence, (1.1) turns out to be solvable whenever, e.g., the multifunction $(x,z,w) \mapsto F(x,z,w)$ is bounded in z, as the next result shows.

Theorem 3.2. *Suppose* (h_1)–(h_2) *hold true and, moreover,*

(h_3') *there exists* $a \in L^q(\Omega, \mathbb{R}_0^+)$, $c \geq 0$ *satisfying*

$$\inf_{y \in F(x,z,w)} |y| < a(x) + c|w|^{p-1} \quad \text{in} \quad \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

where for some $\varepsilon > 0$ *it holds*

$$q \geq \max\left\{p', \frac{N}{p}(1+\varepsilon)\right\}$$

Then Problem (1.1) *admits at least one solution.*

Proof. Keep the same notation introduced in the proof of Theorem 3.1. Assumption (h_3') yields

$$|g(w)(x)| \leq a(x) + c|\nabla w(x)|^{p-1} \quad \forall\, w \in W_0^{1,p}(\Omega). \tag{3.7}$$

So, if $u = \sigma T(u)$ for some $u \in W_0^{1,p}(\Omega)$, $\sigma \in [0,1]$ then

$$\Delta_p u + c|\nabla u|^{p-1} + a = -\sigma^{p-1} g(u) + c|\nabla u|^{p-1} + a \geq 0.$$

Recall that $q \geq N(1+\varepsilon)/p$. By [15, Theorems 6.1.4–6.1.5] this entails

$$\sup_{x\in\Omega} u(x) \leq C,$$

with $C > 0$ depending only on c and $\|a\|_{L^q(\Omega)}$. Likewise, since

$$\Delta_p(-u) + c|\nabla(-u)|^{p-1} + a = \sigma^{p-1}g(u) + c|\nabla(-u)|^{p-1} + a \geq 0$$

we get

$$\inf_{x\in\Omega} u(x) \geq -C.$$

Therefore, $\|u\|_{L^\infty(\Omega)} \leq C$. Letting $v := u$ in (3.6) and exploiting (3.7) besides (2.8) provides

$$\begin{aligned}\|u\|_p^p &\leq \int_\Omega a(x)|u(x)|\,dx + c\int_\Omega |\nabla u(x)|^{p-1}|u(x)|\,dx \\ &\leq C\left(\|a\|_{L^1(\Omega)} + c\,m(\Omega)^{1/p}\|u\|_p^{p-1}\right),\end{aligned}$$

whence a uniform bound for $\|u\|_p$ follows at once. Now the proof goes on exactly as that of Theorem 3.1. □

Maximum principles for *single-valued* elliptic equations of the form

$$-\Delta_p u = f(x,u) \quad \text{in} \quad \Omega$$

are available in the literature, but most of them require suitable monotonicity assumptions on $z \mapsto f(x,z)$; see, e.g., [17]. It seems an interesting task to find conditions for the *multi-valued* analogue

$$-\Delta_p u \in F(x,u) \quad \text{in} \quad \Omega,$$

or on the relevant abstract equation (3.5), which allows to prove some kind of maximum principle, thus obtaining an a-priori estimate on the solutions.

On the other hand, the Schauder Fixed Point Theorem might be employed to solve (1.1) in each case where a maximum principle is not readily available. With this aim, let us recall before the following a priori estimate on $\|\nabla u\|_{L^\infty(\Omega,\mathbb{R}^N)}$; cf. [8, Theorem 1.3].

Proposition 3.3. *Suppose $q > N$. Then there exists a constant $\hat{C} > 0$, depending on p, q, and Ω, such that*

$$\|\nabla u\|_{L^\infty(\Omega,\mathbb{R}^N)} \leq \hat{C}\|\Delta_p u\|_{L^q(\Omega)}^{1/(p-1)}. \tag{3.8}$$

Theorem 3.4. *Let* $q > N$ *and let* (h_1)–(h_2) *be satisfied. If, moreover,*

- (h_3'') *for appropriate* $a \in L^q(\Omega, \mathbb{R}_0^+)$ *and* $\psi : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *nondecreasing with respect to each variable separately one has*

$$\inf_{y \in F(x,z,w)} |y| < a(x) + \psi(|z|, |w|) \quad in \quad \Omega \times \mathbb{R} \times \mathbb{R}^N, \tag{3.9}$$

- (h_4) *there exists* $R > 0$ *such that*

$$\|a\|_{L^q(\Omega)} + m(\Omega)^{1/q} \psi\big(\delta_\Omega \hat{C} R^{1/(p-1)}, \hat{C} R^{1/(p-1)}\big) \le R, \tag{3.10}$$

where $\delta_\Omega := \operatorname{diam}(\Omega)$ *while* $\hat{C}$ *is given by Proposition 3.3,*

then Problem (1.1) *possesses at least one solution.*

Proof. Since $q > N > p^{*\prime}$, the embedding $j_q : L^q(\Omega) \to W^{-1,p'}(\Omega)$ is compact. To shorten notation, write

$$B_R := \{v \in L^q(\Omega) : \|v\|_{L^q(\Omega)} \le R\}, \tag{3.11}$$

$$A_{pq} := A_p^{-1} \circ j_q, \quad X_R := \overline{\mathrm{co}}(A_{pq}(B_R)). \tag{3.12}$$

Obviously, X_R turns out to be a convex compact subset of $W_0^{1,p}(\Omega)$; see [9, Theorem V.2.6]. Inequalities (3.8) and (5) at [4, p. 269] yield, after a standard point-wise convergence argument,

$$\|\nabla u\|_{L^\infty(\Omega,\mathbb{R}^N)} \le \hat{C} R^{1/(p-1)}, \quad \|u\|_{L^\infty(\Omega)} \le \delta_\Omega \hat{C} R^{1/(p-1)} \quad \forall\, u \in X_R.$$

Now, if $\varphi(x, z, w)$ denotes the right-hand side of (3.9) while $G(x, z, w)$ is as in (3.3) then, by simply adapting the reasoning made to prove Lemma 2.2, we see that the multifunction $\mathbb{G} : X_R \to 2^{L^q(\Omega)}$ defined via (2.2) takes decomposable closed values and is lower semi-continuous. Thus, Theorem 2.1 gives a continuous selection $g : X_R \to L^q(\Omega)$ of $\mathbb{G}$, which turns out to be bounded, because

$$\begin{aligned}\|g(u)\|_{L^q(\Omega)} &\le \|a\|_{L^q(\Omega)} + m(\Omega)^{1/q} \psi\big(\|u\|_{L^\infty(\Omega)}, \|\nabla u\|_{L^\infty(\Omega,\mathbb{R}^N)}\big) \\ &\le \|a\|_{L^q(\Omega)} + m(\Omega)^{1/q} \psi\big(\delta_\Omega \hat{C} R^{1/(p-1)}, \hat{C} R^{1/(p-1)}\big)\end{aligned}$$

for any $u \in X_R$. Hence, the nonlinear operator $T := A_{pq} \circ g$ is compact and, due to (h_4), complies with $T(X_R) \subseteq X_R$. By the Schauder Fixed Point Theorem [5, Theorem 4.4] there exists $u \in X_R \subseteq W_0^{1,p}(\Omega)$ such that $u = T(u)$, whence $A_p(u) = g(u)$ in $W^{-1,p'}(\Omega)$. This immediately leads to the conclusion. □

Let us finally treat the case $q \le N$. A simple verification shows that

$$p^{*\prime} < N, \qquad p^{*\prime} < q \;\Leftrightarrow\; p < q^*(p-1).$$

Therefore, if $p^{*\prime} < q \le N$ then the embedding $j_q : L^q(\Omega) \to W^{-1,p'}(\Omega)$ turns out to be compact. So, with the notation (3.11)–(3.12), the operator A_{pq} is compact too. Moreover, for every $v \in L^q(\Omega)$ we can find a unique $u \in W_0^{1,p}(\Omega)$ such that $A_p(u) = v$. Thanks to [6, Theorem 1.8] (cf. also [11, Theorem 2]) one has

$$\begin{cases} \|u\|_\sigma \le \bar{C}\|v\|_{L^q(\Omega)}^{1/(p-1)} & \text{provided } 1 \le \sigma \le q^*(p-1) \text{ and } q < N, \\ \\ \|u\|_\sigma \le \bar{C}\|v\|_{L^N(\Omega)}^{1/(p-1)} & \text{whatever } \sigma > 1, \text{ if } q = N, \end{cases} \tag{3.13}$$

where the constant $\bar{C} > 0$ depends only on p, q, Ω, besides σ. Consequently, A_{pq} actually ranges over $W_0^{1,r}(\Omega)$ for any $r \in]p, q^*(p-1)]$.

Lemma 3.5. *Suppose* $q \in \left]p^{*\prime}, N\right]$ *and* $r \in]p, q^*(p-1)[$. *Then the operator* $A_{pq} : L^q(\Omega) \to W_0^{1,r}(\Omega)$ *is compact.*

Proof. Let $q < N$. Pick a bounded sequence $\{v_n\} \subseteq L^q(\Omega)$. By eventually taking a subsequence, we may assume that

$$v_n \rightharpoonup v \quad \text{in} \quad L^q(\Omega), \quad A_{pq}(v_n) \to A_{pq}(v) \quad \text{in} \quad W_0^{1,p}(\Omega). \tag{3.14}$$

Write $u_n := A_{pq}(v_n)$, $u := A_{pq}(v)$. From (3.13) it evidently follows both $u_n, u \in W_0^{1,q^*(p-1)}(\Omega)$ and

$$\|u_n\|_{q^*(p-1)} \le C \quad \forall\, n \in \mathbb{N}, \tag{3.15}$$

with appropriate $C > 0$. To show that $u_n \to u$ in $W_0^{1,r}(\Omega)$, we shall interpolate between $W_0^{1,p}(\Omega)$ and $W_0^{1,q^*(p-1)}(\Omega)$ via Hölder's inequality on the gradient. Setting

$$\theta := \frac{1 - \frac{p}{r}}{1 - \frac{p'}{q^*}}$$

one has $0 < \theta < 1$. Thus, on account of (3.15),

$$\begin{aligned} \|u_n - u\|_r &\le \|u_n - u\|_{q^*(p-1)}^{\theta} \|u_n - u\|_p^{1-\theta} \\ &\le (C + \|u\|_{q^*(p-1)})^{\theta} \|u_n - u\|_p^{1-\theta}, \quad n \in \mathbb{N}. \end{aligned}$$

By (3.14) this entails $u_n \to u$ in $W_0^{1,r}(\Omega)$.

Finally, if $q = N$, we fix any $\sigma > r$ and proceed as before, but with σ instead of $q^*(p-1)$. □

Theorem 3.6. *Let* $q \in \left]p^{*\prime}, N\right]$, $r \in]p, q^*(p-1)[$, *and let* F *satisfy* (h$_1$)–(h$_2$). *If, moreover,*

(h$_3'''$) *there exist* $a \in L^q(\Omega, \mathbb{R}_0^+)$, $b, c \geq 0$ *such that*

$$\inf_{y \in F(x,z,w)} |y| < a(x) + b|z|^{r/q} + c|w|^{r/q} \quad in \quad \Omega \times \mathbb{R} \times \mathbb{R}^N.$$

(h$_4'$) *for some* $R > 0$ *one has*

$$\|a\|_{L^q(\Omega)} + \left(\frac{b}{\lambda_{1,r}^{1/q}} + c \right) \bar{C}^{r/q} R^{r/q(p-1)} \leq R, \tag{3.16}$$

with $\bar{C}$ *given in* (3.13),

then Problem (1.1) *admits at least one solution.*

Proof. We only sketch the proof, because it is similar to that of Theorem 3.4. By Lemma 3.5 the operator $A_{pq} : L^q(\Omega) \to W_0^{1,r}(\Omega)$ turns out to be compact. Hence, $X_R := \overline{\mathrm{co}}(A_{pq}(B_R))$, with B_R as in (3.11), is a compact convex subset of $W_0^{1,r}(\Omega)$. On the other hand, the multifunction $\mathbb{G} : X_R \to 2^{L^q(\Omega)}$ defined by (2.2), where G comes from (3.3) with $\varphi(x, z, w) := a(x) + b|z|^{r/q} + c|w|^{r/q}$, takes decomposable closed values and is lower semi-continuous. Consequently, it has a continuous selection $g : X_R \to L^q(\Omega)$. Put $T := A_{pq} \circ g$. Like in the above-mentioned proof, the conclusion is achieved once we know that $T(X_R) \subseteq X_R$. So, pick $u \in X_R$. Exploiting (h$_3'''$), (p$_4$), (3.13), and (3.16) yields

$$\begin{aligned} \|g(u)\|_{L^q(\Omega)} &\leq \|a\|_{L^q(\Omega)} + b\|u\|_{L^r(\Omega)}^{r/q} + c\|\nabla u\|_{L^r(\Omega, \mathbb{R}^N)}^{r/q} \\ &\leq \|a\|_{L^q(\Omega)} + \left(\frac{b}{\lambda_{1,r}^{1/q}} + c \right) \|u\|_r^{r/q} \\ &\leq \|a\|_{L^q(\Omega)} + \left(\frac{b}{\lambda_{1,r}^{1/q}} + c \right) \bar{C}^{r/q} R^{r/q(p-1)} \leq R. \end{aligned}$$

As u was arbitrary, this easily furnishes $T(X_R) \subseteq X_R$. □

It may be useful to make some comments on the growth conditions adopted above. For the sake of simplicity, consider the case when

$$F(x, z, w) := a(x) + F_0(z), \quad (x, z, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

where $a(x)$ denotes the forcing term while $F_0(z)$ represents the multi-valued nonlinearity.

Semi-linear differential inclusions or equations with p-Laplacian usually exhibit two types of nonlinear terms:

(i) the $(p-1)$-*sub-linear* ones, namely

$$\inf_{y\in F_0(z)} |y| \simeq b|z|^\gamma \quad \text{with} \quad \gamma \le p-1;$$

(ii) the $(p-1)$-*super-linear* ones, i.e.,

$$\inf_{y\in F_0(z)} |y| \simeq b|z|^\gamma \quad \text{with} \quad \gamma > p-1.$$

Case (i) might fruitfully be investigated through Theorem 3.1. Assumption (3.1) looks almost optimal. Indeed, regarding the classical linear elliptic problem

$$\begin{cases} -\Delta u = \lambda u + a & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

it guarantees the existence of solutions for any $a \in L^2(\Omega)$ provided $\lambda < \lambda_{1,2}$, which represents a well-known conclusion.

On the other hand, Theorems 3.4–3.6 reveal useful for treating Case (ii). As in the single-valued framework, a smallness condition on the forcing term $a(x)$ is taken to overcome the presence of a strong nonlinearity $F_0(z)$. Both (3.10) and (3.16) usually work for small $\|a\|_{L^q(\Omega)}$ and small R. Moreover, (3.10) is more general but requires a smallness hypothesis for the forcing term with respect to a stronger L^q-norm, because $q \ge N$. Condition (3.16) allows better controls (i.e., in smaller L^q-norms, being $q > N$). However, weaker nonlinear terms can be treated. Indeed, since $\gamma := r/q$ and $\gamma > p-1$, the inequality $r < q^*(p-1)$ forces $\gamma < q^*/q(p-1)$. Hence, the constant γ is arbitrary large if $q = N$, whereas for $q \to p^{*\prime}$ from the right it turns out to be no greater than $(p-1) + p/N$.

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