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Multiple interaction strategies in networks related to graph spectra and dominant sets

Marianna Bolla, Ahmed Elbanna

Abstract: An interaction network is a collection of agents with pairwise connections described by an graph. Our objective is to maximize the payoff of the agents simultaneously. In the classical strategic complements or substitutes setup, the objective function has a linear and a quadratic part, and maximized under linear constraints.

To address this task, we use quadratic objective functions on linear or quadratic constraints. We will show how existing results of combinatorial graph theory and spectral clustering can be used to solve the optimization problems, where solutions are closely related to dominant sets or spectral clusters. Our primary focus is on the graph and show how certain model parameters can be built into the edge-weight matrix to get a new objective, thus modifying the interactions between the agents.

Keywords: strategic complements and substitutes, edge-weighted graphs, dominant sets, eigenvalues, spectral clusters.

MSC2010: 05C69, 91B16

1 Introduction

We consider edge-weighted graphs and extend existing results on strategic interactions [1, 9] to them. In the classical papers there are unweighted interactions between the agents, and their actions, we are looking for, are nonnegative real numbers. However, without exact meaning, the scaling and the actual values of these actions do not carry too much information for the physical or economic features of them. In fact, they are rather compared with respect to the agents, and in this way, give important information about agent groups that follow similar strategies, and hence about the overall structure of the network from the point of view of the underlying activity towards which the strategies are considered.

Here we rather investigate the problem from the point of the view of the graph. Based on the spectral properties of a graph based matrix, we are able to tell how many and what kind of strategies can be optimal for the agents, and find agent groups following similar strategies. Since the agents form a social network, the optimal or nearly optimal strategies should inevitably be adapted to the structure of the underlying graph. Together with clustering, we also use evaluation of the vertices and edges, which give optima of potential functions, sometimes related to eigenvectors or weighted characteristic vectors of dominant sets.

The structure of the paper is as follows. In Section 2 we introduce the basic notions, and the classical models of strategic complements and substitutes. In Section 3 we consider quadratic objective function over linear constraints. If we optimize over the standard simplex, we can use the results of Motzkin and Straus [13] to unweighted and those of Pavan and Pelillo [16] to edge-weighted graphs. In this way, unweighted and weighted indicator vectors of maximal cliques and dominant sets enter into the solution. In Section 4 quadratic constraints are considered, under which our quadratic optimization has an explicit solution based on eigenvalues and eigenvectors of graph based matrices. Here we use multiple strategies and spectral clustering tools of [4]. We will show that the existence of large positive eigenvalues makes rise to a complementary, whereas that of outstanding negative eigenvalues to a substitute strategy. Some coordinates of the multi-dimensional strategies of some agents can be negative here, but with appropriate rotations the strategy vectors can be substituted by vectors close to weighted indicator vectors of agent groups. Simulation results on generalized random graphs are also presented. We close the paper with a short discussion in Section 5.

2 Preliminaries

2.1 Notation

Let $G = (V, \mathbf{W})$ be *edge-weighted graph* with vertex-set $V = \{1, \dots, n\}$ and $n \times n$ symmetric edge-weight matrix $\mathbf{W}$ of nonnegative entries and zero diagonal. The vertices correspond to the agents, while the weights represent their pairwise similarity or connectedness. The diagonal is zero, as there are not self-loops at the moment.

Let $d_i = \sum_{j=1}^n w_{ij}$ be the *generalized degree* of vertex i; the degrees are sometimes collected in the *degree-vector* $\mathbf{d} = (d_1, \dots, d_n)^T$ or in the diagonal *degree-matrix* $\boldsymbol{D} = \operatorname{diag}(\mathbf{d})$. In the edge-weighted case we assume that $\sum_{i=1}^n d_i = 1$, since the normalized edge-weight matrix, $\boldsymbol{D}^{-1/2}\mathbf{W}\boldsymbol{D}^{-1/2}$, is not affected by this normalization. In the unweighted case, $\mathbf{W}$ has 1-0 entries depending on whether two agents are connected or not, so it is the usual adjacency matrix of a simple graph, and is denoted by $\mathbf{A}$.

2.2 Game of strategic complements

Based on [1, 9], the strategic complements setup is the following. We generalize their model to an edge-weighted graph $G = (V, \mathbf{W})$; the agents correspond to the vertices and they act with continuous strategies: $x_i \geq 0$ $(i = 1, \dots, n)$, $\mathbf{x} := (x_1, \dots, x_n)^T$. The payoff of player i is

$$u_i(\mathbf{x}) = \alpha x_i - \frac{1}{2} x_i^2 + \phi \sum_{j=1}^{n} w_{ij} x_i x_j,$$

where α and ϕ are given positive constants. The first term is the benefit of agent i using strategy x_i, the second is the cost of agent i, and the last term is the utility (under strategic complementarity in efforts), i.e., the payoff due to his/her collaboration with the neighbors (the neighbors of i are vertices of the set $\{j : w_{ij} > 0\}$, and they are connected to i with strengths proportional to the edge-weights). The players are equivalent, only their network positions differ.

Agents want to maximize their payoffs at the same time, but they can rule only their own strategies. Therefore, we have to maximize $u_i(\mathbf{x})$ with respect to x_i for $i = 1, \dots, n$. Via

$$\frac{\partial u_i(\mathbf{x})}{\partial x_i} = \alpha - x_i + \phi \sum_{j=1}^{n} w_{ij} x_j = 0, \quad i = 1, \dots, n,$$

for the optimal $\mathbf{x}^*$ we have $\mathbf{x}^* = \alpha \mathbf{1} + \phi \mathbf{W} \mathbf{x}^*$, or equivalently, $(\mathbf{I} - \phi \mathbf{W}) \mathbf{x}^* = \alpha \mathbf{1}$, where $\mathbf{1}$ is the all 1's vector, and the vectors are column vectors. Consequently,

$$\mathbf{x}^* = \alpha (\mathbf{I} - \phi \mathbf{W})^{-1} \mathbf{1} \tag{2.1}$$

is a unique and inner solution (equilibrium) if $\mathbf{I} - \phi \mathbf{W}$ is positive definite, see also the forthcoming potential function view of (2.2). Denoting by $\lambda_1 \geq \dots \geq \lambda_n$ the eigenvalues of $\mathbf{W}$, this condition holds if and only if $1 - \|\mathbf{W}\| > 0$, or equivalently, $\phi < \frac{1}{\lambda_1}$.

Here we used that $\mathbf{W}$ is a Frobenius type matrix, therefore λ_1 is the maximum absolute value eigenvalue of $\mathbf{W}$ with eigenvector of nonnegative coordinates. Since $\mathrm{tr}\,(\mathbf{W}) = 0$, $\lambda_n = \lambda_{min}(\mathbf{W}) < 0$ and $|\lambda_n| < \lambda_1$. In this case, the following expansion works:

$$(\mathbf{I} - \phi \mathbf{W})^{-1} = \sum_{k=0}^{\infty} \phi^k \mathbf{W}^k = \mathbf{I} + \phi \mathbf{W} + \phi^2 \mathbf{W}^2 + \dots .$$

Consequently, when $\alpha > 0$, then

$$\mathbf{x}^* = \alpha (\sum_{k=0}^{\infty} \phi^k \mathbf{W}^k) \mathbf{1}.$$

Note that the ith coordinate of $\mathbf{W1}$ is d_i, whereas the ith coordinate of $\mathbf{W}^k\mathbf{1}$, denoted by $d_i(k,\mathbf{W})$, is the sum of the positive edge-weights of walks of length k emanating from vertex i; in particular, $d_i(1,\mathbf{W}) = d_i$. Hence, all the coordinates of $\mathbf{x}^*$ are positive:

$$x_i^* = \alpha \left(1 + \sum_{k=1}^{\infty} \phi^k d_i(k,\mathbf{W}) \right), \quad i = 1,\dots,n.$$

When $\mathbf{W}$ is the usual 0-1 adjacency matrix $\mathbf{A}$ of an unweighted graph, then $d_i(k,\mathbf{A})$ is the number of walks of length k emanating from i, and $1+\sum_{k=1}^{\infty} \phi^k d_i(k,\mathbf{A})$ is called the *Katz–Bonacich centrality* of vertex i. Therefore, $x_i^* \ge \alpha$, and equality holds if and only if $\phi = 0$. Observe that now d_i is the usual degree of vertex i, and as a consequence of the Frobenius theory, $d_{min} \le \lambda_1 \le d_{max}$, therefore $\lambda_1 \ge 1$ and $\phi < 1$. The closer ϕ to 0, the more rapidly ϕ^k decreases, and the shorter walks dominate this centrality.

If $\alpha = 0$ (no individual benefit, the payoff is only due to collaboration with others), then $x_i^* = \phi \sum_{j=1}^n w_{ij} x_j^*$, and the payoff is maximal when ϕ is the largest eigenvalue λ_1 of $\mathbf{W}$ and $\mathbf{x}^*$ is the corresponding eigenvector (with nonnegative coordinates, due to the Frobenius theory).

An equivalent way of reasoning is via *potential function* (the sum of the utilities corrected by a term which takes into account the network extremalities exerted by each player) as follows:

$$P(\mathbf{x}) = \sum_{i=1}^n u_i(\mathbf{x}) - \frac{\phi}{2} \sum_{i=1}^n \sum_{j=1}^n w_{ij} x_i x_j = \alpha \mathbf{x}^T \mathbf{1} - \frac{1}{2} \mathbf{x}^T (\mathbf{I} - \phi \mathbf{W}) \mathbf{x}. \tag{2.2}$$

It is easy to verify that $\frac{\partial u_i(\mathbf{x})}{\partial x_i} = \frac{\partial P(\mathbf{x})}{\partial x_i}$ for $i = 1,\dots,n$. The above $P(\mathbf{x})$ has a unique interior maximum if P is strictly concave, i.e., $-(\mathbf{I} - \phi\mathbf{W})$ is negative definite. Equivalently, $\mathbf{I} - \phi\mathbf{W}$ is positive definite, for which fact a necessary and sufficient condition is that $\phi\lambda_1 < 1$. After differentiating P with respect to $\mathbf{x}$, we get back (2.1).

2.3 Game of strategic substitutes

This type of an interaction, defined in [1], is computationally less tractable, but indicates real competition between the agents, where agents want to use their neighbors' benefit instead of their own actions; in particular, free-riders. We adapt the setup of [9] to edge-weighted graphs, with the strategies $x_i \ge 0$ ($i = 1,\dots,n$) and given positive parameters α, δ. In view of this model, the payoff (utility) of player i is

$$u_i(\mathbf{x}) = \alpha x_i - \frac{1}{2} x_i^2 - \delta \sum_{j=1}^n w_{ij} x_i x_j,$$

where the first term is the benefit of agent i using strategy x_i, the second is the cost of agent i, and the last term is his/her utility (under strategic substitute in efforts), i.e., the payoff due to the competition with the neighbors. Here efforts are decreased by the actions of the neighbors; for example, one do not want to borrow a book if their friends have it, or farmers do not want to plant the same crop as their neighbors do.

Agents again want to maximize their payoffs $u_i(\mathbf{x})$ with respect to x_i at the same time ($i = 1, \dots, n$). This is equivalent to maximizing the potential function:

$$P(\mathbf{x}) = \sum_{i=1}^{n} u_i(\mathbf{x}) + \frac{\delta}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} x_i x_j = \alpha \mathbf{x}^T \mathbf{1} - \frac{1}{2} \mathbf{x}^T (\mathbf{I} + \delta \mathbf{W}) \mathbf{x}.$$

Via differentiation, we get

$$\frac{\partial P(\mathbf{x})}{\partial x_i} = \alpha - x_i - \delta \sum_{j=1}^{n} a_{ij} x_j = 0, \quad i = 1, \dots, n.$$

This yields the system of equations

$$\mathbf{x}^* = \alpha \mathbf{1} - \delta \mathbf{W} \mathbf{x}^* \quad \text{if} \quad \mathbf{x}^* \geq \mathbf{0}. \tag{2.3}$$

P has a unique interior maximum if it is strictly concave, i.e., $-(\mathbf{I}+\delta\mathbf{W})$ (the Hessian of P) is negative definite. Equivalently, $\mathbf{I} + \delta \mathbf{W}$ is positive definite, for which fact a necessary and sufficient condition is that $\delta < \frac{-1}{\lambda_n} = \frac{1}{|\lambda_n|}$. However, we have to ensure that the coordinates of the optimizing $\mathbf{x}^*$ are nonnegative. Hence we get the quadratic programming task:

$$\begin{aligned} &\text{maximize} \quad P(\mathbf{x}) = \alpha \mathbf{x}^T \mathbf{1} - \frac{1}{2} \mathbf{x}^T (\mathbf{I} + \delta \mathbf{W}) \mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \geq 0, \end{aligned}$$

where $\mathbf{x} \geq \mathbf{0}$ means that $x_i \geq 0$, $i = 1, \dots, n$. In accord with [2] and Lemma 1 of [9]: $\mathbf{x}$ is a Nash equilibrium of the substitute game if and only if $\mathbf{x}$ satisfies the following Kuhn–Tucker conditions:

$$\frac{\partial P}{\partial x_i} = 0 \quad \text{and} \quad x_i > 0 \qquad \text{or} \qquad \frac{\partial P}{\partial x_i} \leq 0 \quad \text{and} \quad x_i = 0.$$

By Proposition 1 of [9], in the Nash equilibrium, there will be active agents with $x_i > 0$ ($i \in U$), and inactive ones with $x_i = 0$ ($i \in \overline{U}$); such an $\mathbf{x}$ is called *corner* solution with *support* U. Then the above conditions are equivalent to

$$(\mathbf{I}_U + \delta \mathbf{W}_U) \mathbf{x}_U = \alpha \mathbf{1} \quad \text{and} \quad \delta \mathbf{W}_{\overline{U},U} \mathbf{x}_U \geq \alpha \mathbf{1},$$

where the set in the lower index indicates the corresponding segment of the vector or matrix. The authors of [9] recommend maximizing over all subsets U, but it is computationally intractable. In Section 3 we will show how corner solutions are obtained, at least approximately, by using iterative algorithms.

In [9], it is also shown how partial transformations between substitutes and complements can be applied when δ is 'small'. Based on this, they distinguish between different types of solutions according to the range of δ. Actually, local substitutes can be changed into global substitutes and local complements in the following way; we adapt their reasoning to an edge-weighted graph in the case when $0 \le w_{ij} \le 1$ $(i \ne j)$. Let $\overline{G} = (V, \overline{\mathbf{W}})$ denote the *complementary graph* of $G = (V, \mathbf{W})$ with edge-weights $\bar{w}_{ij} = 1 - w_{ij}$ for $i \ne j$ and $\bar{w}_{ii} = 0$ for $i = 1, \dots, n$. If $\mathbf{C}$ is the adjacency matrix of the complete graph K_n, i.e., $\mathbf{C} = \mathbf{1}\mathbf{1}^T - \mathbf{I}$, then $\overline{\mathbf{W}} = \mathbf{C} - \mathbf{W}$. Therefore,

$$
\begin{aligned}
u_i(\mathbf{x}) &= \alpha x_i - \frac{1}{2}x_i^2 - \delta \sum_{j=1}^n [1 - (1 - w_{ij})] x_i x_j \\
&= \alpha x_i - \frac{1}{2}x_i^2 - \delta \sum_{j=1}^n x_i x_j + \delta \sum_{j=1}^n (1 - w_{ij}) x_i x_j \\
&= \alpha x_i - \frac{1}{2}x_i^2 - \delta \sum_{j=1}^n x_i x_j + \delta \sum_{j \ne i} (1 - w_{ij}) x_i x_j + \delta x_i^2 \qquad (2.4) \\
&= \alpha x_i - \frac{1}{2}x_i^2 - \delta \sum_{j=1}^n x_i x_j + \delta \sum_{j=1}^n \bar{w}_{ij} x_i x_j + \delta x_i^2 \\
&= \alpha x_i - \frac{1}{2}(1 - 2\delta) x_i^2 - \delta \sum_{j=1}^n x_i x_j + \delta \sum_{j=1}^n \bar{w}_{ij} x_i x_j,
\end{aligned}
$$

which is a game of global substitutes and local complements investigated by [1]. Here the complementarities are realized via $\overline{G}$.

In view of Theorem 1 of [1], there is a unique equilibrium if $1 - \delta > \delta \lambda_{max}(\overline{\mathbf{W}})$. Hence,

$$
\delta < \frac{1}{1 + \lambda_{max}(\overline{\mathbf{W}})}. \qquad (2.5)
$$

For finding the unique equilibrium $\mathbf{x}^*$, the constant of the Katz-Bonacich centrality is $\lambda^* = \frac{\delta}{1-\delta}$. Now, let us solve (2.3), i.e., $\alpha\mathbf{1} - \mathbf{x} - \delta\mathbf{W}\mathbf{x} = \mathbf{0}$. Making use of the previous transformations,

$$
\alpha\mathbf{1} - \mathbf{x} - \delta\mathbf{C}\mathbf{x} + \delta(\mathbf{C} - \mathbf{W})\mathbf{x} = \alpha\mathbf{1} - \delta\mathbf{I}\mathbf{x} - (1 - \delta)\mathbf{x} - \delta\mathbf{C}\mathbf{x} + \delta\overline{\mathbf{W}}\mathbf{x} = \mathbf{0}
$$

and

$$
\alpha\mathbf{1} - \delta(\mathbf{I} + \mathbf{C})\mathbf{x} - (1 - \delta)(\mathbf{I} - \frac{\delta}{1 - \delta}\overline{\mathbf{W}})\mathbf{x} = \mathbf{0}.
$$

We will use that

$$(\mathbf{I}+\mathbf{C})\mathbf{x} = (\mathbf{1}\mathbf{1}^T)\mathbf{x} = (\mathbf{1}^T\mathbf{x})\mathbf{1} = x\mathbf{1},$$

where $x = \mathbf{1}^T\mathbf{x} = \sum_{i=1}^n x_i$. Therefore,

$$(\alpha - \delta x)\mathbf{1} = (1-\delta)(\mathbf{I} - \frac{\delta}{1-\delta}\overline{\mathbf{W}})\mathbf{x},$$

consequently,

$$\mathbf{x} = \frac{\alpha - \delta x}{1-\delta}(\mathbf{I} - \frac{\delta}{1-\delta}\overline{\mathbf{W}})^{-1}\mathbf{1} = \frac{\alpha-\delta x}{1-\delta}\mathbf{y}.$$

The inverse exists under (2.5), and can be expanded like the Katz-Bonacich centrality. However, the right hand side also depends on $\mathbf{x}$ through x. To get rid of this dependence, we introduce $\mathbf{y}$ and $y = \sum_{i=1}^n y_i$. Summing up the coordinates, $x = \frac{\alpha-\delta x}{1-\delta} y$, consequently, $x = \frac{\alpha y}{1-\delta+\delta y}$. This implies that

$$\mathbf{x} = \frac{\alpha - \frac{\delta \alpha y}{1-\delta+\delta y}}{1-\delta}\mathbf{y} = \frac{\alpha}{1-\delta+\delta y}\mathbf{y},$$

where

$$\mathbf{y} = (\mathbf{I} - \frac{\delta}{1-\delta}\overline{\mathbf{W}})^{-1}\mathbf{1} = \left[\sum_{k=0}^{\infty}\left(\frac{\delta}{1-\delta}\right)^k \overline{\mathbf{W}}^k\right]\mathbf{1}.$$

Therefore,

$$x_i^* = \frac{\alpha}{1-\delta+\delta y}\left[1 + \sum_{k=1}^{\infty}\left(\frac{\delta}{1-\delta}\right)^k d_i(k, \overline{\mathbf{W}})\right],$$

where $d_i(k, \overline{\mathbf{W}})$ is the sum of the positive edge-weight of walks of length k emanating from vertex i of $\overline{G}$. Since $\frac{\delta}{1-\delta} < 1$ (it decreases with δ), it suffices to consider the first terms. Consequently, x_i^* is 'large' if i has 'strong' connections in the complement graph, or equivalently, 'weak' connections in the original graph. Hence, it seems reasonable, that a set close to the maximal independent one carries the leading strategies.

Summarizing, the following cases of [9] apply in the edge-weighted setup too:

- If $\delta < \frac{1}{1+\lambda_{max}(\overline{\mathbf{W}})}$, then a unique inner equilibrium exists $x_i > 0$ $(i = 1, \dots, n)$.
- If $\frac{1}{1+\lambda_{max}(\overline{\mathbf{W}})} \le \delta < -\frac{1}{\lambda_{min}(\mathbf{W})}$, then a unique equilibrium exits which is a corner or inner point.
- If $-\frac{1}{\lambda_{min}(\mathbf{W})} \le \delta < 1$, then there are multiple equilibria among those there are corners. In this case, only corner equilibria can be stable.

If $\delta = 1$ (see [7, 8], the stable equilibrium is corner: $\mathbf{x}_U = \mathbf{1}$, where U is a maximal independent set of G. Note that the maximal independent sets of G are the maximal cliques of $\overline{G}$, and to find them we recommend algorithms in Section 3.

We remark that the lower range of δ can be made wider and the middle range $\frac{1}{1+\lambda_{max}(\overline{\mathbf{A}})} \le \delta < -\frac{1}{\lambda_{min}(\mathbf{A})}$ narrower by using results of [10, 17], when we have an unweighted graph $G = (V, \mathbf{A})$ at the beginning. In view of these, we are able to find an edge-weighted graph $(V, \mathbf{W})$ with the same skeleton as G, i.e., $w_{ij} = 0$ whenever $a_{ij} = 0$, for which $\lambda_{min}(\mathbf{W})$ is the largest possible. Likewise, for the complementary graph $\overline{G} = (V, \overline{\mathbf{A}})$ we are also able to find an edge-weighted graph $(V, \overline{\mathbf{W}})$, with the same skeleton as $\overline{G}$, for which $\lambda_{max}(\overline{\mathbf{W}})$ is the smallest possible. To find the optimal edge-weights, the authors of [10, 11, 17] suggest theory and algorithms. Roughly speaking, we have to decompose the underlying graph into odd cycles and balanced bipartite graphs, and assign symmetric evaluations to their vertices, which in turn give the optimal evaluations of the edges.

3 Optimizing over the unit simplex

3.1 Maximal cliques and interactions

First, let us consider the simplest case of an edge-weighted graph $G = (V, \mathbf{W})$ when the agents have only mutual benefits and there are complementarities between them. Then the utility of agent i is

$$u_i(\mathbf{x}) = b \sum_{j=1}^{n} w_{ij} x_i x_j$$

with positive normalizing constant b, and we maximize it with respect to x_i, the strategy of agent i, for $i = 1, \dots, n$ over the simplex

$$S = \{x_i \ge 0\, (i = 1, \dots, n), \quad \sum_{i=1}^{n} x_i = 1\} = \{\mathbf{x} \ge \mathbf{0}, \quad \mathbf{x}^T \mathbf{1} = 1\}.$$

This is equivalent to the following quadratic programming task:

$$\begin{aligned} &\text{maximize} \quad P(\mathbf{x}) = \frac{1}{2} b \mathbf{x}^T \mathbf{W} \mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \in S. \end{aligned} \tag{3.1}$$

Apart from the constant $b > 0$, the quadratic form $\mathbf{x}^T \mathbf{W} \mathbf{x}$ maximizes the cohesiveness of a cluster of vertices with fuzzy membership vector $\mathbf{x} \ge 0$ under the simplex constraint. With a 'small' value of the coordinate x_i, vertex i is weakly, while with a 'large' value, it is strongly associated with the cluster. Under cluster we understand internal homogeneity and external inhomogeneity of the vertices included in it.

Motzkin and Strauss were the first to consider this quadratic programming task for simple graphs as the continuous relaxation of the *maximal clique* problem. A clique $C \subset V$ (complete subgraph) of the simple graph $G = (V, \mathbf{A})$ is maximal if no strict superset of C is a clique. A maximal clique C is *strictly maximal* if no vertex i external to C has the property that that the enlarged set $C \cup \{i\}$ contains a clique of the size $|C|$. Maximal cliques can be several (even overlapping), and to find all of them is NP-hard. A *maximum clique* is a maximal clique with largest cardinality. The characteristic vector of a vertex-subset $U \subset V$ is denoted $\mathbf{x}^U$ and is defined with the following coordinates: $x_i^U = \frac{1}{|U|}$ if $i \in U$ and 0 otherwise.

Theorem 3.1 (Motzkin–Strauss theorem as formulated in [6])**.** *Let $G = (V, \mathbf{A})$ be a simple graph and $C \subset V$. Then $(\mathbf{x}^C)^T \mathbf{A} (\mathbf{x}^C) = 1 - \frac{1}{|C|}$ if and only if C is a clique. Moreover,*

- *$\mathbf{x}^C$ is a strict local maximizer of $\mathbf{x}^T \mathbf{A} \mathbf{x}$ over S if and only if C is a strictly maximal clique.*
- *$\mathbf{x}^C$ is a global maximizer of $\mathbf{x}^T \mathbf{A} \mathbf{x}$ over S if and only if C is a maximum clique.*

In case of an unweighted graph G, Motzkin and Straus [13] further generalized the maximization problem to what they called *non-square-free quadratic forms.* Their theorem solves the problem of maximizing the utility function

$$u_i(\mathbf{x}) = d_i x_i^2 + \sum_{j \sim i} x_j^2 + b x_i \sum_{j \sim i} x_j$$

with respect to x_i $(i = 1, \dots, n)$ over S. Here the first term is the benefit of the agent i proportional to his/her number of ties (d_i is the degree of vertex i), the second term is the sum of the benefits of the neighbors, while the last term is the mutual benefit due to collaboration multiplied with the constant $b > 0$. This model may not be applicable in economy, but in cultural collaborations and co-authorships, where personal costs are not counted and the agents are glad with the success of their neighbors, it indeed has rational. Since

$$u_i(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (x_i^2 + x_j^2 + b x_i x_j),$$

in the potential function context this is equivalent to

$$\begin{aligned} &\text{maximize} \quad P(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T (\boldsymbol{D} + \frac{b}{2} \mathbf{A}) \mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \in S, \end{aligned} \tag{3.2}$$

where $\boldsymbol{D}$ is the diagonal degree-matrix. The solution is given in [13], and depending on the relation of d_{max} and $\frac{b}{2}$, local maxima are again related to characteristic vectors supported by maximal cliques or vertices having maximal degree.

Theorem 3.2 (Theorem 4 of [13]). *Let $G = (V, \mathbf{A})$ be unweighted graph, and let d_{max} denote its maximal vertex degree. Then a strict local maximum of (3.2) is the following:*

- *If $d_{max} > \frac{b}{2}$ then $\max_S P(\mathbf{x}) = d_{max}$, and the maximum is attained at an $\mathbf{x}$ which is the characteristic vector of a vertex of degree d_{max}.*
- *If $d_{max} = \frac{b}{2}$, then $\max_S P(\mathbf{x}) = d_{max}$ and the maximum is attained at the weighted characteristic vector of a complete subgraph, all of whose vertices have degree d_{max}.*
- *If $d_{max} < \frac{b}{2}$, then $\max_S P(\mathbf{x}) = \frac{b}{2} - \frac{c}{2}$ with $\frac{1}{c} = \max_{G'} \sum_{G'} (b - 2d_i)^{-1}$, where G' ranges over the cliques of G; the maximum is attained at an $\mathbf{x}$ with coordinates $x_i = \frac{c}{b-2d_i}$ for $i \in G'$ and $x_j = 0$ for $j \notin G'$.*

3.2 Dominant sets and weighted characteristic vectors

Now let $G = (V, \mathbf{W})$ be an edge-weighted graph. We will use the notion of a dominant set as introduced by Pavan and Pelillo [16] as follows. Let $U \subset V$ and $j \notin U$. Then

$$\varphi_U(i, j) = w_{ij} - \frac{1}{|U|} \sum_{l \in S} w_{il}, \quad i \in U$$

is the relative similarity between vertices i and j with respect to the average similarity between vertex i and its neighbors in U, where the second term is the average weighted degree of i with respect to vertices of U. Note that $\varphi_U(i, j)$ is positive if the connection between vertices i and j is stronger than the connection between vertex i and its neighbors in U, and it is negative, otherwise. Using their relative similarity, the weight of vertex i with respect to U is defined by the following recursive formula:

$$\mathbf{w}_U(i) = \begin{cases} 1, & \text{if } |U| = 1 \\ \displaystyle\sum_{l \in U \setminus \{i\}} \varphi_{U \setminus \{i\}}(l, i) \mathbf{w}_{U \setminus \{i\}}(l), & \text{otherwise.} \end{cases}$$

The total weight of U is $W(U) = \sum_{i \in U} \mathbf{w}_U(i)$. The function $\mathbf{w}_U(i)$ measures the relative similarity between vertex i and the vertices of $U \setminus \{i\}$ with respect to the overall similarity among the vertices in $U \setminus \{i\}$.

Definition 3.3. If $W(T) > 0$ for any nonempty $T \subseteq U$, $U \subseteq V$, then U is a dominant set if

- $\mathbf{w}_U(i) > 0$, for all $i \in U$,
- $\mathbf{w}_{U\cup\{i\}}(i) < 0$, for all $i \notin U$.

These two conditions correspond to the main properties of a cluster: internal homogeneity and external inhomogeneity. The first condition ensures that vertices in U are strongly connected to each other, i.e., U induces a strongly connected subgraph, while the second condition ensures that the set U induces the most strongly connected subset in G. This definition shows that in a dominant set, the overall similarity among its vertices is higher than the similarity between its vertices and the rest of the vertices in V. Note that in an unweighted graph (with 0-1 weights) dominant sets correspond to the strictly maximal cliques. The quadratic programming task

$$\begin{aligned} &\text{maximize} \quad P(\mathbf{x}) = \mathbf{x}^T\mathbf{W}\mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \in S, \end{aligned} \tag{3.3}$$

is the generalization of the problem (3.1) and it favors pairs of vertices with similar coordinates in $\mathbf{x}$ that also have strong connection in $\mathbf{W}$. Pavan and Pelillo [16] characterized the strict local maxima of the above task by means of weighted characteristic vectors.

Definition 3.4. The weighted characteristic vector of a set U, also denoted by $\mathbf{x}^U$, has the following coordinates:

$$x_i^U = \begin{cases} \frac{\mathbf{w}_U(i)}{W(U)}, & \text{if } i \in U \\ 0, & \text{otherwise.} \end{cases}$$

Note that the weighted characteristic vector satisfies the simplex constrains, and it also corresponds to a corner solution in Section 2.3.

Theorem 3.5 (Theorem 1 of [16])**.** *Let $G = (V, \mathbf{W})$ be an edge-weighted graph.*

- *If U is a dominant set of G, then its weighted characteristic vector $\mathbf{x}^U$ is a strict local solution of the program (3.3).*
- *Conversely, if $\mathbf{x}^*$ is a strict local solution of the program (3.3), then its support $\sigma = \{i : x_i^* \neq 0\}$ is a dominant set, provided that $\mathbf{w}_{\sigma\cup\{i\}}(i) \neq 0$ for all $i \notin \sigma$.*

In [15, 16], the authors recommend the so-called *replicator dynamics* to solve the problem (3.3). Namely, they used the following iteration:

$$x_i(t+1) = x_i(t)\frac{(\mathbf{W}\mathbf{x}(t))_i}{\mathbf{x}(t)^T\mathbf{W}\mathbf{x}(t)} \tag{3.4}$$

for $i = 1, \dots, n$ and $t = 0, 1, 2, \dots$, until convergence. The simplex S is invariant under the above dynamics, which means that every trajectory starting in S will remain in S for the eternity. Further, if $\mathbf{W}$ is symmetric, the objective function is strictly increasing along any nonconstant trajectory of (3.4), and its asymptotically stable points are in one-to-one correspondence to the strict local solutions of (3.3).

To avoid spurious solutions (that are not characteristic vectors), in the unweighted case (0-1 weights) Bomze et al. [6] suggested the following regularization of (3.1) with introducing a positive parameter α:

$$\begin{aligned} &\text{maximize} \quad \mathbf{x}^T(\mathbf{A}+\alpha\mathbf{I})\mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \in S. \end{aligned} \tag{3.5}$$

They proved the following.

Theorem 3.6 (Theorem 10 of [6]). *Let $G = (V, \mathbf{A})$ be an unweighted graph and $0 < \alpha < 1$. Then*

- *the only strict local maximizers of $\mathbf{x}^T(\mathbf{A}+\alpha\mathbf{I})\mathbf{x}$ over S (i.e., the only attracting stationary points under the replicator dynamics with $\mathbf{A}+\alpha\mathbf{I}$ instead of $\mathbf{A}$) are characteristic vectors $\mathbf{x}^C$ where C is a maximal clique of G;*
- *conversely, if C is a maximal clique of G, then $\mathbf{x}^C$ represents a strict local maximizer.*

Therefore, when selecting an $\alpha \in (0, 1)$, e.g., $\alpha = \frac{1}{2}$, all local maximizers of (3.5) are strict and are in one-to-one correspondence with the characteristic vectors of the maximal cliques of the unweighted graph $G = (V, \mathbf{A})$.

It is an open question, what kind of regularization is useful when we have an edge-weighted graph $G = (V, \mathbf{W})$. Since the argmax $\mathbf{x}^T\mathbf{W}\mathbf{x}$ is invariant under scaling the entries of $\mathbf{W}$, we may assume that $0 \le w_{ij} \le 1$ $(i \ne j)$. We conjecture that the regularization with $\alpha \in (0, 1)$ will have the same effect. Alternatively, without normalizing $\mathbf{W}$, we could run the dynamics for $\mathbf{W} + \alpha\mathbf{I}$, where $0 < \alpha < \max_{i \ne j} w_{ij}$.

3.3 Interactions and dominant sets

First, let us consider the simplest case when the agents have individual costs and mutual benefit based on complementaries between them. The connections between the agents is described by the edge-weighted graph $G = (V, \mathbf{W})$. The utility of agent i is

$$u_i(\mathbf{x}) = \beta x_i \sum_{j \sim i} x_j - \alpha x_i^2 = \beta \sum_{j=1}^{n} w_{ij} x_i x_j - \alpha x_i^2 \tag{3.6}$$

with positive constants α and β, balancing between the benefit of agent i due to collaborations and its individual quadratic cost; further, we maximize it with respect to x_i for $i = 1, \ldots, n$ over the simplex S.

In potential function view, (3.6) is equivalent to the following quadratic programming task:

$$\begin{aligned} &\text{maximize} \quad P(\mathbf{x}) = \frac{1}{2}\beta \mathbf{x}^T \mathbf{W}\mathbf{x} - \frac{1}{2}\alpha \mathbf{x}^T \mathbf{I}\mathbf{x} = \frac{1}{2}\mathbf{x}^T(\beta \mathbf{W} - \alpha \mathbf{I})\mathbf{x} \\ &\text{subject to} \quad \mathbf{x} \in S. \end{aligned} \tag{3.7}$$

Using the ideas of [15], the solutions of (3.7) remain the same if the matrix $\beta\mathbf{W} - \alpha\mathbf{I}$ is replaced with $\beta\mathbf{W} - \alpha\mathbf{I} + \kappa\mathbf{1}\mathbf{1}^T$, where κ is an arbitrary real number. Indeed, $\kappa\mathbf{x}^T\mathbf{1}\mathbf{1}^T\mathbf{x} = \kappa(\mathbf{x}^T\mathbf{1})^2 = \kappa$, since $\mathbf{x}^T\mathbf{1} = 1$ due to $\mathbf{x} \in S$. In particular, if $\kappa = \alpha$, the resulting matrix has nonnegative entries and zero diagonal. Therefore, Theorem 3.5 is applicable to it, and implies that the strict local maxima of (3.7) are weighted characteristic vectors of dominant sets for the scaled edge-weight matrix $\beta\mathbf{W} + \alpha(\mathbf{1}\mathbf{1}^T - \mathbf{I})$ having zero diagonal and off-diagonal entries equal to $\beta w_{ij} + \alpha \geq 0$ $(i \neq j)$. Let us denote by G' this new edge-weighted graph: $G' = (V, \beta\mathbf{W} + \alpha(\mathbf{1}\mathbf{1}^T - \mathbf{I}))$.

In [15, 16], the authors adapted the replicator dynamics (3.4) to maximize (3.7) over S. Namely, they recommended the following iteration:

$$x_i(t+1) = x_i(t)\frac{(\beta\mathbf{W}\mathbf{x}(t))_i - \alpha x_i(t)}{\mathbf{x}(t)^T(\beta\mathbf{W} - \alpha\mathbf{I})\mathbf{x}(t)} \tag{3.8}$$

for $i = 1, \ldots, n$ and $t = 0, 1, 2, \ldots$, until convergence.

However, α could basically change the scale that would result in excluding dominant sets under a certain size. When α is large, namely $\alpha > \beta\lambda_{max}(\mathbf{W})$, then the regularization term dominates, and the only solution is an $\mathbf{x}$ having all positive coordinates, and hence, being the weighted characteristic vector of the whole V. If α gets smaller, but $\alpha > \beta\lambda_{max}(\mathbf{W}_U)$, where $\mathbf{W}_U$ is the edge-weight matrix of the induced subgraph of G on the vertex-set $U \subset V$, then there is no maximizing $\mathbf{x}$ with support which is the subset or equal to U. Therefore, if one wants to avoid too small clusters, we select an α according to this rule. Starting with $\alpha = \beta(n-1) \geq \beta\lambda_{max}(\mathbf{W})$, we can decrease α one by one to obtain smaller and smaller clusters, which support the weighted characteristic vector of the solution. However, if $\alpha > \beta(m-1)$, then we exclude characteristic vectors of dominant sets with $|U| \leq m$. Nonetheless, if α

is very small, the effect of regularization becomes negligible and dominant sets of $G = (V, \beta \mathbf{W})$, or equivalently, those of $G = (V, \mathbf{W})$ will enter into the solution.

Summarizing, the constants α and β are built into the edge-weight matrix of G, hence reshaping its structure, and suppressing the ties w_{ij}'s if α is large relative to β. The smaller α, the smaller dominant sets of agents will pursue a non-zero strategy (with the coordinates of the support of their weighted characteristic vectors). This means that if the individual costs are large compared to the mutual benefit, then larger sets of agents can collaborate fruitfully. On the contrary, when the individual costs are small compared to the mutual benefits, then the effect of the original edge-weights dominates, and smaller dominant sets – close to the ones of the original graph – of agents maximize their payoffs at the same time. However, in this case, a larger number of agents is rendered to have zero strategy.

We illustrate this process on a so-called generalized random graph.

Definition 3.7. Let n be a natural number and $k \le n$ be a positive integer. The graph $G_n(\mathbf{P}, \mathcal{P}_k)$ is a generalized random graph with probability matrix $\mathbf{P}$ and proper k-partition $\mathcal{P}_k = (V_1, \dots, V_k)$ of the vertices if it satisfies the following. The vertex set is V, $|V| = n$; the $k \times k$ symmetric matrix $\mathbf{P}$ is such that its entries satisfy $0 \le p_{ij} \le 1$ $(1 \le i \le j \le k)$. Then vertices of V_i and V_j are connected independently, with probability p_{ij}, $1 \le i \le j \le k$.

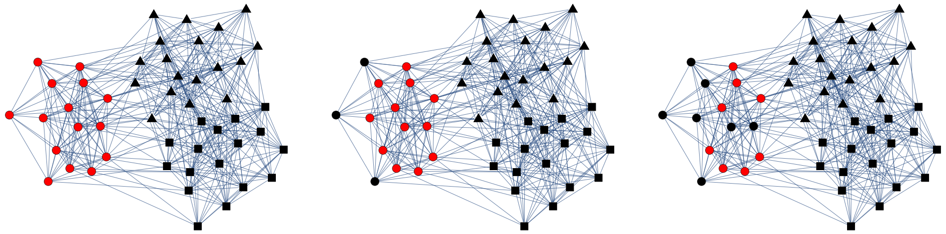
With the probability matrix

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.1 & 0.15 \\ 0.1 & 0.75 & 0.2 \\ 0.15 & 0.2 & 0.7 \end{pmatrix}$$

a random graph on 50 vertices was generated, where the vertices formed three loosely connected clusters; particularly, Cluster 1 (V_1)is loosely connected to Cluster 2 (V_2) and Cluster 3 (V_3). Depending on the initialization, we obtained indicator vectors of subsets of V_1, V_2, or V_3. The support of them is indicated by red points in Figures 1,2,3. With $\beta = 1$ and decreasing values of α, smaller and smaller supports appeared, but they were concentrated on one of the clusters. The weighted characteristic vectors supported on parts of the first cluster appeared soon, whereas those supported on parts of the second and third clusters were separated later. With $\alpha = -0.5$, the result of Theorem 3.6 is applicable, and we indeed obtain the support of a strongly maximal clique within one of the clusters.

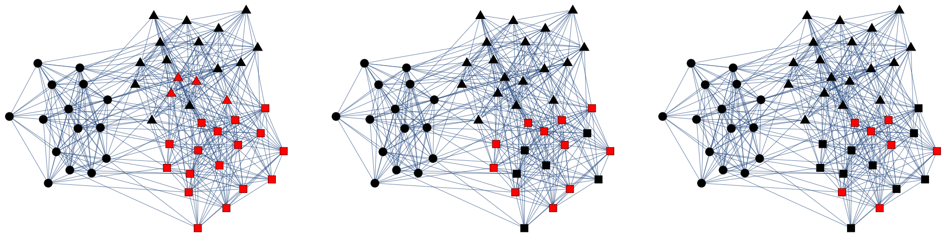
4 Optimizing over spheres and ellipsoids

From now on, we consider *multiple strategies*. The k-dimensional strategies of the agents can be thought of as intensities of buying/selling k different stocks or borro-



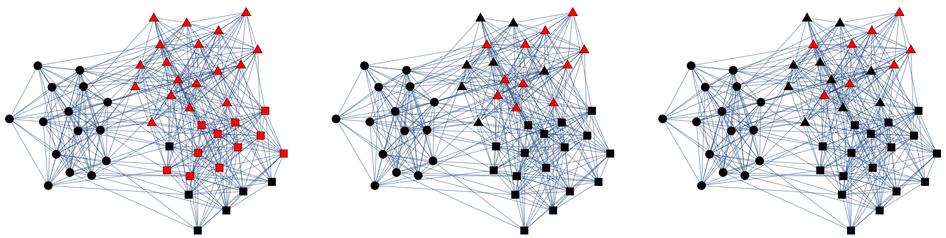
$\alpha \in (3.5, 6.5)$ $\alpha = 1$ $\alpha = -0.5$

Figure 1: Dominant sets with weighted characteristic vectors concentrated on the first cluster. Vertices of the three clusters are denoted by $\text{O}, \square, \triangle$ and red dots indicate the support of the weighted characteristic vector obtained by the dynamics with the actual values of α.



$\alpha = 6.5$ $\alpha \in (1.5, 2)$ $\alpha = -0.5$

Figure 2: Dominant sets with weighted characteristic vectors concentrated on the second cluster. Vertices of the three clusters are denoted by $\text{O}, \square, \triangle$ and red dots indicate the support of the weighted characteristic vector obtained by the dynamics with the actual values of α.



$\alpha = 6.5$ $\alpha = 1.5$ $\alpha \in (-0.5, 0.5)$

Figure 3: Dominant sets with weighted characteristic vectors concentrated on the third cluster. Vertices of the three clusters are denoted by $\text{O}, \square, \triangle$ and red dots indicate the support of the weighted characteristic vector obtained by the dynamics with the actual values of α.

wing/lending k different goods (they may have negative coordinates).

Now the quadratic objective function of Section 3 or its multidimensional extension will be maximized with respect to quadratic constraints. Here we have exact solutions: the maxima are given in terms of the bottom or top eigenvalues of the transformed edge-weight matrix, whereas the optimal multiple strategies are derived by means of the corresponding eigenvectors. The two extremes, corresponding to strategic complements or substitutes are unified into a multiway clustering problem, where we are looking for groups of agents following similar strategies with respect to the other groups, and in this case, strategies can be assigned to the agents, depending on their group memberships.

We saw that in the classical setup of strategic complements (see Section 2.2) when the parameter δ is small ($\delta < \frac{1}{1+\lambda_{max}(\bar{\mathbf{G}})}$), a unique inner equilibrium exists ($\forall x_i > 0$), and it can be found by matrix inversion, also using the Katz–Bonacich centrality. However, in the case of strategic substitutes (see Section 2.3), for larger δ's corner equilibria appear, and these are the only stable equilibria. To find corner equilibria, in [9] the authors define an algorithm which examines all subsets of vertices for possible corner solutions. This is computationally not tractable if the number of vertices is very large, since it is NP-complete. Instead, we may approximate corner equilibria by spectral clustering tools of [4] in polynomial time.

4.1 When there are complementarities between the agents

The utility function of agent i is defined by

$$u_i(\mathbf{X}) = \sum_{\ell=1}^{k} \alpha s_{i\ell}^2 - \frac{1}{2}\sum_{\ell=1}^{k} s_{i\ell}^2 + \phi \sum_{\ell=1}^{k}\sum_{j=1}^{n} w_{ij} s_{i\ell} s_{j\ell} \tag{4.1}$$

where α and ϕ are given positive constants. The first term is the benefit of agent i using strategy x_i, the second is the cost of agent i, and the last term is the utility (under strategic complementarity in efforts), i.e., the payoff due to his/her collaboration with the neighbors. The k-dimensional strategies $\mathbf{s}_1, \dots, \mathbf{s}_n \in \mathbb{R}^k$ of the agents are collected as row vectors of the $n \times k$ matrix $\mathbf{X}$. The coordinate $s_{i\ell}$ of $\mathbf{s}_i$ denotes the strategy of agent i towards the ℓ-th subject. The constant α now scales the quadratic gain of the agents. We assume that $0 < \alpha \le \frac{1}{2}$, so the gain would not exceed the costs for solitary agents; further, $\phi > 0$ is a constant that serves to regulate the effect of complementarities.

The simultaneous maximization of $u_i(\mathbf{X})$'s with respect to $\mathbf{s}_1, \dots, \mathbf{s}_n$ subject to $\sum_{i=1}^{n} \mathbf{s}_i \mathbf{s}_i^T = \mathbf{X}^T\mathbf{X} = \mathbf{I}_k$ is equivalent to maximizing the following potential function under the same constraint:

$$P(\mathbf{X}) = \sum_{i=1}^{n} u_i(\mathbf{X}) - \frac{\phi}{2} \sum_{i=1}^{n} \sum_{\ell=1}^{k} \sum_{j=1}^{n} w_{ij} s_{i\ell} s_{j\ell}$$

$$= \frac{1}{2} \sum_{\ell=1}^{k} \mathbf{x}_\ell^T [(2\alpha - 1)\mathbf{I} + \phi \mathbf{W}] \mathbf{x}_\ell = \frac{1}{2} \mathrm{tr}\, \mathbf{X}^T [(2\alpha - 1)\mathbf{I} + \phi \mathbf{W}] \mathbf{X},$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_k$ denote the column vectors of the suborthogonal matrix $\mathbf{X}$. The maximum of $P(\mathbf{X})$ subject to $\mathbf{X}^T \mathbf{X} = \mathbf{I}_k$ is taken on with the $\mathbf{X}$ that maximizes

$$\mathrm{tr}\, \mathbf{X}^T [(2\alpha - 1)\mathbf{I} + \phi \mathbf{W}] \mathbf{X}$$

on the constraint $\mathbf{X}^T \mathbf{X} = \mathbf{I}_k$. Irrespective of the definiteness of the matrix in brackets, the maximum is attained by an $\mathbf{X}^*$ which contains pairwise orthogonal, unit-norm eigenvectors, corresponding to the k largest eigenvalues of $(2\alpha - 1)\mathbf{I} + \phi \mathbf{W}$ in its columns, and the maximum is $\sum_{l=1}^{k} (2\alpha - 1 + \phi \lambda_l)$, where $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\mathbf{W}$, and it is attained by the corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ as columns of $\mathbf{X}^*$. These may contain negative coordinates, but they can be approximated by stepwise constant vectors of mainly nonnegative coordinates if the following condition is met: the subspace of these partition-vectors is close to the subspace spanned by $\mathbf{u}_1, \ldots, \mathbf{u}_k$. This is the case if there is a gap between λ_k and λ_{k+1}. In this case, the squared distance between these two subspaces is the k-variance of the clusters S_k^2 (see [4]), which is the minimum of the objective function of the k-means algorithm. Hence, the clusters of agents following similar strategies are obtained by applying the k-means algorithm to the optimum strategy vectors, row vectors of the optimum $\mathbf{X}$. Note, that the representatives can as well be rotated so that the column vectors of the matrix $\mathbf{X}^*$ are near to characteristic vectors of the optimizing vertex clusters, giving the same representation, but resulting in near zero or positive strategies. In this way, a k-partition of the vertices is obtained, so that each cluster of the partition is specialized to a strategy out of the k ones. Members of the same cluster pursue the same strategy with the same (positive) intensity , and the others do almost nothing. There are different groups responsible for different strategies (it is possible, since the number of clusters is equal to the number of strategies). In view of [12], when there is a remarkable gap between λ_k and λ_{k+1} these clusters are loosely connected, but themselves define dense subgraphs. Consequently, neighbors, or agents with strong connections will follow similar strategies in all the k respects. In Tables 1,2,3, one rotated eigenvector is concentrated on one cluster, and after suitable normalization it shows good agreement with the weighted characteristic vector obtained in Section 3.3, in terms of the MSE.

Ev1	0.034	0.058	0.044	0.025	0.058	0.006	0.055	0.041	0.003	0.066	-0.007	0.005	0.019	0.065	0.002
Ev2	-0.022	-0.067	-0.043	0.016	-0.085	0.032	-0.065	0.009	0.035	-0.050	0.016	-0.019	-0.019	-0.086	-0.020
Ev3	0.245	0.188	0.281	0.243	0.266	0.275	0.253	0.269	0.290	0.242	0.226	0.259	0.242	0.272	0.196
Wcv	0.065	0.040	0.081	0.063	0.075	0.073	0.069	0.077	0.081	0.065	0.054	0.069	0.060	0.079	0.040

Table 1: Coordinates of the three rotated leading eigenvectors corresponding to the first cluster. The third one (Ev3) is concentrated on Cluster 1, and the MSE between its normalized version and the weighted characteristic vector (Wcv) of this cluster (its non-zero coordinates are in the last row) is 0.0674341.

Ev1	0.263	0.262	0.288	0.190	0.178	0.226	0.201	0.230	0.248
Ev2	-0.105	0.008	-0.111	0.021	-0.087	-0.006	-0.004	0.025	-0.065
Ev3	-0.04	-0.048	-0.033	-0.040	-0.0008	-0.03	-0.016	-0.064	-0.052
Wcv	0.082	0.100	0.103	0.027	0.023	0.032	0.073	0.066	0.068

Ev1	0.232	0.214	0.216	0.295	0.22	0.191	0.21	0.234
Ev2	-0.008	-0.013	0.005	-0.039	-0.047	-0.014	-0.047	-0.039
Ev3	-0.023	-0.013	-0.031	-0.069	0.005	0.022	-0.053	-0.068
Wcv	0.065	0.064	0.106	0.0216	0.046	0.031	0	0.084

Table 2: Coordinates of the three rotated leading eigenvectors corresponding to Cluster 2. The first one (Ev1) is concentrated on Cluster 2, and the MSE between its normalized version and the weighted characteristic vector (Wcv) of this cluster (its non-zero coordinates are in the last row, except the last coordinate, instead of which we have a non-zero coordinate corresponding to a vertex of Cluster 3) is 0.135404.

Ev1	0.066	0.019	0.057	-0.008	0.015	0.037	0.076	0.050	-0.006
Ev2	0.224	0.270	0.205	0.260	0.213	0.268	0.270	0.187	0.299
Ev3	0.045	-0.020	0.011	-0.016	0.026	-0.004	0.006	0.081	0.011
Wcv	0.061	0.079	0.051	0.072	0.032	0.085	0.089	0.036	0.093

Ev1	0.058	0.053	0.139	0.076	0.056	0.077	0.024	0.024	0.003
Ev2	0.190	0.150	0.120	0.122	0.253	0.135	0.172	0.296	0.290
Ev3	0.003	0.029	-0.008	0.009	0.034	0.044	0.015	0.006	0.070
Wcv	0.057	0.011	0.038	0.080	0.023	0.018	0.087	0	0.080

Table 3: Coordinates of the three rotated leading eigenvectors corresponding to Cluster 3. The second one (Ev2) is concentrated on Cluster 3, and the MSE between its normalized version and the weighted characteristic vector (Wcv) of this cluster (its non-zero coordinates are in the last row, except the last coordinate, instead of which we have a non-zero coordinate corresponding to a vertex of Cluster 2) is 0.129198.

In the case of $k = 1$ we optimize over the sphere $\|\mathbf{x}\|=1$, and the above maximum is $2\alpha - 1 + \phi\lambda_1$, which is positive if and only if $\phi > \frac{1-2\alpha}{\lambda_1}$, in view of $\lambda_1 > 0$ (since $\mathbf{W}$ is a Frobenius-type matrix). Because of $0 < \alpha < \frac{1}{2}$, this gives a positive lower bound for ϕ. Consequently, the above maximum is positive.

When $k > 1$ is such that $\lambda_1 \geq \cdots \geq \lambda_k > 0$, then $\sum_{\ell=1}^{k}(2\alpha - 1 + \phi\lambda_\ell) > 0$ holds if $\phi > \frac{k(1-2\alpha)}{\sum_{\ell=1}^{k}\lambda_\ell}$. Therefore, the number of strategies cannot exceed the number of positive eigenvalues of $\mathbf{W}$ to get a positive optimum. However, when the size of G is large, it suffices to select a k such that $\lambda_k > 0$ and it is much 'larger' that λ_{k+1}.

The utility function can be further generalized to

$$u_i(\mathbf{X}) = \sum_{\ell=1}^{k} \alpha s_{i\ell}^2 - \frac{1}{2}\sum_{\ell=1}^{k} s_{i\ell}^2 + \sum_{\ell=1}^{k} \phi_\ell \sum_{j=1}^{n} a_{ij} s_{i\ell} s_{j\ell}, \tag{4.2}$$

when the potential function becomes

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=1}^{n} u_i(\mathbf{X}) - \sum_{\ell=1}^{k} \frac{\phi_\ell}{2} \sum_{j=1}^{n} w_{ij}^{(\ell)} s_{i\ell} s_{j\ell} \\ &= \frac{1}{2}\sum_{l=1}^{k} \mathbf{x}_l^T [(2\alpha - 1)\mathbf{I} + \phi_\ell \mathbf{W}^{(\ell)}]\mathbf{x}_\ell, \end{aligned}$$

where $\mathbf{W}^{(\ell)}$ is the edge-weight matrix of the agents under strategy ℓ, $\ell = 1, \dots, k$ (these connections are given, and they may differ for different strategies). For maximizing the sum of the inhomogeneous quadratic forms we introduced an algorithm in [5]. In particular, when $\mathbf{W}^{(1)} = \cdots = \mathbf{W}^{(k)} = \mathbf{W}$, i.e., the matrices in the brackets commute, we select their largest eigenvalues (assuming that ϕ_ℓ's are different) with the corresponding eigenvectors.

Another possibility is to take into consideration the vertex degrees in G. Then the modified utility is

$$u_i(\mathbf{X}) = \sum_{\ell=1}^{k} \alpha d_i s_{i\ell}^2 - \frac{1}{2}\sum_{\ell=1}^{k} d_i s_{i\ell}^2 + \phi \sum_{\ell=1}^{k}\sum_{j=1}^{n} a_{ij} s_{i\ell} s_{j\ell}. \tag{4.3}$$

The simultaneous maximization of $u_i(\mathbf{X})$'s ($i = 1, \dots, k$) subject to $\mathbf{X}^T \boldsymbol{D} \mathbf{X} = \mathbf{I}_k$ is equivalent to maximizing the following potential function under the same constraint:

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=1}^{n} u_i(\mathbf{X}) - \frac{\phi}{2} \sum_{\ell=1}^{k}\sum_{j=1}^{n} a_{ij} s_{i\ell} s_{j\ell} \\ &= \frac{1}{2}\sum_{\ell=1}^{k} (\boldsymbol{D}^{1/2}\mathbf{x}_\ell)^T [(2\alpha - 1)\mathbf{I} + \phi \mathbf{W}_D](\boldsymbol{D}^{1/2}\mathbf{x}_\ell) \\ &= \frac{1}{2} \mathrm{tr}\, (\boldsymbol{D}^{1/2}\mathbf{X})^T [(2\alpha - 1)\mathbf{I} + \phi \mathbf{W}_D](\boldsymbol{D}^{1/2}\mathbf{X}). \end{aligned}$$

Its maximum subject to $\mathbf{X}^T \boldsymbol{D} \mathbf{X} = \mathbf{I}_k$ (ellipsoid) is taken on with the $\mathbf{X}$ that maximizes

$$\mathrm{tr}\, (\boldsymbol{D}^{1/2}\mathbf{X})^T [(2\alpha - 1)\mathbf{I} + \phi \mathbf{W}_D](\boldsymbol{D}^{1/2}\mathbf{X})$$

on the constraint $\mathbf{X}^T \boldsymbol{D}\mathbf{X} = \mathbf{I}_k$. Irrespective whether the matrix in brackets is positive semidefinite, it is attained by an $\boldsymbol{D}^{-1/2}\mathbf{X}^*$, where the columns of $\mathbf{X}^*$ are pairwise orthogonal, unit-norm eigenvectors, corresponding to the k largest eigenvalues of $(2\alpha-1)\mathbf{I}+\phi\mathbf{W}_D$ (see Section 2), and the maximum is $\sum_{l=1}^{k}(2\alpha-1+\phi\lambda'_l)$, where $\lambda'_1 \geq \cdots \geq \lambda'_n$ are the eigenvalues of $\mathbf{W}_D$, and it is attained by the corresponding eigenvectors $\mathbf{u}'_1, \dots, \mathbf{u}'_k$ as columns of $\mathbf{X}^*$. Since the eigenvalues of $\mathbf{W}_D$ are in the [-1,1] interval and $0 \leq 2\alpha - 1 \leq 1$, the eigenvalues of $(2\alpha-1)\mathbf{I}+\phi\mathbf{W}_D$ are in the $[-\phi-1, \phi]$ interval.

4.2 When there are substitutes between the agents

The utility function of agent i is now defined by

$$u_i(\mathbf{X}) = \sum_{\ell=1}^{k} \alpha s_{i\ell}^2 - \frac{1}{2}\sum_{\ell=1}^{k} s_{i\ell}^2 - \delta \sum_{\ell=1}^{k}\sum_{j=1}^{n} a_{ij} s_{i\ell} s_{j\ell} \tag{4.4}$$

with constants $0 < \alpha \leq \frac{1}{2}$ and $\delta > 0$ to regulate the effect of substitutes. The simultaneous maximization of $u_i(\mathbf{X})$'s subject to $\sum_{i=1}^{n} \mathbf{s}_i\mathbf{s}_i^T = \mathbf{X}^T\mathbf{X} = \mathbf{I}_k$ is equivalent to maximizing the following potential function under the same constraint:

$$\begin{aligned} P(\mathbf{X}) &= \sum_{i=1}^{n} u_i(\mathbf{X}) + \frac{\delta}{2}\sum_{i=1}^{n}\sum_{\ell=1}^{k}\sum_{j=1}^{n} a_{ij} s_{i\ell} s_{j\ell} \\ &= -\frac{1}{2}\sum_{\ell=1}^{k} \mathbf{x}_\ell^T[(1-2\alpha)\mathbf{I}+\delta\mathbf{W}]\mathbf{x}_\ell \\ &= -\frac{1}{2}\mathrm{tr}\,\mathbf{X}^T[(1-2\alpha)\mathbf{I}+\delta\mathbf{W}]\mathbf{X}. \end{aligned}$$

Its maximum subject to $\mathbf{X}^T\mathbf{X} = \mathbf{I}_k$ is taken on with the same $\mathbf{X}$ that gives the minimum of

$$\mathrm{tr}\,\mathbf{X}^T[(1-2\alpha)\mathbf{I}+\delta\mathbf{W}]\mathbf{X}$$

on the same constraint. Irrespective of the definiteness of the matrix in brackets, the minimum is attained at an $\mathbf{X}^*$ which contains pairwise orthogonal, unit-norm eigenvectors, corresponding to the k smallest eigenvalues of $(1-2\alpha)\mathbf{I}+\delta\mathbf{W}$ in its columns, and the minimum is $\sum_{\ell=1}^{k}(1-2\alpha+\delta\lambda_{n-\ell+1})$, where $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\mathbf{G}$, and it is attained by the corresponding eigenvectors $\mathbf{u}_n, \dots, \mathbf{u}_{n-k+1}$ as columns of $\mathbf{X}$. These may contain negative coordinates, but they can be approximated by stepwise constant vectors of nonnegative coordinates. The subspace of these partition-vectors is close to the subspace spanned by $\mathbf{u}_n, \dots, \mathbf{u}_{n-k+1}$ if there is a gap between λ_{n-k+1} and λ_{n-k}. In this case, the clusters of agents following similar strategies are obtained by applying the k-means algorithm to the optimum strategy vectors, row vectors of the optimum $\mathbf{X}$.

In the case of $k=1$, this minimum is $1-2\alpha+\delta\lambda_1$, which is negative if and only if $\delta > \frac{2\alpha-1}{\lambda_n}$, in view of $\lambda_n<0$ and $0<\alpha<\frac{1}{2}$. It means that the above maximum is positive.

The inequality $\delta > \frac{2\alpha-1}{\lambda_n}$ can be restricted to the range of δ where corner equilibria are stable. The corresponding 2-partition of the vertices is obtained by the k-means algorithm applied for the coordinates of $\mathbf{u}_1$. In the $k>1$ case the same holds with applying the k-means algorithm with the optimal $\mathbf{s}_1^*,\dots,\mathbf{s}_n^*$ as row vectors of the $n\times k$ matrix $\mathbf{X}^*$.

When $k>1$ is such that $\lambda_n \le \dots \le \lambda_{n-k+1} < 0$, then $\sum_{\ell=1}^k (1-2\alpha+\delta\lambda_{n-\ell+1}) < 0$ holds if $\delta > \frac{k(2\alpha-1)}{\sum_{\ell=1}^k \lambda_{n-\ell+1}}$. Therefore, the number of strategies cannot exceed the number of negative eigenvalues of G to get a positive optimum. However, when the size of G is large, it suffices to select a k such that $\lambda_{n-k+1}<0$ and it is much 'smaller' that λ_{n-k}.

The potential function can be also generalized to

$$\begin{aligned}P(\mathbf{X}) &= \sum_{i=1}^n u_i(\mathbf{X}) - \sum_{\ell=1}^k \frac{\delta_\ell}{2} \sum_{i=1}^n \sum_{j=1}^n a_{ij} s_{i\ell} s_{j\ell} \\ &= \frac{1}{2}\sum_{\ell=1}^k \mathbf{x}_\ell^T [(2\alpha-1)\mathbf{I} + \delta_l \mathbf{G}^{(l)}]\mathbf{x}_\ell\end{aligned}$$

as in Section 4.1.

Since the eigenvectors not always have positive coordinates, we approximate them by partition vectors. In this way, clusters of agents, following similar strategy are found. In the substitute case, these clusters have sparse within- and dense between-cluster connections.

When the similarity matrix depends on the actual strategy, the potential function can be further generalized to

$$\begin{aligned}P(\mathbf{X}) &= \sum_{i=1}^n u_i(\mathbf{X}) + \frac{\delta}{2} \sum_{l=1}^k \sum_{i=1}^n \sum_{j=1}^n w_{ij}^{(\ell)} s_{i\ell} s_{j\ell} \\ &= \frac{1}{2}\sum_{\ell=1}^k \mathbf{x}_\ell^T [(2\alpha-1)\mathbf{I} + \delta \mathbf{G}^{(l)}]\mathbf{x}_\ell,\end{aligned}$$

where $W^{(\ell)}$ is the connection matrix of the agents under strategy ℓ ($\ell=1,\dots,k$), and $\mathbf{x}_1,\dots,\mathbf{x}_k$ form an orthonormal set. The solution is given by the compromise vectors of the symmetric matrices in brackets. This generalization corresponds to the real-life situation when the agents have different connections with respect to different strategies (e.g., for buying different kinds of stocks or planting different kinds of crops).

5 Discussion

When maximizing the mutual utility of agents in a network of interactions, we consider edge-weighted graphs describing pairwise relations of the agents. We show how the graph structure determines the optimal strategies with respect to quadratic objective functions maximized on linear or quadratic constraints. Under simplex constraints, dominant sets of an edge-weighted graph will give the solution, where the model parameters are built into the edge-weights. Under quadratic constraints, the spectrum of the unnormalized or normalized edge-weight matrix decides which strategy to follow. Large positive eigenvalues favor complementary strategies in as many respect as the number of the structural positive eigenvalues; while negative eigenvalues of large absolute value favor substitute strategies in as many respect as the number of the structural negative eigenvalues. This is also supported by social network studies, see, e.g., [3, 14]. Note that an eigenvector-based feature organization is also discussed in [18].

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Refining Jensen's Integral Inequality for Partitions of Weights

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Abstract: In this paper we establish a refinement and some reverses for Jensen's inequality for the general Lebesgue integral on measurable spaces and partitions of weights. Applications for discrete inequalities and weighted means of positive numbers are also given.

Keywords: Jensen's inequality,Convex functions, Lebesgue integral, Weighted means.

MSC2010: Primary 26D15; Secondary 94A17.

1 Introduction

Let C be a convex subset of the linear space X and f a convex function on C. If $\mathbf{p} = (p_1, \ldots, p_n)$ is a *probability sequence,* i.e. $p_1, \ldots, p_n \geq 0$ with $\sum_{i=1}^{n} p_i = 1$ and $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

is well known in the literature as *Jensen's inequality.*

The Jensen inequality for convex functions plays a crucial role in the Theory of Inequalities due to the fact that other inequalities such as the arithmetic mean-geometric mean inequality, Hölder and Minkowski's inequalities, Ky Fan's inequality etc. can be obtained as particular cases of it.

In 1989, J. Pečarić and the author obtained the following refinement of (1.1) (see [14]):

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &\leq \sum_{i_1,\ldots,i_{k+1}=1}^{n} p_{i_1}\ldots p_{i_{k+1}} f\left(\frac{x_{i_1}+\cdots+x_{i_{k+1}}}{k+1}\right) \\ &\leq \sum_{i_1,\ldots,i_k=1}^{n} p_{i_1}\ldots p_{i_k} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \leq \cdots \leq \sum_{i=1}^{n} p_i f(x_i), \end{aligned} \tag{1.2}$$

for $k \geq 1$ and $\mathbf{p}$, $\mathbf{x}$ as above.

If $q_1, \dots, q_k \geq 0$ with $\sum_{j=1}^{k} q_j = 1$, then the following weighted refinement obtained in 1994 by the author also holds (see [5]):

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1} \dots p_{i_k} f\left(\frac{x_{i_1} + \dots + x_{i_k}}{k}\right) \tag{1.3}$$
$$\leq \sum_{i_1,\dots,i_k=1}^{n} p_{i_1} \dots p_{i_k} f\left(q_1 x_{i_1} + \dots + q_k x_{i_k}\right) \leq \sum_{i=1}^{n} p_i f(x_i),$$

where $1 \leq k \leq n$ and $\mathbf{p}$, $\mathbf{x}$ are as above.

More recently the author obtained a different refinement of Jensen's inequality incorporated in (see [7]):

$$f\left(\sum_{j=1}^{n} p_j x_j\right) \leq \min_{k\in\{1,\dots,n\}} \left[(1-p_k) f\left(\frac{\sum_{j=1}^{n} p_j x_j - p_k x_k}{1-p_k}\right) + p_k f(x_k)\right] \tag{1.4}$$
$$\leq \frac{1}{n}\left[\sum_{k=1}^{n} (1-p_k) f\left(\frac{\sum_{j=1}^{n} p_j x_j - p_k x_k}{1-p_k}\right) + \sum_{k=1}^{n} p_k f(x_k)\right]$$
$$\leq \max_{k\in\{1,\dots,n\}} \left[(1-p_k) f\left(\frac{\sum_{j=1}^{n} p_j x_j - p_k x_k}{1-p_k}\right) + p_k f(x_k)\right]$$
$$\leq \sum_{j=1}^{n} p_j f(x_j),$$

where f, x_k and p_k are as above.

For other refinements and applications related to Ky Fan's inequality, the arithmetic mean-geometric mean inequality, the generalized triangle inequality, the f-Divergence measure etc., see [1], [2]-[12], [13] and [15]-[16]

Motivated by the above results, we investigate in this paper the integral version of Jensen inequality and establish some refinements and reverses of interest for applications.

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ − algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$. For the μ-integrable positive μ-a.e. weight w consider the Lebesgue space

$$L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, w(t)\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$ etc..

For the μ-integrable positive μ-a.e. weight w and a given $n \geq 2$ we consider the set $\mathfrak{P}_n(w)$ all possible n-tuples of μ-integrable positive μ-a.e. weights $\overline{w} = (w_1, ..., w_n)$ with the property that $\sum_{i=1}^n w_i = w$. The n-tuple $\overline{w} = (w_1, ..., w_n)$ it is called *a partition* of the weight w. It is clear that $\sum_{i=1}^n \int_\Omega w_i d\mu = \int_\Omega w d\mu$ for any $(w_1, ..., w_n) \in \mathfrak{P}_n(w)$ and $\int_\Omega w_i d\mu > 0$.

For a convex function $\Phi : [m, M] \to \mathbb{R}$, a μ-measurable function $f : \Omega \to [m, M]$ such that f, $\Phi \circ f \in L_w(\Omega, \mu)$ we define the functional $\psi(\Phi, f, \cdot) : \mathfrak{P}_n(w) \to \mathbb{R}$ by

$$\psi(\Phi, f, \overline{w}) := \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^n \Phi\left(\frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu}\right) \int_\Omega w_i d\mu. \tag{1.5}$$

In the next section we establish some results concerning this functional that are related to Jensen's integral inequality. Applications for discrete inequalities and weighted means are provided in the third section. In the last section some applications for univariate functions are also given.

2 Main Results

The following basic result holds:

Theorem 2.1. *Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m, M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega, \mu)$. Then for any $\overline{w} \in \mathfrak{P}_n(w)$ we have*

$$\frac{\int_\Omega (\Phi \circ f) w d\mu}{\int_\Omega w d\mu} \geq \psi(\Phi, f, \overline{w}) \geq \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right), \tag{2.1}$$

where $n \geq 2$.

Proof. From Jensen's integral inequality we have

$$\int_\Omega (\Phi \circ f) w_i d\mu \geq \Phi\left(\frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu}\right) \int_\Omega w_i d\mu \tag{2.2}$$

for any $i \in \{1, ..., n\}$.

If we sum the inequality (2.2) over i from 1 to n we get

$$\sum_{i=1}^n \int_\Omega (\Phi \circ f) w_i d\mu \geq \sum_{i=1}^n \Phi\left(\frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu}\right) \int_\Omega w_i d\mu \tag{2.3}$$

and since

$$\sum_{i=1}^n \int_\Omega (\Phi \circ f) w_i d\mu = \int_\Omega (\Phi \circ f) \left(\sum_{i=1}^n w_i\right) d\mu = \int_\Omega (\Phi \circ f) w d\mu$$

then from (2.3) we get the first part of (2.1).

Let

$$p_i = \int_\Omega w_i d\mu > 0 \; z_i = \frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu} \in [m, M], \; i \in \{1, ..., n\}.$$

Then

$$P_n := \sum_{i=1}^n p_i = \int_\Omega w d\mu,$$

and

$$\sum_{i=1}^n p_i z_i = \sum_{i=1}^n \int_\Omega f w_i d\mu = \int_\Omega f w d\mu.$$

From Jensen's discrete inequality

$$\frac{1}{P_n} \sum_{i=1}^n p_i \Phi(z_i) \geq \Phi\left(\frac{\sum_{i=1}^n p_i z_i}{P_n}\right)$$

we have

$$\frac{1}{\int_\Omega w d\mu} \sum_{i=1}^n \Phi\left(\frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu}\right) \int_\Omega w_i d\mu \geq \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right)$$

and the second inequality in (2.1) is also proved. □

Remark 2.2. The double inequality (2.1) is equivalent to

$$\frac{\int_\Omega (\Phi \circ f) w d\mu}{\int_\Omega w d\mu} \geq \sup_{\overline{w} \in \mathfrak{P}_n(w)} \psi(\Phi, f, \overline{w}) \tag{2.4}$$

and

$$\inf_{\overline{w} \in \mathfrak{P}_n(w)} \psi(\Phi, f, \overline{w}) \geq \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right), \tag{2.5}$$

where $n \geq 2$.

We use the following lemma [9].

Lemma 2.3. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ is the interior of I. If $g : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq g(x) \leq M < \infty \text{ for } \mu\text{-a.e. } x \in \Omega$$

and such that $g, \Phi \circ g \in L_p(\Omega,\mu)$, *where* $p \geq 0$ μ*-a.e. on* Ω *with* $\int_\Omega p d\mu = 1$, *then*

$$\begin{aligned} 0 &\leq \int_\Omega p(\Phi \circ f)\, d\mu - \Phi\left(\int_\Omega p f d\mu\right) \quad (2.6) \\ &\leq \frac{\left(M - \int_\Omega p f d\mu\right)\left(\int_\Omega p f d\mu - m\right)}{M-m} \sup_{t\in(m,M)} \Psi_\Phi(t;m,M) \\ &\leq \left(M - \int_\Omega p f d\mu\right)\left(\int_\Omega p f d\mu - m\right) \frac{\Phi'_-(M) - \Phi'_+(m)}{M-m} \\ &\leq \frac{1}{4}(M-m)\left[\Phi'_-(M) - \Phi'_+(m)\right], \end{aligned}$$

where $\Psi_\Phi(\cdot;m,M) : (m,M) \to \mathbb{R}$ *is defined by*

$$\Psi_\Phi(t;m,M) = \frac{\Phi(M) - \Phi(t)}{M-t} - \frac{\Phi(t) - \Phi(m)}{t-m}.$$

We have the following reverse of the first inequality in (2.1).

Theorem 2.4. *Let* $\Phi : [m,M] \to \mathbb{R}$ *be a convex function,* $f : \Omega \to [m,M]$ *a* μ*-measurable function such that* f, $\Phi \circ f \in L_w(\Omega,\mu)$. *Then for any* $\overline{w} \in \mathfrak{P}_n(w)$, $n \geq 2$ *we have*

$$\begin{aligned} 0 &\leq \frac{\int_\Omega (\Phi \circ f) w d\mu}{\int_\Omega w d\mu} - \psi(\Phi, f, \overline{w}) \quad (2.7) \\ &\leq \frac{1}{M-m} \sup_{t\in(m,M)} \Psi_\Phi(t;m,M) \\ &\times \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^n \left(\int_\Omega w_i d\mu\right)\left(M - \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu}\right)\left(\frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m\right) \\ &\leq \frac{1}{M-m} \sup_{t\in(m,M)} \Psi_\Phi(t;m,M) \left(M - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right)\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - m\right). \end{aligned}$$

Proof. From the second inequality in (2.6) for $g = f$ and $p = \frac{w_i}{\int_\Omega w_i d\mu}$, $i \in \{1, ..., n\}$ we have

$$\begin{aligned} 0 &\leq \frac{1}{\int_\Omega w_i d\mu} \int_\Omega w_i (\Phi \circ f)\, d\mu - \Phi\left(\frac{1}{\int_\Omega w_i d\mu} \int_\Omega w_i f d\mu\right) \quad (2.8) \\ &\leq \frac{1}{M-m}\left(M - \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu}\right)\left(\frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m\right) \sup_{t\in(m,M)} \Psi_\Phi(t;m,M) \end{aligned}$$

for any $i \in \{1, ..., n\}$.

If we multiply by $\int_\Omega w_i d\mu > 0$ and sum over i from 1 to n we get

$$0 \le \sum_{i=1}^{n} \int_\Omega w_i (\Phi \circ f)\, d\mu - \sum_{i=1}^{n} \Phi\left(\frac{1}{\int_\Omega w_i d\mu} \int_\Omega w_i f d\mu\right) \int_\Omega w_i d\mu \tag{2.9}$$
$$\le \frac{1}{M-m} \sup_{t\in(m,M)} \Psi_\Phi(t;m,M)$$
$$\times \sum_{i=1}^{n} \left(\int_\Omega w_i d\mu\right)\left(M - \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu}\right)\left(\frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m\right),$$

which proves the second inequality in (2.7).

Now, observe that the function $\Psi : [m,M] \to [0,\infty)$, $\Psi(t) = (M-t)(t-m)$ is concave and by Jensen's inequality for concave functions with

$$p_i = \int_\Omega w_i d\mu > 0 \;\; z_i = \frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu} \in [m,M],\; i \in \{1,...,n\}$$

we have

$$\frac{1}{\int_\Omega w d\mu} \sum_{i=1}^{n} \left(\int_\Omega w_i d\mu\right)\left(M - \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu}\right)\left(\frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m\right)$$
$$\le \left(M - \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^{n} \left(\int_\Omega w_i d\mu\right) \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu}\right)$$
$$\times \left(\frac{1}{\int_\Omega w d\mu} \sum_{i=1}^{n} \left(\int_\Omega w_i d\mu\right) \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m\right)$$
$$= \left(M - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right)\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - m\right),$$

which proves the last part of (2.7). □

Remark 2.5. Since, as shown in [9],

$$\sup_{t\in(m,M)} \Psi_\Phi(t;m,M) \le \Phi'_-(M) - \Phi'_+(m)$$

then we have the following simpler inequality

$$0 \le \frac{\int_\Omega (\Phi \circ f)\, w d\mu}{\int_\Omega w d\mu} - \psi(\Phi, f, \overline{w}) \tag{2.10}$$
$$\le \frac{\Phi'_-(M) - \Phi'_+(m)}{M-m}$$
$$\times \sum_{i=1}^{n} \frac{\int_\Omega w_i d\mu}{\int_\Omega w d\mu} \left(M - \frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} \right) \left(\frac{\int_\Omega w_i f d\mu}{\int_\Omega w_i d\mu} - m \right)$$
$$\le \frac{\Phi'_-(M) - \Phi'_+(m)}{M-m} \left(M - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} \right) \left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - m \right).$$

If we use Lemma 2.3 for the discrete measure, we can state the following result:

Lemma 2.6. *Let $\Phi : [a,b] \to \mathbb{R}$ be a convex function on the interval of real numbers $[a,b]$, $z_i \in [a,b]$, $p_i \ge 0$, $i \in \{1, ..., n\}$ and $\sum_{i=1}^n p_i = 1$. Then*

$$0 \le \sum_{i=1}^{n} p_i \Phi(z_i) - \Phi\left(\sum_{i=1}^{n} p_i z_i\right) \tag{2.11}$$
$$\le \frac{\left(b - \sum_{i=1}^n p_i z_i\right)\left(\sum_{i=1}^n p_i z_i - a\right)}{b-a} \sup_{t \in (a,b)} \Psi_\Phi(t; a, b)$$
$$\le \left(b - \sum_{i=1}^{n} p_i z_i\right)\left(\sum_{i=1}^{n} p_i z_i - a\right) \frac{\Phi'_-(b) - \Phi'_+(a)}{b-a}$$
$$\le \frac{1}{4}(b-a)\left[\Phi'_-(b) - \Phi'_+(a)\right].$$

The following reverse of the second inequality in (2.1) holds:

Theorem 2.7. *Let $\Phi : [m,M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m,M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega, \mu)$. Then for any $\overline{w} \in \mathfrak{P}_n(w)$, $n \ge 2$ we have*

$$0 \le \psi(\Phi, f, \overline{w}) - \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right) \tag{2.12}$$
$$\le \frac{\left(L(\overline{w}) - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right)\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - l(\overline{w})\right)}{L(\overline{w}) - l(\overline{w})} \sup_{t \in (l(\overline{w}), L(\overline{w}))} \Psi_\Phi(t; l(\overline{w}), L(\overline{w}))$$
$$\le \frac{1}{4}\left(L(\overline{w}) - l(\overline{w})\right) \sup_{t \in (l(\overline{w}), L(\overline{w}))} \Psi_\Phi(t; l(\overline{w}), L(\overline{w})),$$

where

$$l(\overline{w}) := \min_{i\in\{1,\dots,n\}} \left\{ \frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu} \right\},\ L(\overline{w}) := \max_{i\in\{1,\dots,n\}} \left\{ \frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu} \right\}. \tag{2.13}$$

Proof. If we write the first two inequalities in (2.11) for

$$p_i = \frac{\int_\Omega w_i d\mu}{\int_\Omega w d\mu} > 0\ z_i = \frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu},\ i \in \{1, ..., n\}$$

and for $a = l(\overline{w}), b = L(\overline{w})$ as above we have

$$\begin{aligned} 0 &\le \frac{1}{\int_\Omega w d\mu} \sum_{i=1}^n \left(\int_\Omega w_i d\mu \right) \Phi\left(\frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu} \right) - \Phi\left(\frac{\sum_{i=1}^n \int_\Omega f w_i d\mu}{\int_\Omega w d\mu} \right) \\ &\le \frac{\sup_{t\in(l(\overline{w}),L(\overline{w}))} \Psi_\Phi(t; l(\overline{w}), L(\overline{w}))}{L(\overline{w}) - l(\overline{w})} \\ &\times \left(L(\overline{w}) - \sum_{i=1}^n \frac{\int_\Omega w_i d\mu}{\int_\Omega w d\mu} \cdot \frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu} \right) \left(\sum_{i=1}^n \frac{\int_\Omega w_i d\mu}{\int_\Omega w d\mu} \cdot \frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu} - l(\overline{w}) \right) \\ &= \frac{\left(L(\overline{w}) - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} \right) \left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - l(\overline{w}) \right)}{L(\overline{w}) - l(\overline{w})} \sup_{t\in(l(\overline{w}),L(\overline{w}))} \Psi_\Phi(t; l(\overline{w}), L(\overline{w})), \end{aligned} \tag{2.14}$$

which proves the second inequality in (2.12).

The last part in (2.12) follows by the elementary inequality

$$\alpha\beta \le \left(\frac{\alpha + \beta}{2} \right)^2,\ \alpha, \beta \in \mathbb{R}.$$

□

Remark 2.8. Since

$$\Psi_\Phi(t; l(\overline{w}), L(\overline{w})) \le \Phi'_-(L(\overline{w})) - \Phi'_+(l(\overline{w})),$$

then from (2.12) we have the simpler inequalities

$$\begin{aligned} 0 &\le \psi(\Phi, f, \overline{w}) - \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} \right) \\ &\le \frac{\Phi'_-(L(\overline{w})) - \Phi'_+(l(\overline{w}))}{L(\overline{w}) - l(\overline{w})} \left(L(\overline{w}) - \frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} \right) \left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu} - l(\overline{w}) \right) \\ &\le \frac{1}{4} (L(\overline{w}) - l(\overline{w})) \left[\Phi'_-(L(\overline{w})) - \Phi'_+(l(\overline{w})) \right]. \end{aligned} \tag{2.15}$$

The following reverse of Jensen inequality also holds [9]:

Lemma 2.9. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ is the interior of I. If $g : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq g(x) \leq M < \infty \text{ for } \mu\text{-a.e. } x \in \Omega$$

and such that $g, \Phi \circ g \in L_p(\Omega, \mu)$, where $p \geq 0$ μ-a.e. on Ω with $\int_\Omega p d\mu = 1$, then

$$\begin{aligned} 0 &\leq \int_\Omega p(\Phi \circ f)\, d\mu - \Phi\left(\int_\Omega pf d\mu\right) \\ &\leq \left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right]\left(1 + \frac{2}{M-m}\left|\int_\Omega pf d\mu - \frac{m+M}{2}\right|\right) \\ &\leq 2\left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right]. \end{aligned} \tag{2.16}$$

We have the following result:

Theorem 2.10. *Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m, M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega, \mu)$. Then for any $\overline{w} \in \mathfrak{P}_n(w)$, $n \geq 2$ we have*

$$\begin{aligned} 0 &\leq \frac{\int_\Omega (\Phi \circ f)\, w d\mu}{\int_\Omega w d\mu} - \psi(\Phi, f, \overline{w}) \\ &\leq \left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\ &\quad \times \left(1 + \frac{2}{M-m}\sum_{i=1}^{n}\left|\frac{1}{\int_\Omega w d\mu}\int_\Omega w_i\left(f - \frac{m+M}{2}\right) d\mu\right|\right) \\ &\leq \left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\ &\quad \times \left(1 + \frac{2}{M-m}\frac{1}{\int_\Omega w d\mu}\int_\Omega w\left|f - \frac{m+M}{2}\right| d\mu\right). \end{aligned} \tag{2.17}$$

Proof. From the second inequality in (2.16) for $g = f$ and $p = \frac{w_i}{\int_\Omega w_i d\mu}$, $i \in \{1, ..., n\}$

we have

$$
\begin{aligned}
0 &\leq \frac{1}{\int_\Omega w_i d\mu}\int_\Omega w_i(\Phi\circ f)\,d\mu-\Phi\left(\frac{1}{\int_\Omega w_i d\mu}\int_\Omega w_i f d\mu\right) \\
&\leq \left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right] \\
&\times\left(1+\frac{2}{M-m}\left|\frac{1}{\int_\Omega w_i d\mu}\int_\Omega w_i f d\mu-\frac{m+M}{2}\right|\right)
\end{aligned} \tag{2.18}
$$

for any $i\in\{1,...,n\}$.

If we multiply the inequality (2.18) by $\int_\Omega w_i d\mu>0$ and sum over i from 1 to n we get

$$
\begin{aligned}
0 &\leq \sum_{i=1}^n\int_\Omega w_i(\Phi\circ f)\,d\mu-\sum_{i=1}^n\Phi\left(\frac{1}{\int_\Omega w_i d\mu}\int_\Omega w_i f d\mu\right)\int_\Omega w_i d\mu \\
&\leq \left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\sum_{i=1}^n\int_\Omega w_i d\mu+\frac{2}{M-m}\sum_{i=1}^n\left(\int_\Omega w_i d\mu\right)\left|\frac{1}{\int_\Omega w_i d\mu}\int_\Omega w_i f d\mu-\frac{m+M}{2}\right|\right) \\
&= \left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\int_\Omega w d\mu+\frac{2}{M-m}\sum_{i=1}^n\left|\int_\Omega w_i\left(f-\frac{m+M}{2}\right)d\mu\right|\right)
\end{aligned}
$$

any by dividing with $\int_\Omega w d\mu=\sum_{i=1}^n\int_\Omega w_i d\mu$, we get the second inequality in (2.17).

By the properties of modulus we have

$$
\begin{aligned}
\sum_{i=1}^n\left|\frac{1}{\int_\Omega w d\mu}\int_\Omega w_i\left(f-\frac{m+M}{2}\right)d\mu\right| &\leq \frac{1}{\int_\Omega w d\mu}\sum_{i=1}^n\int_\Omega w_i\left|f-\frac{m+M}{2}\right|d\mu \\
&= \frac{1}{\int_\Omega w d\mu}\int_\Omega w\left|f-\frac{m+M}{2}\right|d\mu
\end{aligned}
$$

and the last part of (2.17) is proved. □

If we use Lemma 2.3 for the discrete measure, we can state the following result:

Lemma 2.11. *Let $\Phi : [a,b] \to \mathbb{R}$ be a convex function on the interval of real numbers $[a,b]$, $z_i \in [a,b]$, $p_i \geq 0$, $i \in \{1,...,n\}$ and $\sum_{i=1}^{n} p_i = 1$. Then*

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n} p_i \Phi(z_i) - \Phi\left(\sum_{i=1}^{n} p_i z_i\right) \\
&\leq \left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right]\left(1 + \frac{2}{b-a}\left|\sum_{i=1}^{n} p_i z_i - \frac{a+b}{2}\right|\right) \\
&\leq 2\left[\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right].
\end{aligned} \tag{2.19}
$$

Using this lemma we can state and prove the following result as well:

Theorem 2.12. *Let $\Phi : [m,M] \to \mathbb{R}$ be a convex function, $f : \Omega \to [m,M]$ a μ-measurable function such that f, $\Phi \circ f \in L_w(\Omega,\mu)$. Then for any $\overline{w} \in \mathfrak{P}_n(w)$, $n \geq 2$ we have*

$$
\begin{aligned}
0 &\leq \psi(\Phi, f, \overline{w}) - \Phi\left(\frac{\int_\Omega f w d\mu}{\int_\Omega w d\mu}\right) \\
&\leq \left[\frac{\Phi(l(\overline{w})) + \Phi(L(\overline{w}))}{2} - \Phi\left(\frac{l(\overline{w}) + L(\overline{w})}{2}\right)\right] \\
&\times \left(1 + \frac{2}{L(\overline{w}) - l(\overline{w})}\left|\frac{1}{\int_\Omega w d\mu}\int_\Omega w f d\mu - \frac{l(\overline{w}) + L(\overline{w})}{2}\right|\right) \\
&\leq 2\left[\frac{\Phi(l(\overline{w})) + \Phi(L(\overline{w}))}{2} - \Phi\left(\frac{l(\overline{w}) + L(\overline{w})}{2}\right)\right],
\end{aligned} \tag{2.20}
$$

where $l(\overline{w})$, $L(\overline{w})$ are defined by (2.13).

Proof. If we write the first two inequalities in (2.11) for

$$
p_i = \frac{\int_\Omega w_i d\mu}{\int_\Omega w d\mu} > 0 \; z_i = \frac{\int_\Omega f w_i d\mu}{\int_\Omega w_i d\mu}, \; i \in \{1,...,n\}
$$

and for $a = l(\overline{w})$, $b = L(\overline{w})$ as above we have

$$
\begin{aligned}
0 &\leq \frac{1}{\int_{\Omega} w d\mu} \sum_{i=1}^{n} \left(\int_{\Omega} w_i d\mu \right) \Phi\left(\frac{\int_{\Omega} f w_i d\mu}{\int_{\Omega} w_i d\mu} \right) \\
&- \Phi\left(\sum_{i=1}^{n} \frac{\int_{\Omega} w_i d\mu}{\int_{\Omega} w d\mu} \cdot \frac{\int_{\Omega} f w_i d\mu}{\int_{\Omega} w_i d\mu} \right) \\
&\leq \left[\frac{\Phi(l(\overline{w})) + \Phi(L(\overline{w}))}{2} - \Phi\left(\frac{l(\overline{w}) + L(\overline{w})}{2} \right) \right] \\
&\times \left(1 + \frac{2}{L(\overline{w}) - l(\overline{w})} \left| \sum_{i=1}^{n} \frac{\int_{\Omega} w_i d\mu}{\int_{\Omega} w d\mu} \frac{\int_{\Omega} f w_i d\mu}{\int_{\Omega} w_i d\mu} - \frac{l(\overline{w}) + L(\overline{w})}{2} \right| \right) \\
&= \left[\frac{\Phi(l(\overline{w})) + \Phi(L(\overline{w}))}{2} - \Phi\left(\frac{l(\overline{w}) + L(\overline{w})}{2} \right) \right] \\
&\times \left(1 + \frac{2}{L(\overline{w}) - l(\overline{w})} \left| \frac{1}{\int_{\Omega} w d\mu} \int_{\Omega} \left(\sum_{i=1}^{n} w_i \right) f d\mu - \frac{l(\overline{w}) + L(\overline{w})}{2} \right| \right)
\end{aligned}
$$

that proves the required inequalities in (2.20). □

3 Discrete Case and Some Applications

Let $\Phi : [a, b] \to \mathbb{R}$ be a convex function on the interval of real numbers $[a, b]$, $x_k \in [a, b]$, $w_k > 0$, $k \in \{1, ..., m\}$. Let $w_{ki} > 0$ for $k \in \{1, ..., m\}$, $i \in \{1, ..., n\}$ with m, $n \geq 2$ and $\sum_{i=1}^{n} w_{ki} = w_k$ for any $k \in \{1, ..., m\}$.

We consider the functional associated with the matrix $W := \{w_{ki}\}_{k \in \{1,...,m\}, i \in \{1,...,n\}}$

$$
\psi(\Phi, \overline{x}, W) := \frac{1}{\sum_{k=1}^{m} w_k} \sum_{i=1}^{n} \Phi\left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}} \right) \sum_{k=1}^{m} w_{ki}, \tag{3.1}
$$

where $\overline{x} = (x_1, ..., x_m) \in [a, b]^m$.

Using the results from the previous section we have

$$
\frac{\sum_{k=1}^{m} \Phi(x_k) w_k}{\sum_{k=1}^{m} w_k} \geq \psi(\Phi, \overline{x}, W) \geq \Phi\left(\frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} \right), \tag{3.2}
$$

$$
\begin{aligned}
0 &\le \frac{\sum_{k=1}^{m}\Phi(x_k)\,w_k}{\sum_{k=1}^{m}w_k}-\psi(\Phi,\overline{x},W) \\
&\le \frac{\Phi'_-(b)-\Phi'_+(a)}{b-a} \\
&\times \frac{1}{\sum_{k=1}^{m}w_k}\sum_{i=1}^{n}\sum_{k=1}^{m}w_{ki}\left(b-\frac{\sum_{k=1}^{m}x_kw_{ki}}{\sum_{k=1}^{m}w_{ki}}\right)\left(\frac{\sum_{k=1}^{m}x_kw_{ki}}{\sum_{k=1}^{m}w_{ki}}-a\right) \\
&\le \frac{\Phi'_-(b)-\Phi'_+(a)}{b-a}\left(b-\frac{\sum_{k=1}^{m}x_kw_k}{\sum_{k=1}^{m}w_k}\right)\left(\frac{\sum_{k=1}^{m}x_kw_k}{\sum_{k=1}^{m}w_k}-a\right),
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
0 &\le \frac{\sum_{k=1}^{m}\Phi(x_k)\,w_k}{\sum_{k=1}^{m}w_k}-\psi(\Phi,\overline{x},W) \\
&\le \left[\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right] \\
&\times \left(1+\frac{2}{b-a}\sum_{i=1}^{n}\left|\frac{1}{\sum_{k=1}^{m}w_k}\sum_{k=1}^{m}w_{ki}\left(x_k-\frac{a+b}{2}\right)\right|\right) \\
&\le \left[\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right] \\
&\times \left(1+\frac{2}{b-a}\frac{1}{\sum_{k=1}^{m}w_k}\sum_{k=1}^{m}w_k\left|x_k-\frac{a+b}{2}\right|\right).
\end{aligned}
\tag{3.4}
$$

Define

$$
l(W):=\min_{i\in\{1,\dots,n\}}\left\{\frac{\sum_{k=1}^{m}w_{ki}x_k}{\sum_{k=1}^{m}w_{ki}}\right\},\;\; L(W):=\max_{i\in\{1,\dots,n\}}\left\{\frac{\sum_{k=1}^{m}w_{ki}x_k}{\sum_{k=1}^{m}w_{ki}}\right\}. \tag{3.5}
$$

Then we also have

$$
\begin{aligned}
0 &\le \psi(\Phi,\overline{x},W)-\Phi\left(\frac{\sum_{k=1}^{m}x_kw_k}{\sum_{k=1}^{m}w_k}\right) \\
&\le \frac{\Phi'_-(L(W))-\Phi'_+(l(W))}{L(W)-l(W)} \\
&\times \left(L(W)-\frac{\sum_{k=1}^{m}x_kw_k}{\sum_{k=1}^{m}w_k}\right)\left(\frac{\sum_{k=1}^{m}x_kw_k}{\sum_{k=1}^{m}w_k}-l(W)\right) \\
&\le \frac{1}{4}(L(W)-l(W))\left[\Phi'_-(L(W))-\Phi'_+(l(W))\right]
\end{aligned}
\tag{3.6}
$$

and

$$0 \le \psi(\Phi, \overline{x}, W) - \Phi\left(\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}\right) \tag{3.7}$$
$$\le \left[\frac{\Phi(l(W)) + \Phi(L(W))}{2} - \Phi\left(\frac{l(W) + L(W)}{2}\right)\right]$$
$$\times \left(1 + \frac{2}{L(W) - l(W)}\left|\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k} - \frac{l(W) + L(W)}{2}\right|\right)$$
$$\le 2\left[\frac{\Phi(l(W)) + \Phi(L(W))}{2} - \Phi\left(\frac{l(W) + L(W)}{2}\right)\right].$$

We consider the convex function $\Phi : (0, \infty) \to (0, \infty)$, $\Phi(t) = t^p$ with $p \in (-\infty, 0) \cup (1, \infty)$. Then

$$\psi(\Phi, \overline{x}, W) := \frac{1}{\sum_{k=1}^m w_k} \sum_{i=1}^n \left(\frac{\sum_{k=1}^m x_k w_{ki}}{\sum_{k=1}^m w_{ki}}\right)^p \sum_{k=1}^m w_{ki} \tag{3.8}$$
$$= \frac{1}{\sum_{k=1}^m w_k} \sum_{i=1}^n \left(\sum_{k=1}^m x_k w_{ki}\right)^p \left(\sum_{k=1}^m w_{ki}\right)^{1-p}$$

where $x_k > 0$, $w_k > 0$, $w_{ki} > 0$ for $k \in \{1, ..., m\}$, $i \in \{1, ..., n\}$ with $m, n \ge 2$ and $\sum_{i=1}^n w_{ki} = w_k$ for any $k \in \{1, ..., m\}$.

From (3.2) we have

$$\frac{\sum_{k=1}^m x_k^p w_k}{\sum_{k=1}^m w_k} \ge \frac{1}{\sum_{k=1}^m w_k} \sum_{i=1}^n \left(\sum_{k=1}^m x_k w_{ki}\right)^p \left(\sum_{k=1}^m w_{ki}\right)^{1-p} \tag{3.9}$$
$$\ge \left(\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}\right)^p.$$

If we set $a = \min\{x_k\}_{k \in \{1,...,m\}}$ and $b = \max\{x_k\}_{k \in \{1,...,m\}}$ then from (3.3) and (3.4) we have

$$0 \le \frac{\sum_{k=1}^m x_k^p w_k}{\sum_{k=1}^m w_k} - \frac{1}{\sum_{k=1}^m w_k} \sum_{i=1}^n \left(\sum_{k=1}^m x_k w_{ki}\right)^p \left(\sum_{k=1}^m w_{ki}\right)^{1-p} \tag{3.10}$$
$$\le p\frac{b^{p-1} - a^{p-1}}{b - a}$$
$$\times \frac{1}{\sum_{k=1}^m w_k} \sum_{i=1}^n \sum_{k=1}^m w_{ki}\left(b - \frac{\sum_{k=1}^m x_k w_{ki}}{\sum_{k=1}^m w_{ki}}\right)\left(\frac{\sum_{k=1}^m x_k w_{ki}}{\sum_{k=1}^m w_{ki}} - a\right)$$
$$\le p\frac{b^{p-1} - a^{p-1}}{b - a}\left(b - \frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k}\right)\left(\frac{\sum_{k=1}^m x_k w_k}{\sum_{k=1}^m w_k} - a\right),$$

and

$$0 \le \frac{\sum_{k=1}^{m} x_k^p w_k}{\sum_{k=1}^{m} w_k} - \frac{1}{\sum_{k=1}^{m} w_k} \sum_{i=1}^{n} \left(\sum_{k=1}^{m} x_k w_{ki} \right)^p \left(\sum_{k=1}^{m} w_{ki} \right)^{1-p} \tag{3.11}$$
$$\le \left[\frac{a^p + b^p}{2} - \left(\frac{a+b}{2} \right)^p \right]$$
$$\times \left(1 + \frac{2}{b-a} \sum_{i=1}^{n} \left| \frac{1}{\sum_{k=1}^{m} w_k} \sum_{k=1}^{m} w_{ki} \left(x_k - \frac{a+b}{2} \right) \right| \right)$$
$$\le \left[\frac{a^p + b^p}{2} - \left(\frac{a+b}{2} \right)^p \right]$$
$$\times \left(1 + \frac{2}{b-a} \frac{1}{\sum_{k=1}^{m} w_k} \sum_{k=1}^{m} w_k \left| x_k - \frac{a+b}{2} \right| \right).$$

If $l(W)$ and $L(W)$ are defined as in (3.5), then from (3.6) and (3.7) we also have

$$0 \le \frac{1}{\sum_{k=1}^{m} w_k} \sum_{i=1}^{n} \left(\sum_{k=1}^{m} x_k w_{ki} \right)^p \left(\sum_{k=1}^{m} w_{ki} \right)^{1-p} - \left(\frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} \right)^p \tag{3.12}$$
$$\le p \frac{(L(W))^{p-1} - (l(W))^{p-1}}{L(W) - l(W)}$$
$$\times \left(L(W) - \frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} \right) \left(\frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} - l(W) \right)$$
$$\le \frac{1}{4} (L(W) - l(W)) \left[(L(W))^{p-1} - (l(W))^{p-1} \right]$$

and

$$0 \le \frac{1}{\sum_{k=1}^{m} w_k} \sum_{i=1}^{n} \left(\sum_{k=1}^{m} x_k w_{ki} \right)^p \left(\sum_{k=1}^{m} w_{ki} \right)^{1-p} - \left(\frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} \right)^p \tag{3.13}$$
$$\le \left[\frac{(l(W))^p + (L(W))^p}{2} - \left(\frac{l(W) + L(W)}{2} \right)^p \right]$$
$$\times \left(1 + \frac{2}{L(W) - l(W)} \left| \frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} - \frac{l(W) + L(W)}{2} \right| \right)$$
$$\le 2 \left[\frac{(l(W))^p + (L(W))^p}{2} - \left(\frac{l(W) + L(W)}{2} \right)^p \right].$$

Further on, consider the convex function $\Phi : (0, \infty) \to (0, \infty)$, $\Phi(t) = -\ln t$ then from (3.2) for $x_k > 0$, $w_k > 0$, $w_{ki} > 0$ for $k \in \{1, ..., m\}$, $i \in \{1, ..., n\}$ with m,

$n \geq 2$ and $\sum_{i=1}^{n} w_{ki} = w_k$ for any $k \in \{1, ..., m\}$ we get

$$\left(\prod_{k=1}^{m} x_k^{w_k}\right)^{\frac{1}{\sum_{k=1}^{m} w_k}} \leq \left(\prod_{i=1}^{n}\left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}}\right)^{\sum_{k=1}^{m} w_{ki}}\right)^{\frac{1}{\sum_{k=1}^{m} w_k}} \leq \frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k}. \tag{3.14}$$

By (3.3) and (3.4) we have

$$1 \leq \frac{\left(\prod_{i=1}^{n}\left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}}\right)^{\sum_{k=1}^{m} w_{ki}}\right)^{\frac{1}{\sum_{k=1}^{m} w_k}}}{\left(\prod_{k=1}^{m} x_k^{w_k}\right)^{\frac{1}{\sum_{k=1}^{m} w_k}}}$$
$$\leq \exp\left[\frac{1}{ba\sum_{k=1}^{m} w_k}\sum_{i=1}^{n}\sum_{k=1}^{m} w_{ki}\left(b - \frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}}\right)\left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}} - a\right)\right]$$
$$\leq \exp\left[\frac{1}{ab}\left(b - \frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k}\right)\left(\frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} - a\right)\right] \tag{3.15}$$

and

$$1 \leq \frac{\left(\prod_{i=1}^{n}\left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}}\right)^{\sum_{k=1}^{m} w_{ki}}\right)^{\frac{1}{\sum_{k=1}^{m} w_k}}}{\left(\prod_{k=1}^{m} x_k^{w_k}\right)^{\frac{1}{\sum_{k=1}^{m} w_k}}} \tag{3.16}$$
$$\leq \left(\frac{a+b}{2\sqrt{ab}}\right)^{1+\frac{2}{b-a}\sum_{i=1}^{n}\left|\frac{1}{\sum_{k=1}^{m} w_k}\sum_{k=1}^{m} w_{ki}\left(x_k - \frac{a+b}{2}\right)\right|}$$
$$\leq \left(\frac{a+b}{2\sqrt{ab}}\right)^{1+\frac{2}{b-a}\frac{1}{\sum_{k=1}^{m} w_k}\sum_{k=1}^{m} w_k\left|x_k - \frac{a+b}{2}\right|}.$$

From (3.6) and (3.7) we also have

$$1 \leq \frac{\frac{1}{\sum_{k=1}^{m} w_k}\sum_{k=1}^{m} x_k w_k}{\left(\prod_{i=1}^{n}\left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}}\right)^{\sum_{k=1}^{m} w_{ki}}\right)^{\frac{1}{\sum_{k=1}^{m} w_k}}} \tag{3.17}$$

$$\leq \exp\left[\frac{1}{L(W)\,l(W)}\left(L(W) - \frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k}\right)\left(\frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} - l(W)\right)\right]$$
$$\leq \frac{1}{4}\frac{\left(L(W) - l(W)\right)^2}{L(W)\,l(W)}$$

and

$$1 \leq \frac{\frac{1}{\sum_{k=1}^{m} w_k} \sum_{k=1}^{m} x_k w_k}{\left(\prod_{i=1}^{n} \left(\frac{\sum_{k=1}^{m} x_k w_{ki}}{\sum_{k=1}^{m} w_{ki}}\right)^{\sum_{k=1}^{m} w_{ki}}\right)^{\frac{1}{\sum_{k=1}^{m} w_k}}} \tag{3.18}$$

$$\leq \left(\frac{l(W)+L(W)}{2\sqrt{l(W)L(W)}}\right)^{1+\frac{2}{L(W)-l(W)}\left|\frac{\sum_{k=1}^{m} x_k w_k}{\sum_{k=1}^{m} w_k} - \frac{l(W)+L(W)}{2}\right|}$$

$$\leq \left(\frac{l(W)+L(W)}{2\sqrt{l(W)L(W)}}\right)^{2}.$$

4 Applications for Univariate Functions

Let $\Phi : [m, M] \to \mathbb{R}$ be a convex function and $f : [0, \pi] \to [m, M]$ an integrable function. Since $\sin^2 t + \cos^2 t = 1$ for any $t \in [0, \pi]$ then $\overline{w} = (\sin^2, \cos^2)$ is a partition of the unity. We then have

$$\begin{aligned}
&\psi(\Phi, f, \overline{w}) \\
&:= \frac{1}{\pi}\left[\Phi\left(\frac{\int_0^{\pi} f(t)\sin^2 t dt}{\int_0^{\pi} \sin^2 t dt}\right)\int_0^{\pi} \sin^2 t dt + \Phi\left(\frac{\int_0^{\pi} f(t)\cos^2 t dt}{\int_0^{\pi} \cos^2 t dt}\right)\int_0^{\pi} \cos^2 t dt\right] \\
&= \frac{1}{2}\left[\Phi\left(\frac{2}{\pi}\int_0^{\pi} f(t)\sin^2 t dt\right) + \Phi\left(\frac{2}{\pi}\int_0^{\pi} f(t)\cos^2 t dt\right)\right].
\end{aligned}$$

By the inequality (2.1) we have

$$\begin{aligned}
\frac{\int_0^{\pi} (\Phi \circ f)(t) dt}{\pi} &\geq \frac{1}{2}\left[\Phi\left(\frac{2}{\pi}\int_0^{\pi} f(t)\sin^2 t dt\right) + \Phi\left(\frac{2}{\pi}\int_0^{\pi} f(t)\cos^2 t dt\right)\right] \\
&\geq \Phi\left(\frac{\int_0^{\pi} f(t) dt}{\pi}\right),
\end{aligned} \tag{4.1}$$

while from (2.10) we have

$$\begin{aligned}
0 \leq &\frac{\int_0^{\pi} (\Phi \circ f)(t) dt}{\pi} \\
&- \frac{1}{2}\left[\Phi\left(\frac{2}{\pi}\int_0^{\pi} f(t)\sin^2 t dt\right) + \Phi\left(\frac{2}{\pi}\int_0^{\pi} f(t)\cos^2 t dt\right)\right]
\end{aligned} \tag{4.2}$$

$$\leq \frac{1}{2}\frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}$$
$$\left[\left(M-\frac{2}{\pi}\int_{0}^{\pi}f(t)\sin^{2}tdt\right)\left(\frac{2}{\pi}\int_{0}^{\pi}f(t)\sin^{2}tdt-m\right)\right.$$
$$\left.+\left(M-\frac{2}{\pi}\int_{0}^{\pi}f(t)\cos^{2}tdt\right)\left(\frac{2}{\pi}\int_{0}^{\pi}f(t)\cos^{2}tdt-m\right)\right]$$
$$\leq \frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m}\left(M-\frac{\int_{0}^{\pi}f(t)\,dt}{\pi}\right)\left(\frac{\int_{0}^{\pi}f(t)\,dt}{\pi}-m\right).$$

Now, let

$$\begin{aligned} l(\overline{w}) \quad &: = \frac{2}{\pi}\min\left\{\int_{0}^{\pi}f(t)\sin^{2}tdt, \int_{0}^{\pi}f(t)\cos^{2}tdt\right\} \\ &= \frac{1}{\pi}\left[\int_{0}^{\pi}f(t)\,dt-\left|\int_{0}^{\pi}f(t)\cos 2tdt\right|\right] \end{aligned}$$

and

$$\begin{aligned} L(\overline{w}) \quad &: = \frac{2}{\pi}\max\left\{\int_{0}^{\pi}f(t)\sin^{2}tdt, \int_{0}^{\pi}f(t)\cos^{2}tdt\right\} \\ &= \frac{1}{\pi}\left[\int_{0}^{\pi}f(t)\,dt+\left|\int_{0}^{\pi}f(t)\cos 2tdt\right|\right] \end{aligned}$$

then by (2.15) we have

$$0 \leq \frac{1}{2}\left[\Phi\left(\frac{2}{\pi}\int_{0}^{\pi}f(t)\sin^{2}tdt\right)+\Phi\left(\frac{2}{\pi}\int_{0}^{\pi}f(t)\cos^{2}tdt\right)\right] \tag{4.3}$$
$$-\Phi\left(\frac{\int_{0}^{\pi}f(t)\,dt}{\pi}\right)$$
$$\leq \frac{1}{2\pi}\left|\int_{0}^{\pi}f(t)\cos 2tdt\right|$$
$$\times\left[\Phi'_{-}\left(\frac{1}{\pi}\left[\int_{0}^{\pi}f(t)\,dt+\left|\int_{0}^{\pi}f(t)\cos 2tdt\right|\right]\right)\right.$$
$$\left.-\Phi'_{+}\left(\frac{1}{\pi}\left[\int_{0}^{\pi}f(t)\,dt-\left|\int_{0}^{\pi}f(t)\cos 2tdt\right|\right]\right)\right].$$

Similar inequalities may be obtained if someone would also use the inequalities (2.17) and (2.20). The details are omitted.

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Hilfer and Hilfer-Hadamard Fractional Differential Equations with Random Effects

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Abstract: This paper deals with some existence and Ulam stability results for some functional differential equations of Hilfer and Hilfer-Hadamard type. An application is made of Itoh's random fixed point theorem for the existence of random solutions. Next we prove that our problems are generalized Ulam-Hyers-Rassias stable.

Keywords: Functional Random differential equation, left-sided mixed Riemann-Liouville integral of fractional order, left-sided mixed Hadamard integral of fractional order, Hilfer fractional derivative, Hilfer-Hadamard fractional derivative, existence, Ulam stability, random solution, fixed point.

MSC2010: 26A33

1 Introduction

The fractional calculus deals with extensions of derivatives and integrals to non-integer orders. It represents a powerful tool in applied mathematics to study many problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [14, 29]. For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs of Abbas *et al.* [7, 8, 9], Samko *et al.* [28], Kilbas *et al.* [22] and Zhou [33, 34], the papers by Abbas *et al.* [1, 4, 5, 10, 11], and the references therein.

The stability of functional equations was originally raised by Ulam [31]). next by Hyers [15]. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [25] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations; one can see the monographs of [9, 17], and the papers of Abbas *et al.* [1, 2, 3, 4, 6, 10, 11], Petru *et al.* [23], and Rus [26, 27] discussed the Ulam-Hyers stability for operatorial equations and inclusions. More details from historical point of view, and recent developments of such stabilities are reported in [18, 26].

Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative; see [12, 13, 14, 19, 30, 32]. Motivated by the above papers, in this article we discuss the existence and the Ulam stability of solutions for the following problem of Random Hilfer fractional differential equations of the form

$$\begin{cases} (D_0^{\alpha,\beta}u)(t,w) = f(t,u(t,w),w);\ t \in I := [0,T], \\ \\ (I_0^{1-\gamma}u)(t,w)|_{t=0} = \phi(w), \end{cases} \quad w \in \Omega, \tag{1.1}$$

where $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha+\beta-\alpha\beta,\ T > 0$, $(\Omega,\mathcal{A})$ is a measurable space, $\phi : \Omega \to \mathbb{R}$ is a measurable function, $f : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, $I_0^{1-\gamma}$ is the left-sided mixed Riemann-Liouville integral of order $1-\gamma$, and $D_0^{\alpha,\beta}$ is the Hilfer fractional derivative of order α and type β.

Next, we consider the following problem of random Hilfer-Hadamard fractional differential equations of the form

$$\begin{cases} ({}^HD_1^{\alpha,\beta}u)(t,w) = g(t,u(t,w),w);\ t \in [1,T], \\ \\ ({}^HI_1^{1-\gamma}u)(1,w) = \phi_0(w), \end{cases} \quad w \in \Omega, \tag{1.2}$$

where $\alpha \in (0,1),\ \beta \in [0,1],\ \gamma = \alpha+\beta-\alpha\beta,\ T > 1,\ \phi_0 : \Omega \to \mathbb{R}$ is a measurable function, $g : [1,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, ${}^HI_1^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1-\gamma$, and ${}^HD_1^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order α and type β.

The present paper initiates the Ulam stability for random differential equations involving Hilfer and Hilfer-Hadamard fractional derivatives.

2 Preliminaries

Let C be the Banach space of all continuous functions v from I into $\mathbb{R}$ with the supremum (uniform) norm

$$\|v\|_\infty := \sup_{t\in I} |v(t)|.$$

As usual, $AC(I)$ denotes the space of absolutely continuous functions from I into $\mathbb{R}$. We denote by $AC^1(I)$ the space defined by

$$AC^1(I) := \{w : I \to \mathbb{R} : \frac{d}{dt}w(t) \in AC(I)\}.$$

By $L^1(I)$, we denote the space of Lebesgue-integrable functions $v : I \to \mathbb{R}$ with the norm

$$\|v\|_1 = \int_0^T |v(t)|dt.$$

Let $L^\infty(I)$ be the Banach space of measurable functions $u : I \to \mathbb{R}$ which are essentially bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf\{c > 0 : |u(t)| \leq c, \ a.e. \ t \in I\}.$$

By $C_\gamma(I)$ and $C^1_\gamma(I)$, we denote the weighted spaces of continuous functions defined by

$$C_\gamma(I) = \{w : (0, T] \to \mathbb{R} : t^{1-\gamma}w(t) \in C\},$$

with the norm

$$\|w\|_{C_\gamma} := \sup_{t \in I} |t^{1-\gamma}w(t)|,$$

and

$$C^1_\gamma(I) = \{w \in C : \frac{dw}{dt} \in C_\gamma\},$$

with the norm

$$\|w\|_{C^1_\gamma} := \|w\|_\infty + \|w'\|_{C_\gamma}.$$

Throughout this paper, we denote $\|w\|_{C_\gamma}$ by $\|w\|_C$.

Definition 2.1. A function $T : \Omega \times \mathbb{R} \to \mathbb{R}$ is called jointly measurable if $T(\cdot, u)$ is measurable for all $u \in \mathbb{R}$ and $T(w, \cdot)$ is continuous for all $w \in \Omega$.

Definition 2.2. A function $f : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ is called random Carathéodory if the following conditions are satisfied:

(i) The map $(t, w) \to f(t, u, w)$ is jointly measurable for all $u \in \mathbb{R}$, and

(ii) The map $u \to f(t, u, w)$ is continuous for all $t \in I$ and $w \in \Omega$.

Let E be a Banach space and $T : \Omega \times E \to E$ be a mapping. Then T is called a random operator if $T(w, u)$ is measurable in w for all $u \in E$ and it expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on E. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w,u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [16].

Definition 2.3. Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w,y) : w \in \Omega,\ y \in C(w)\} \to Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w,y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called random (stochastic) fixed point of T if for almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Now, we give some results and properties of fractional calculus.

Definition 2.4. [8, 22, 28] The left-sided mixed Riemann-Liouville integral of order $r > 0$ of a function $w \in L^1(I)$ is defined by

$$(I_0^r w)(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} w(s)ds;\ for\ a.e.\ t \in I,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt;\ \xi > 0.$$

Notice that for all $r, r_1, r_2 > 0$ and each $w \in C$, we have $I_0^r w \in C$, and

$$(I_0^{r_1} I_0^{r_2} w)(t) = (I_0^{r_1+r_2} w)(t);\ for\ a.e.\ t \in I.$$

Definition 2.5. [8, 22, 28] The Riemann-Liouville fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned} (D_0^r w)(t) &= \left(\frac{d}{dt} I_0^{1-r} w\right)(t) \\ &= \frac{1}{\Gamma(1-r)} \frac{d}{dt} \int_0^t (t-s)^{-r} w(s)ds;\ for\ a.e.\ t \in I. \end{aligned}$$

Let $r \in (0,1]$, $\gamma \in [0,1)$ and $w \in C_{1-\gamma}(I)$. Then the following expression leads to the left inverse operator as follows.

$$(D_0^r I_0^r w)(t) = w(t);\ for\ all\ t \in (0,T].$$

Moreover, if $I_0^{1-r}w \in C^1_{1-\gamma}(I)$, then the following composition is proved in [28]

$$(I_0^r D_0^r w)(t) = w(t) - \frac{(I_0^{1-r}w)(0^+)}{\Gamma(r)} t^{r-1};\ for\ all\ t \in (0,T].$$

Definition 2.6. [8, 22, 28] The Caputo fractional derivative of order $r \in (0,1]$ of a function $w \in L^1(I)$ is defined by

$$\begin{aligned} ({}^cD_0^r w)(t) &= \left(I_0^{1-r}\frac{d}{dt}w\right)(t) \\ &= \frac{1}{\Gamma(1-r)}\int_0^t (t-s)^{-r}\frac{d}{ds}w(s)ds;\ for\ a.e.\ t \in I. \end{aligned}$$

In [14], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and the Caputo derivatives as specific cases (see also [19, 30].

Definition 2.7. (Hilfer derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $w \in L^1(I)$, $I_0^{(1-\alpha)(1-\beta)}w \in AC^1(I)$. The Hilfer fractional derivative of order α and type β of w is defined as

$$(D_0^{\alpha,\beta}w)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{(1-\alpha)(1-\beta)}w\right)(t);\ for\ a.e.\ t \in I. \tag{2.1}$$

Properties. Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, and $w \in L^1(I)$.
1. The operator $(D_0^{\alpha,\beta}w)(t)$ can be written as

$$(D_0^{\alpha,\beta}w)(t) = \left(I_0^{\beta(1-\alpha)}\frac{d}{dt}I_0^{1-\gamma}w\right)(t) = \left(I_0^{\beta(1-\alpha)}D_0^{\gamma}w\right)(t);\ for\ a.e.\ t \in I.$$

Moreover, the parameter γ satisfies

$$\gamma \in (0,1],\ \gamma \geq \alpha,\ \gamma > \beta,\ 1-\gamma < 1-\beta(1-\alpha).$$

2. The generalization (2.1) for $\beta = 0$, coincides with the Riemann-Liouville derivative and for $\beta = 1$ with the Caputo derivative.

$$D_0^{\alpha,0} = D_0^{\alpha},\ and\ D_0^{\alpha,1} = {}^cD_0^{\alpha}.$$

3. If $D_0^{\beta(1-\alpha)}w$ exists and in $L^1(I)$, then

$$(D_0^{\alpha,\beta}I_0^{\alpha}w)(t) = (I_0^{\beta(1-\alpha)}D_0^{\beta(1-\alpha)}w)(t);\ for\ a.e.\ t \in I.$$

Furthermore, if $w \in C_\gamma(I)$ and $I_0^{1-\beta(1-\alpha)}w \in C^1_\gamma(I)$, then

$$(D_0^{\alpha,\beta}I_0^{\alpha}w)(t) = w(t);\ for\ a.e.\ t \in I.$$

4. If $D_0^\gamma w$ exists and in $L^1(I)$, then

$$(I_0^\alpha D_0^{\alpha,\beta} w)(t) = (I_0^\gamma D_0^\gamma w)(t) = w(t) - \frac{I_0^{1-\gamma}(0^+)}{\Gamma(\gamma)} t^{\gamma-1};\ for\ a.e.\ t \in I.$$

Lemma 2.8. *Let $h \in C_\gamma(I)$. Then the linear Cauchy problem*

$$\begin{cases} (D_0^{\alpha,\beta} u)(t) = h(t);\ t \in I, \\ \\ (I_0^{1-\gamma} u)(t)|_{t=0} = \phi, \end{cases}$$

has a unique solution $u \in L^1(I)$ given by

$$u(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha h)(t).$$

From the above lemma, we concluded the following lemma

Lemma 2.9. *Let $f : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ be such that $f(\cdot, u(\cdot, w), w) \in C_\gamma$ for all $w \in \Omega$, and any $u(w) \in C_\gamma$. Then problem (1.1) is equivalent to the problem of the solutions of the integral equation*

$$u(t, w) = \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} + (I_0^\alpha f(\cdot, u(\cdot, w), w))(t);\ w \in \Omega.$$

Now, we consider the Ulam stability for the problem (1.1). Let $\epsilon > 0$ and $\Phi : I \times \Omega \to [0, \infty)$ be a continuous function. We consider the following inequalities

$$|(D_0^{\alpha,\beta} u)(t, w) - f(t, u(t, w), w)| \le \epsilon;\ t \in I,\ w \in \Omega. \tag{2.2}$$

$$|(D_0^{\alpha,\beta} u)(t, w) - f(t, u(t, w), w)| \le \Phi(t, w);\ t \in I,\ w \in \Omega. \tag{2.3}$$

$$|(D_0^{\alpha,\beta} u)(t, w) - f(t, u(t, w), w)| \le \epsilon\Phi(t, w);\ t \in I,\ w \in \Omega. \tag{2.4}$$

Definition 2.10. *[8, 26]* The problem *(1.1)* is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.2)* there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t, w) - v(t, w)| \le \epsilon c_f;\ t \in I,\ w \in \Omega.$$

Definition 2.11. *[8, 26]* The problem *(1.1)* is generalized Ulam-Hyers stable if there exists $c_f : C([0, \infty), [0, \infty))$ with $c_f(0) = 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.2)* there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t, w) - v(t, w)| \le c_f(\epsilon);\ t \in I,\ w \in \Omega.$$

Definition 2.12. *[8, 26]* The problem *(1.1)* is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.4)* there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t,w) - v(t,w)| \leq \epsilon c_{f,\Phi}\Phi(t,w);\ t \in I,\ w \in \Omega.$$

Definition 2.13. *[8, 26]* The problem *(1.1)* is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each random solution $u : \Omega \to C_\gamma$ of the inequality *(2.3)*, there exists a random solution $v : \Omega \to C_\gamma$ of *(1.1)* with

$$|u(t,w) - v(t,w)| \leq c_{f,\Phi}\Phi(t,w);\ t \in I,\ w \in \Omega.$$

Remark 2.14. It is clear that

(i) Definition 2.10 $\Rightarrow$ Definition 2.11,

(ii) Definition 2.12 $\Rightarrow$ Definition 2.13,

(iii) Definition 2.12 for $\Phi(\cdot,\cdot) = 1$ $\Rightarrow$ Definition 2.10.

One can have similar remarks for the inequalities (2.2) and (2.4).

In the sequel, we employ the following random fixed point theorem.

Theorem 2.15. *(Itoh [16]) Let X be a non-empty, closed convex bounded subset of the separable Banach space E and let $N : \Omega \times X \to X$ be a compact and continuous random operator. Then the random equation $N(w)u = u$ has a random solution.*

3 Hilfer fractional random differential equations

In this section, we are concerned with the existence and the Ulam-Hyers-Rassias stability for problem (1.1). Let us start by defining what we mean by a random solution of the problem (1.1).

Definition 3.1. By a random solution of the problem (1.1) we mean a measurable function $u : \Omega \to C_\gamma$ that satisfies the condition $(I_0^{1-\gamma}u)(0^+, w) = \phi(w)$, and the equation $(D_0^{\alpha,\beta}u)(t,w) = f(t, u(t,w), w)$ on $I \times \Omega$.

The following hypotheses will be used in the sequel.

(H_1) The function f is random Carathéodory on $I \times \mathbb{R} \times \Omega$,

(H_2) There exist a measurable and bounded function $p : \Omega \to L^\infty(I, [0, \infty))$, such that

$$|f(t, u, w)| \leq \frac{p(t, w)|u|}{1 + |u|}; \; for\ a.e.\ t \in I,\ and\ each\ u \in \mathbb{R},\ w \in \Omega.$$

Set

$$p^* = \sup_{w \in \Omega} \|p(w)\|_{L^\infty},\ and\ \phi^* = \sup_{w \in \Omega} |\phi(w)|.$$

Now, we shall prove the following theorem concerning the existence of random solutions of problem (1.1).

Theorem 3.2. *Assume that the hypotheses* (H_1) *and* (H_2) *hold. Then the problem (1.1) has at least one random solution defined on* $I \times \Omega$.

Proof. Define a mapping $N : \Omega \times C_\gamma \to C_\gamma$ by:

$$(N(w)u)(t) = \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma - 1} + \int_0^t (t - s)^{\alpha - 1} \frac{f(s, u(s, w), w)}{\Gamma(\alpha)} ds. \tag{3.1}$$

The map ϕ is measurable for all $w \in \Omega$. Again, as the indefinite integral is continuous on I, then $N(w)$ defines a mapping $N : \Omega \times C_\gamma \to C_\gamma$. Thus u is a random solution for the problem (1.1) if and only if $u = N(w)u$.
Next, for any $u \in C_\gamma$, and each $t \in I$ and $w \in \omega$, we have

$$\begin{aligned} |t^{1-\gamma}(N(w)u)(t)| &\leq \frac{|\phi(w)|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u(s, w), w)| ds \\ &\leq \frac{|\phi(w)|}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s, w) ds \\ &\leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^* T^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}. \end{aligned}$$

Thus

$$\|N(w)u\|_C \leq \frac{\phi^*}{\Gamma(\gamma)} + \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)} := R. \tag{3.2}$$

This proves that $N(w)$ transforms the ball $B_R := B(0, R) = \{u \in C_\gamma : \|u\|_C \leq R\}$ into itself. We shall show that the operator $N : \Omega \times B_R \to B_R$ satisfies all the assumptions of Theorem 2.15. The proof will be given in several steps.

Step 1. $N(w)$ is a random operator on $\Omega \times B_R$ into B_R.
Since $f(t,u,w)$ is random Carathéodory, the map $w \to f(t,u,w)$ is measurable in view of Definition 2.1. Similarly, the product $(t-s)^{\alpha-1}f(s,u(s,w),w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w \mapsto \frac{\phi(w)}{\Gamma(\gamma)}t^{\gamma-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s,w),w)ds,$$

is measurable. As a result, $N(w)$ is a random operator on $\Omega \times B_R$ into B_R.

Step 2. $N(w)$ *is continuous in* u.
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_R. Then, for each $t \in I$, and $w \in \Omega$, we have

$$\begin{aligned} &|t^{1-\gamma}(N(w)u_n)(t) - t^{1-\gamma}(N(w)u)(t)| \\ &\le \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|f(s,u_n(s,w),w) - f(s,u(s,w),w)|ds. \end{aligned} \tag{3.3}$$

Since $u_n \to u$ as $n \to \infty$ and f is random Carathéeodory, then by the Lebesgue dominated convergence theorem, equation (3.3) implies

$$\|N(w)u_n - N(w)u\|_C \to 0 \quad \text{as } n \to \infty.$$

Step 3. $N(w)B_R$ *is uniformly bounded.*
This is clear since $N(w)B_R \subset B_R$ and B_R is bounded.

Step 4. $N(w)B_R$ *is equicontinuous.*
Let $t_1, t_2 \in I$, $t_1 < t_2$ and let $u \in B_R$. Then, for each $w \in \Omega$, we have

$$\begin{aligned} &|t_2^{1-\gamma}(N(w)u)(t_2) - t_1^{1-\gamma}(N(w)u)(t_1)| \\ &\le \left| t_2^{1-\gamma}\int_0^{t_2}(t_2-s)^{\alpha-1}\frac{f(s,u(s,w),w)}{\Gamma(\alpha)}ds - t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}\frac{f(s,u(s,w),w)}{\Gamma(\alpha)}ds \right| \\ &\le t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{|f(s,u(s,w),w)|}{\Gamma(\alpha)}ds \\ &\quad + \int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{|f(s,u(s,w),w)|}{\Gamma(\alpha)}ds \\ &\le t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\frac{p(s,w)}{\Gamma(\alpha)}ds \\ &\quad + \int_0^{t_1}|t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}|\frac{p(s.w)}{\Gamma(\alpha)}ds. \end{aligned}$$

Thus, we get

$$|t_2^{1-\gamma}(N(w)u)(t_2) - t_1^{1-\gamma}(N(w)u)(t_1)| \leq \frac{p^* T^{1-\gamma+\alpha}}{\Gamma(1+\alpha)}(t_2 - t_1)^\alpha$$
$$+ \frac{p^*}{\Gamma(\alpha)} \int_0^{t_1} |t_2^{1-\gamma}(t_2 - s)^{\alpha-1} - t_1^{1-\gamma}(t_1 - s)^{\alpha-1}|ds.$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of steps 1 to 4 together with the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \times B_R \to B_R$ is continuous and compact. From an application of Theorem 2.15, we deduce that the operator equation $N(w)u = u$ has a random solution. This implies that the random problem (1.1) has a random solution. □

Now, we are concerned with the generalized Ulam-Hyers-Rassias stability of our problem (1.1).

Theorem 3.3. *Assume that the hypotheses* (H_1), (H_2) *and the following hypotheses hold.*

(H_3) *There exists* $\lambda_\Phi > 0$ *such that for each* $t \in I$, *and* $w \in \Omega$, *we have*

$$(I_0^\alpha \Phi)(t, w) \leq \lambda_\Phi \Phi(t, w),$$

(H_4) *There exists* $q \in C(I, [0, \infty))$ *such that for each* $t \in I$, *and* $w \in \Omega$, *we have*

$$p(t, w) \leq q(t)\Phi(t, w).$$

Then the problem (1.1) is generalized Ulam-Hyers-Rassias stable.

Proof. Consider the operator N defined in (3.1). Let u be a random solution of the inequality (2.3), and let us assume that v is a random solution of problem (1.1). Thus, we have

$$v(t, w) = \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} + \int_0^t (t - s)^{\alpha-1} \frac{f(s, v(s, w), w)}{\Gamma(\alpha)} ds.$$

From the inequality (2.3) for each $t \in I$, and $w \in \Omega$, we have

$$\left| u(t, w) - \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} - \int_0^t (t - s)^{\alpha-1} \frac{f(s, u(s, w), w)}{\Gamma(\alpha)} ds \right| \leq (I_0^\alpha \Phi)(t, w).$$

Set

$$q^* = \sup_{t\in I} q(t).$$

From hypotheses (H_3) and (H_4), for each $t \in I$, and $w \in \Omega$, we get

$$\begin{aligned}
|u(t,w) - v(t,w)| &\leq \left| u(t,w) - \frac{\phi(w)}{\Gamma(\gamma)} t^{\gamma-1} - \int_0^t (t-s)^{\alpha-1} \frac{f(s,u(s,w),w)}{\Gamma(\alpha)} ds \right| \\
&+ \int_0^t (t-s)^{\alpha-1} \frac{|f(s,u(s,w),w) - f(s,v(s,w),w)|}{\Gamma(\alpha)} ds \\
&\leq (I_0^\alpha \Phi)(t,w) + \int_0^t (t-s)^{\alpha-1} \frac{2q^*\Phi(s,w)}{\Gamma(\alpha)} ds \\
&\leq (I_0^\alpha \Phi)(t) + 2q^*(I_0^\alpha \Phi)(t,w) \\
&\leq [1+2q^*]\lambda_\phi \Phi(t,w) \\
&:= c_{f,\Phi}\Phi(t,w).
\end{aligned}$$

Hence, the problem (1.1) is generalized Ulam-Hyers-Rassias stable. □

In the sequel, we will use of the following Theorem.

Theorem 3.4. *[27] Let (Ω, d) be a generalized complete metric space and $\Theta : \Omega \to \Omega$ a strictly contractive operator with a Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Theta^{k+1}x, \Theta^k x) < \infty$ for any $x \in \Omega$, then the following propositions hold true:*

(A) *The sequence $(\Theta^k x)_{n\in N}$ converges to a fixed point x^* of Θ;*

(B) *x^* is the unique fixed point of Θ in $\Omega^* = \{y \in \Omega \mid d(\Theta^k x, y) < \infty\}$;*

(C) *If $y \in \Omega^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(y, \Theta x)$.*

Let $X = X(I, \mathbb{R})$ be the metric space, with the metric

$$d(u,v) = \sup_{t\in I} \frac{t^{1-\gamma}\|u(t,w) - v(t,w)\|}{\Phi(t,w)}.$$

Theorem 3.5. *Assume that (H_3) and the following hypothesis hold.*

(H_5) *There exists $\varphi \in C(I, [0,\infty))$ such that for each $t \in I$, $w \in \Omega$, and all $u, v \in \mathbb{R}$, we have*

$$|f(t,u,w) - f(t,v,w)| \leq t^{1-\gamma}\varphi(t)\Phi(t,w)|u-v|.$$

If

$$L := T^{1-\gamma}\varphi^*\lambda_\phi < 1, \tag{3.4}$$

where $\varphi^* = \sup_{t\in I}\varphi(t)$, *then there exists a unique random solution* u_0 *of problem (1.1), and the problem (1.1) is generalized Ulam-Hyers-Rassias stable. Furthermore, we have*

$$|u(t,w) - u_0(t,w)| \leq \frac{\Phi(t,w)}{1-L},$$

for any solution u *of (2.3).*

Proof. Let N be the operator defined in (3.1). Apply Theorem 3.4, we have

$$\begin{aligned}
|(N(w)u)(t) - (N(w)v)(t)| &\leq \int_0^t (t-s)^{\alpha-1}\frac{|f(s,u(s,w),w) - f(s,v(s,w),w)|}{\Gamma(\alpha)}ds \\
&\leq \int_0^t (t-s)^{\alpha-1}\frac{\varphi(s)\Phi(s,w)|s^{1-\gamma}u(s,w) - s^{1-\gamma}v(s,w)|}{\Gamma(\alpha)}ds \\
&\leq \int_0^t (t-s)^{\alpha-1}\frac{\varphi^*\Phi(s,w)\|u(w) - v(w)\|_C}{\Gamma(\alpha)}ds \\
&\leq \varphi^*(I_0^\alpha\Phi)(t,w)\|u(w) - v(w)\|_C \\
&\leq \varphi^*\lambda_\phi\Phi(t)\|u(w) - v(w)\|_C.
\end{aligned}$$

Thus

$$|t^{1-\gamma}(N(w)u)(t) - t^{1-\gamma}(N(w)v)(t)| \leq T^{1-\gamma}\varphi^*\lambda_\phi\Phi(t,w)\|u(w) - v(w)\|_C.$$

Hence, we get

$$d(N(u),N(v)) \leq L\|u(w) - v(w)\|_C,$$

from which we conclude the theorem. □

4 Hilfer-Hadamard fractional random differential equations

Now, we are concerned with the existence and the Ulam-Hyers-Rassias stability for problem (1.2).

Set $C := C([1,T])$. Denote the weighted space of continuous functions defined by

$$C_{\gamma,\ln}([1,T]) = \{w(t) : (\ln t)^{1-\gamma}w(t) \in C\},$$

with the norm

$$\|w\|_{C_{\gamma,\ln}} := \sup_{t\in[1,T]} |(\ln t)^{1-r} w(t)|.$$

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [22] for a more detailed analysis.

Definition 4.1. [22] (Hadamard fractional integral). The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1,T])$, is defined as

$$({}^H I_1^q g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds,$$

provided the integral exists.

Example 4.2. Let $0 < q < 1$. Then

$${}^H I_1^q \ln t = \frac{1}{\Gamma(2+q)} (\ln t)^{1+q}, \ \ for\ a.e.\ t \in [0,e].$$

Set

$$\delta = x\frac{d}{dx}, \ q > 0, \ n = [q]+1,$$

and

$$AC_\delta^n := \{u : [1,T] \to E : \delta^{n-1}[u(x)] \in AC(I)\}.$$

Analogous to the Riemann-Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way:

Definition 4.3. [22] (Hadamard fractional derivative). The Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as

$$({}^H D_1^q w)(x) = \delta^n ({}^H I_1^{n-q} w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^H D_1^q w)(x) = \delta ({}^H I_1^{1-q} w)(x).$$

Example 4.4. Let $0 < q < 1$. Then

$${}^H D_1^q \ln t = \frac{1}{\Gamma(2-q)} (\ln t)^{1-q}, \ \ for\ a.e.\ t \in [0,e].$$

It has been proved (see e.g. Kilbas [[21], Theorem 4.8]) that in the space $L^1(I,E)$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$({}^H D_1^q)({}^H I_1^q w)(x) = w(x).$$

From Theorem 2.3 of [22], we have

$$({}^H I_1^q)({}^H D_1^q w)(x) = w(x) - \frac{({}^H I_1^{1-q} w)(1)}{\Gamma(q)}(\ln x)^{q-1}.$$

Analogous to the Hadamard fractional calculus, the Caputo-Hadamard fractional derivative is defined in the following way:

Definition 4.5. (Caputo-Hadamard fractional derivative). The Caputo-Hadamard fractional derivative of order $q > 0$ applied to the function $w \in AC_\delta^n$ is defined as

$$({}^{Hc}D_1^q w)(x) = ({}^H I_1^{n-q} \delta^n w)(x).$$

In particular, if $q \in (0,1]$, then

$$({}^{Hc}D_1^q w)(x) = ({}^H I_1^{1-q} \delta w)(x).$$

From the Hadamard fractional integral, the Hilfer-Hadamard fractional derivative (introduced for the first time in [24]) is defined in the following way:

Definition 4.6. (Hilfer-Hadamard fractional derivative). Let $\alpha \in (0,1)$, $\beta \in [0,1]$, $\gamma = \alpha + \beta - \alpha\beta$, $w \in L^1(I)$, and ${}^H I_1^{(1-\alpha)(1-\beta)} w \in AC^1(I)$. The Hilfer-Hadamard fractional derivative of order α and type β applied to the function w is defined as

$$\begin{aligned}({}^H D_1^{\alpha,\beta} w)(t) &= \left({}^H I_1^{\beta(1-\alpha)}({}^H D_1^{\gamma} w)\right)(t)\\ &= \left({}^H I_1^{\beta(1-\alpha)} \delta({}^H I_1^{1-\gamma} w)\right)(t);\ \textit{for a.e. } t \in [1,T].\end{aligned} \tag{4.1}$$

This new fractional derivative (4.1) may be viewed as interpolating the Hadamard fractional derivative and the Caputo-Hadamard fractional derivative. Indeed for $\beta = 0$ this derivative reduces to the Hadamard fractional derivative and when $\beta = 1$, we recover the Caputo-Hadamard fractional derivative.

$${}^H D_1^{\alpha,0} = {}^H D_1^{\alpha},\ \textit{and}\ {}^H D_1^{\alpha,1} = {}^{Hc} D_1^{\alpha}.$$

From Theorem 21 in [20], we concluded the following lemma

Lemma 4.7. *Let $g : I \times \mathbb{R} \times \Omega \to \mathbb{R}$ be such that $g(\cdot, u(\cdot, w), w) \in C_{\gamma,\ln}([1,T])$ for any $u(., w) \in C_{\gamma,\ln}([1,T])$. Then Then problem (1.2) is equivalent to the following volterra integral equation*

$$u(t,w) = \frac{\phi_0(w)}{\Gamma(\gamma)}(\ln t)^{\gamma-1} + ({}^H I_1^\alpha g(\cdot, u(\cdot, w), w))(t);\ w \in \Omega.$$

Definition 4.8. By a random solution of the problem (1.2) we mean a measurable function $u \in C_{\gamma,\ln}$ that satisfies the condition $({}^H I_1^{1-\gamma} u)(1^+, w) = \phi_0(w)$, and the equation $({}^H D_1^{\alpha,\beta} u)(t,w) = g(t, u(t,w), w)$ on $[1,T] \times \Omega$.

Now we give (without proof) existence and Ulam slability results for problem (1.2). The following hypotheses will be used in the sequel.

(H_1') The function g is random Carathéodory on $[1,T] \times \mathbb{R} \times \Omega$,

(H_2') There exists a measurable and bounded function $p_1 : \Omega \to L^\infty([1,T],[0,\infty))$, such that

$$|g(t,u,w)| \le \frac{p_1(t,w)|u|}{1+|u|};\ \textit{for a.e. } t \in [1,T],\ \textit{and each } u \in \mathbb{R},\ w \in \Omega,$$

(H_3') There exists $\lambda_\Phi > 0$ such that for each $t \in [1,T]$, and $w \in \Omega$, we have

$$({}^H I_1^\alpha \Phi)(t,w) \le \lambda_\Phi \Phi(t,w),$$

(H_4') There exists $q_1 \in C(I,[0,\infty))$ such that for each $t \in I$, and $w \in \Omega$, we have

$$p_1(t,w) \le q_1(t)\Phi(t,w),$$

(H_5') There exists $\varphi_1 \in C([1,T],[0,\infty))$ such that for each $t \in [1,T],\ w \in \Omega$, and all $u, v \in \mathbb{R}$, we have

$$|g(t,u,w) - g(t,v,w)| \le (\ln t)^{1-\gamma}\varphi_1(t)\Phi(t,w)|u-v|.$$

Theorem 4.9. *Assume that the hypotheses (H_1') and (H_2') hold. Then the problem (1.2) has at least one random solution defined on $[1,T] \times \Omega$.*

Theorem 4.10. *Assume that the hypotheses $(H_1')(H_4')$ hold. Then the problem (1.2) is generalized Ulam-Hyers-Rassias stable.*

Theorem 4.11. *Assume that the hypotheses* (H_3') *and* (H_5') *hold. If*

$$L_1 := (\ln T)^{1-\gamma} \varphi_1^* \lambda_\phi < 1, \tag{4.2}$$

where $\varphi_1^* = \sup_{t \in [1,T]} \varphi_(t)$, *then there exists a unique random solution* u_1 *of problem (1.2), and the problem (1.2) is generalized Ulam-Hyers-Rassias stable. Furthermore, we have*

$$|u(t,w) - u_1(t,w)| \leq \frac{\Phi(t,w)}{1 - L_1}.$$

5 An Example

As an application of our results we consider the following problem of Hilfer fractional differential equation

$$\begin{cases} (D_0^{\frac{1}{2},\frac{1}{2}} u)(t) = f(t,u(t)); \ t \in [0,1], \\ (I_0^{\frac{1}{4}} u)(t)|_{t=0} = 1, \end{cases} \tag{5.1}$$

where

$$\begin{cases} f(t,u) = \dfrac{ct^{\frac{-1}{4}} \sin t}{64(1+\sqrt{t})(1+|u|)}; \ t \in (0,1] \ u \in \mathbb{R}, \\ f(0,u) = 0; \qquad u \in \mathbb{R}, \end{cases}$$

and $c = \frac{9\sqrt{\pi}}{16}$. Clearly, the function f is Carathéodory.
The hypothesis (H_2) is satisfied with

$$\begin{cases} p(t) = \dfrac{ct^{\frac{-1}{4}} |\sin t|}{64(1+\sqrt{t})}; \ t \in (0,1], \\ p(0) = 0. \end{cases}$$

Hence, Theorem 3.2 implies that the problem (5.1) has at least one solution defined on $[0,1]$. Also, the hypothesis (H_3) is satisfied with

$$\Phi(t) = e^3, \ and \ \lambda_\Phi = \frac{1}{\Gamma(1+\alpha)}.$$

Indeed, for each $t \in [0,1]$ we get

$$\begin{aligned} (I_0^\alpha \Phi)(t) &\leq \frac{e^3}{\Gamma(1+\alpha)} \\ &= \lambda_\Phi \Phi(t). \end{aligned}$$

Consequently, Theorem 3.3 implies that the problem (5.1) is generalized Ulam-Hyers-Rassias stable.

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Jeribi essential spectrum

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Abstract: The aim of this paper is to present some results concerning the Jeribi essential spectrum. We use the notion of measure of weak noncompactness to give a formulae for the Jeribi essential spectral radius and we use the class of tauberian and cotauberian operators to present some relationship between the Jeribi essential spectrum and the other essential spectra.

Keywords: Measure of weak noncompactness, Tauberian operators, essential spectra, Fredholm operators.

MSC2010: 47A13, 47A53, 47H08.

1 Introduction

The Jeribi essential spectrum of an operator $T \in \mathscr{L}(X)$, denoted by $\sigma_j(T)$, was defined in [9, 7, 13] as the intersection of the spectra of all $K \in \mathcal{W}_*(X)$ perturbations of T, where $\mathcal{W}_*(X)$ stands for each one of the sets of weakly compact operators $\mathcal{W}(X)$ and strictly singular operators $\mathcal{S}(X)$. The Jeribi essential spectrum always satisfying the inclusion $\sigma_j(T) \subseteq \sigma_{e_5}(T)$, where $\sigma_{e_5}(T)$ designed the Schechter essential spectrum. One of the crucial question about the Jeribi essential spectrum of a bounded linear operator on Banach spaces is the following : is there any relationship between the Jeribi essential spectrum and the other essential spectra (Kato essential spectrum, Gustafson essential spectrum, Wolf essential spectrum, Weidman essential spectrum, and others essential spectra ....)? A partial answer was given in [8] when X is a reflexive Banach space, then the Jeribi essential spectrum is the smallest essential spectrum in the sense of the inclusion of the Kato, Gustafson, Weidmann, Wolf, Schechter and Browder essential spectra. In $L_1(\Omega, d\mu)$ spaces, where (Ω, Σ, μ) is an arbitrary positive measure space it was proved in A. Jeribi's thesis [7] that the Jeribi essential spectrum coincide with the Schechter essential spectrum. Moreover, if the space X satisfies the Dunford-Pettis property then we have $\sigma_{e_5}(T) = \sigma_j(T)$.

We are interested in this paper by the study of the Jeribi essential spectrum of a bounded linear operator on a Banach space X. By using the De Blasi measure of weak noncompactness, we give a formulae for the Jeribi essential spectral radius and we present the relationship between the Jeribi essential spectrum and the other essential spectra, in the particular case, when the Banach space X has no reflexive infinite dimensional subspaces.

We organize the paper in the following way: In the next section, we gather some results and notations which we deal in this paper. In section 3, we present the main results.

2 Preliminaries

Let X be a Banach space. We denote by $\mathscr{L}(X)$ the set of all bounded linear operators on X and by $\mathcal{K}(X)$ the subspace of compact operators on X. The set of upper semi-Fredholm operators, denoted by $\Phi_+(X)$, consists of the bounded linear operators on X with closed range and finite dimentional kernel. The set of lower semi-Fredholm operators, denoted by $\Phi_-(X)$, consists of the bounded linear operators on X with finite codimensional in X. Operators in $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$ are called semi-Fredholm operators on X while $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ denotes the set of Fredholm operators on X. An operator $T \in \mathscr{L}(X)$ is said to have a left Fredholm inverse, if there exist $T^l \in \mathscr{L}(X)$ and $K \in \mathcal{K}(X)$ such that $T^lT = I_X - K$. Similarly, $T \in \mathscr{L}(X)$ is said to have a right Fredholm inverse, if there exist $T^r \in \mathscr{L}(X)$ and $K \in \mathcal{K}(X)$ such that $TT^r = I_X - K$. Denote by $\Phi^l(X)$ (respectively $\Phi^r(X)$) the set of linear operators which have a left (respectively right) Fredholm inverse. The index, $\text{ind}(T)$, is given by $\text{ind}(T) = \alpha(T) - \beta(T)$ where $\alpha(T)$ is the nullity of T and $\beta(T)$ is the defect of T.

Definition 2.1. An operator $T \in \mathscr{L}(X)$ is said to be strictly singular if the restriction of T to any infinite dimensional subspace of X is not an homeomorphism.

The concept of strictly singular operators was introduced in [11] as a generalization of the notion of compact operators. The set of all strictly singular operators on X, denoted by $\mathcal{S}(X)$, is a closed two-sided ideal of $\mathscr{L}(X)$ containing $\mathcal{K}(X)$ (see [5, 11]).

Definition 2.2. An operator $T \in \mathscr{L}(X, Y)$ is called weakly compact if $T(B)$ is relatively weakly compact in Y, for every bounded subset $B \subset X$.

The set of all weakly compact operators is denoted by $\mathcal{W}(X, Y)$. When $X = Y$, $\mathcal{W}(X)$ is a closed two-sided ideal of $\mathscr{L}(X)$ containing $\mathcal{K}(X)$. For basic properties of these operators we refer to [5, 3].

In order to recall the tauberian and cotauberian operators, let us denote X^{**} the second dual (or bidual) of X, T^* the conjugate of T and T^{**} the second conjugate of T. These classes of operators were introduced and investigated respectively by Wilansky [10] and Tacon [15].

Definition 2.3. An operator $T \in \mathscr{L}(X,Y)$ is said to be tauberian whenever T^{**} preserves the natural embedding of X into its double dual i.e $x \in X^{**}, T^{**}x \in Y$implies $x \in X$.

Denote by $\mathcal{T}(X,Y)$ the set of tauberian operators. It is immediate that a tauberian operator has the property

$$x \in X^{**},\ T^{**}x = 0 \text{ imply } x \in X.$$

This implies that

$$\ker(T) \text{ is reflexive.}$$

So, the upper semi-Fredholm operators are trivial examples of Tauberian operators. In the next proposition we recall some well known properties of tauberian operators.

Proposition 2.4 ([6]). *Let $T \in \mathscr{L}(X,Y)$ and $L \in \mathscr{L}(Y,Z)$. The following statements hold.*

1. *If both T and L are tauberian, then LT is tauberian.*
2. *If LT is tauberian, then T is tauberian.*
3. *T is tauberian and weakly compact if and only if X is reflexive.*
4. *If T is tauberian and $W \in \mathcal{W}(X,Y)$, then $T+W$ is tauberian.*

The following result was established in [10, Corollary 4.3].

Lemma 2.5. *Suppose X has no reflexive infinite dimentional subspace, then for $T \in \mathscr{L}(X,Y)$, T is tauberian if and only if $T \in \Phi_+(X,Y)$.*

Definition 2.6. An operator $T \in \mathscr{L}(X,Y)$ is said to be cotauberian if T^* is tauberian.

Denote by $\mathcal{CT}(X,Y)$ the class of all cotauberian operators.

Proposition 2.7 ([6]). *Let $T \in \mathscr{L}(X,Y)$ and $L \in \mathscr{L}(Y,Z)$. The following statements hold.*

1. *If both T and L are cotauberian, then LT is cotauberian.*
2. *If LT is cotauberian, then L is cotauberian.*
3. *T is cotauberian and weakly compact if and only if Y is reflexive.*
4. *If T is cotauberian and $W \in \mathcal{W}(X,Y)$, then $T+W$ is cotauberian.*

Now, let us recall some definitions of essential spectra of an operator $T \in \mathscr{L}(X)$.

Weidmann	$: \sigma_{e,1}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_-(X)\}$
Gustafson	$: \sigma_{e,2}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_+(X)\}$
Kato	$: \sigma_{e,3}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_\pm(X)\}$
Wolf	$: \sigma_{e,4}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\}$
Schechter	$: \sigma_{e,5}(T) = \mathbb{C} \setminus \Phi_T(X)$
left Fredholm	$: \sigma_{el}(T) = \left\{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi^l(X)\right\}$
right Fredholm	$: \sigma_{er}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi^r(X)\}$
Jeribi	$: \sigma_j(T) = \bigcap\limits_{K \in \mathcal{W}_*(X)} \sigma(T+K),$

where $\Phi_T(X) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi(X) \text{ and } \operatorname{ind}(\lambda I - T) = 0\}$; $\sigma(T+K)$ denote the spectrum of $(T+K)$ and $\mathcal{W}_*(X)$ stands for each one of the sets $\mathcal{S}(X)$ and $\mathcal{W}(X)$. In general, we have the following inclusions

$$\sigma_{e,3}(T) = \sigma_{e,1}(T) \cap \sigma_{e,2}(T) \subseteq \sigma_{e,4}(T) \subseteq \sigma_{e,5}(T), \tag{2.1}$$

$$\sigma_{e,2}(T) \subseteq \sigma_{el}(T) \subseteq \sigma_{e,4}(T), \tag{2.2}$$

$$\sigma_{e,1}(T) \subseteq \sigma_{er}(T) \subseteq \sigma_{e,4}(T), \tag{2.3}$$

and $\sigma_j(T) \subseteq \sigma_{e,5}(T)$. The spectral radius of the essential spectra is defined as

$$r_{e_i}(T) = \sup\{|\lambda| : \lambda \in \sigma_{e_i}(T)\} \qquad i = 1, \ldots, 5$$

and $r_{e_i}(T) = \lim\limits_{n\to\infty} (\chi(T^n))^{\frac{1}{n}}$, where $\chi(T)$ is the Hausdorff measure of noncompactness of T (see [4]).

The Hausdorff measure of noncompactness of a bounded subset A of X, denoted by $\chi(A)$, is defined to be the infimum of the set of all reals $\varepsilon > 0$ such that A can be covered by a finite number of balls of radius $< \varepsilon$. The Hausdorff measure of noncompactness of an operator $T \in \mathscr{L}(X,Y)$, denoted by $\chi(T)$, is given by $\chi(T) = \chi(T(B_X))$ with B_X the closed unit ball in X.

Definition 2.8. Let X be a Banach space and $A \subset X$ be a bounded set of X. The measure of weak noncompactness $\omega(A)$ of the set A is the infimum of the numbers $\varepsilon > 0$ such that A has a weakly compact ε-net in X.

This measure of weak noncompactness and its properties have been studied by De Blasi in [2]. The measure of weak noncompactness of an operator $T \in \mathscr{L}(X,Y)$, $\omega(T)$, is given by $\omega(T) = \omega(T(B_X))$. The most useful properties of this measure of weak noncompactness, $\omega(T)$, are related in the following theorem.

Theorem 2.9. *Let X, Y be two Banach spaces and $T \in \mathscr{L}(X,Y)$. Then we have*

1. *$\omega(T) = 0$ if and only if $T \in \mathcal{W}(X, Y)$.*

2. *$\omega(T) = \omega(T + W)$, for all $W \in \mathcal{W}(X, Y)$.*

3. *$\omega(T) \leq \chi(T)$.*

3 Main results

In [12], the authors proved that, if X is a Banach space, then

$$\sigma_{e,5}(T) = \bigcap_{K \in \mathcal{S}(X)} \sigma(T + K).$$

So, in the definition of Jeribi essential spectrum, we restrict K belonging to $\mathcal{W}(X)$ only. We define the spectral radius of the Jeribi essential spectrum as

$$r_j(T) = \sup \{\lambda \in \mathbb{C} : \lambda \in \sigma_j(T)\}.$$

We start with the following main proposition

Proposition 3.1. *Let $T \in \mathscr{L}(X)$. Then, the following assertions hold:*

i) For all $W \in \mathcal{W}(X)$, $\sigma_{e_i}(T + W) \subset \sigma_j(T) \qquad i = 1, \ldots, 5, l, r$.

ii) $r_j(T^n) = (r_j(T))^n$, for all $n \in \mathbb{N}$.

iii) $r_j(T) \leqslant \omega(T)$.

Proof. Let $\lambda \in \mathbb{C}$ such that $\lambda \notin \sigma_j(T)$. Then we have the following implications

$$\begin{aligned} \lambda \notin \sigma_j(T) \quad &\Rightarrow \quad \exists W \in \mathcal{W}(X) : \lambda \in \rho(T + W) (\text{the resolvent set of } T + W) \\ &\Rightarrow \quad \exists W \in \mathcal{W}(X) : (\lambda I - T - W) \in \Phi(X) \text{ind}(\lambda I - T - W) = 0 \\ &\Rightarrow \quad \exists W \in \mathcal{W}(X) : \lambda \notin \sigma_{e_5}(T + W). \end{aligned}$$

So, for all weakly compact operator W on X, $\lambda \in \sigma_{e_5}(T+W)$ implies that $\lambda \in \sigma_j(T)$. By the inclusions (2.1), (2.2) and (2.3), it follows that $\sigma_{e_i}(T + W) \subset \sigma_j(T)$ for $i = 1, \ldots, 4, l, r$.

Let $n \in \mathbb{N}$ and $W \in \mathcal{W}(X)$. Then $\sigma((T + W)^n) = \sigma(T^n + \sum_{k=0}^{n-1} C_k^n T^k W^{n-k})$ where C_k^n is the binomial coefficient defined by $C_k^n = \dfrac{n!}{k!(n-k)!}$. Since $\mathcal{W}(X)$

is a closed two-sided ideal of $\mathscr{L}(X)$, then $W' = \sum_{k=0}^{n-1} C_k^n T^k W^{n-k} \in \mathcal{W}(X)$. So, $\sigma((T+W)^n) = \sigma(T^n + W')$ and

$$\bigcap_{W\in\mathcal{W}(X)} \sigma((T+W)^n) = \bigcap_{W'\in\mathcal{W}(X)} \sigma(T^n + W') = \sigma_j(T^n).$$

Hence

$$\bigcap_{W\in\mathcal{W}(X)} \sigma((T+W)^n) = \sigma_j(T^n). \tag{3.1}$$

On the other hand, it is well known that $\sigma(T^n) = \{\lambda^n : \lambda \in \sigma(T)\}$, then

$$\bigcap_{W\in\mathcal{W}(X)} \sigma((T+W)^n) = \left\{\lambda^n : \lambda \in \bigcap_{W\in\mathcal{W}(X)} \sigma(T+W)\right\} = \{\lambda^n : \lambda \in \sigma_j(T)\}. \tag{3.2}$$

From the two inequalities (3.1) and (3.2) we see that

$$\sigma_j(T^n) = \{\lambda^n : \lambda \in \sigma_j(T)\}.$$

From the above equality, it follows that $r_j(T^n) = (r_j(T))^n$.

Let $\lambda \in \mathbb{C}$ such that $\chi(T) < |\lambda|$. Then using [4, Theorem 4.4] we get $(\lambda I - T) \in \Phi(X)$ with $\text{ind}(\lambda I - T) = 0$. By the fact that $\omega(T) \leq \chi(T)$, then we have the following implications

$$\begin{aligned} \lambda \in \sigma_{e_5}(T) &\Rightarrow |\lambda| \leq \omega(T) \\ &\Rightarrow r_{e_5}(T) \leq \omega(T). \end{aligned}$$

Hence $r_j(T) \leq \omega(T)$ since $\sigma_j(T) \subseteq \sigma_{e_5}(T)$. □

The following theorem is the main result.

Theorem 3.2. *Let $T \in \mathscr{L}(X)$. Then, the spectral radius of the Jeribi essential spectrum is given by*

$$r_j(T) = \lim_{n\to\infty} [\omega(T^n)]^{\frac{1}{n}}.$$

Proof. The limits of $[\omega(T^n)]^{\frac{1}{n}}$ exists for all $T \in \mathscr{L}(X)$ (see [1]). From the two assertions ii) and iii) of proposition (3.1), it follows immediately that

$$r_j(T) \leq \lim_{n\to\infty} [\omega(T^n)]^{\frac{1}{n}}. \tag{3.3}$$

Let us prove the opposite inequality. Let $W \in \mathcal{W}(X)$. From the assertion i) of proposition (3.1), we have

$$\sigma_{e_1}(T+W) \subset \sigma_j(T).$$

Moreover, $r_{e_1}(T+W) \le r_j(T)$. It is well known that

$$r_{e_1}(T+W) = \lim_{n\to\infty} [\chi((T+W)^n)]^{\frac{1}{n}} ,$$

then using the fact that $\omega(T) = \omega(T+W)$ for any weakly compact operator W on X and $\omega(T+W) \le \chi(T+W)$ we obtain

$$\lim_{n\to\infty} [\omega(T^n)]^{\frac{1}{n}} \le r_j(T). \tag{3.4}$$

From the two inequalities (3.3) and (3.4) we deduce that

$$r_j(T) = \lim_{n\to\infty} [\omega(T^n)]^{\frac{1}{n}} .$$

□

In the next main result, we give a relationship between the Jeribi essential spectrum and the other essential spectra when the Banach space X has no reflexive infinite dimensional subspaces.

Theorem 3.3. *Let X be a Banach space which has no reflexive infinite dimensional subspaces and $T \in \mathscr{L}(X)$. Then we have*

$$\sigma_{e_i}(T) \subset \sigma_j(T) \qquad i = 1, \ldots, 4, l, r.$$

Proof. Take $\lambda \notin \sigma_j(T)$. Then there exists a weakly compact operator W on X such that $\lambda \in \rho(T+W)$ i.e

$$(\lambda I - T - W) \in \Phi(X) \text{ with } \operatorname{ind}(\lambda I - T - W) = 0.$$

In particular, $(\lambda I - T - W) \in \Phi_+(X)$ which is a tauberian operator. Using the stability of tauberian operators under a weakly compact perturbation $(-W)$ (assertion 4 of proposition (2.4)), we get $(\lambda I - T) \in \mathcal{T}(X)$. Since the space X has no reflexive infinite dimensional subspaces, using lemma (2.5), we obtain $(\lambda I - T) \in \Phi_+(X)$. So, $\lambda \notin \sigma_{e_2}(T)$. Hence $\sigma_{e_2}(T) \subset \sigma_j(T)$.

To prove that $\sigma_{e_1}(T) \subset \sigma_j(T)$, let us consider $\lambda \notin \sigma_j(T)$. Then there exists a weakly compact operator W on X such that $\lambda \in \rho(T+W)$ i.e

$$(\lambda I - T - W) \in \Phi(X) \text{ with } \operatorname{ind}(\lambda I - T - W) = 0.$$

In particular, $(\lambda I - T - W) \in \Phi_-(X)$ which is a cotauberian operator. Using the stability of cotauberian operators under a weakly compact perturbation $(-W)$ (assertion 4 of proposition (2.7)), we get $(\lambda I - T) \in \mathcal{CT}(X)$. By the definition of cotauberian operator, we have $(\lambda I - T)^* \in \mathcal{T}(X)$. Since the space X has no reflexive infinite dimensional subspaces, then using lemma (2.5), we obtain $(\lambda I - T)^* \in \Phi_+(X)$. It follows from [14, Theorem 4, p.156] that $(\lambda I - T) \in \Phi_-(X)$ i.e $\lambda \notin \sigma_{e_1}(T)$.

The following two inclusions $\sigma_{e_3}(T) \subset \sigma_j(T)$ and $\sigma_{e_4}(T) \subset \sigma_j(T)$ are immediately since $\sigma_{e_3}(T) = \sigma_{e_1}(T) \cap \sigma_{e_2}(T)$ and $\sigma_{e_4}(T) = \sigma_{e_1}(T) \cup \sigma_{e_2}(T)$.

From the inclusions (2.2) and (2.3), we see that $\sigma_{el}(T) \subset \sigma_j(T)$ and $\sigma_{el}(T) \subset \sigma_j(T)$. □

As an immediate consequence of Theorem (3.3) we have the following corollary.

Corollary 3.4. *Let X be a Banach space such that X has no reflexive infinite dimensional subspaces and $T \in \mathscr{L}(X)$. Then we have the following inclusions*

$$\sigma_{e_3}(T) = \sigma_{e_1}(T) \cap \sigma_{e_2}(T) \subset \sigma_{e_4}(T) \subset \sigma_j(T) \subseteq \sigma_{e_5}(T).$$

Furthermore, $r_j(T) = \lim_{n\to\infty} \chi(T^n)^{\frac{1}{n}}$.

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Comaximal Submodule Graphs of Unitary Modules

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Abstract: In this paper, a new kind of graph on a unitary module A over a commutative ring R with identity, namely the co-maximal submodule graph is defined and studied as a natural generalization of the comaximal ideal graph of a commutative ring R, denoted by $\mathbb{C}(R)$. We use $\mathbb{C}(A)$ to denote this graph, with its vertices the proper submodules of A which are not contained in the Jacobson radical of A, and two vertices B_1 and B_2 are adjacent if and only if $B_1 + B_2 = A$. We show some properties of this graph and compare some of the results of $\mathbb{C}(A)$ and $\mathbb{C}(R)$. For example, this graph is a simple, connected graph with diameter less than or equal to three, and both the clique number and the chromatic number of the graph are equal to the number of maximal submodules of the module A. It is shown that $\mathbb{C}(R)$ is isomorphic to a subgraph of $\mathbb{C}(A)$ when A is a finitely generated cancellation (in particular, a finitely generated free) R-module. We also discuss the conditions under which A is a finite direct sum of simple modules, $\mathbb{C}(A)$ is isomorphic to a finite Boolean graph, and $\mathbb{C}(A)$ and $\mathbb{C}(R)$ are isomorphic graphs.

Keywords: Co-maximal (submodule, ideal) graph, (finitely generated, cancellation, content) module, number of maximal submodules, injector ideal, connectedness and diameter, clique number, chromatic number, weakly perfect, Boolean graph.

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Introduction

The main purpose of this paper is to study the *comaximal submodule graph* of a (finitely generated) *unitary module* A, denoted by $\mathbb{C}(A)$, as a natural extension of the *comaximal ideal graph* of a commutative ring R, denoted by $\mathbb{C}(R)$ [41], by replacing proper ideals of R (not contained in the Jacobson radical of R) with proper submodules of A (not contained in the *Jacobson radical* of A) as the vertex set of the graph $\mathbb{C}(A)$ and edges are defined by *comaximal submodules* of A (Definition 0.1). In [41], Ye and Wu extended the notion of the *comaximal graph* of a commutative ring R (by Sharma and Bhatwadekar [38]) to the comaximal ideal graph of the ring R by replacing *principal ideals* of R with proper ideals of R that are not contained in $J(R)$ (the Jacobson radical of R) as the vertex set of their graph $\mathbb{C}(R)$, where edges are defined by *comaximal ideals* of R. For some recent works on comaximal ideal graphs of commutative rings, see [1, 8, 19, 42, 43] with more algebraic and graph-theoretic properties of R and $\mathbb{C}(R)$ such as *planarity, classification of diameter, and graph perfection* for rings with (infinitely) many maximal ideals. In this work, besides applying direct arguments on the module A for characterizing $\mathbb{C}(A)$, we will use the results and techniques that are related to $\mathbb{C}(R)$ by making a bridge between $\mathbb{C}(R)$ and $\mathbb{C}(A)$ by applying the intrinsic connection between the ring R and its associated module, which is natural in commutative algebra. Notice that in the next section, we provide all necessary results and definitions on modules and graphs that are required in this work, for the sake of completeness, and use them throughout the paper in the sequel.

Throughout this paper, unless otherwise indicated, all rings are assumed to be commutative with identity and modules are unitary. For a ring R [resp. an R-module A], we use (R) [resp. (A)] to denote the set of all *maximal ideals* of R [resp. maximal submodules of A]. A ring R [resp. module A] is said to be *local* if it has a unique maximal ideal [resp. unique maximal submodule (see Example 2.1)] (i.e., $|(R)| = 1$ [resp. $|(A)| = 1$]). Some authors, equivalently, (as in [26]) use "quasi-local" to mean a ring with a unique maximal ideal. We use $J(R)$ and $U(R)$ [resp. $J(A)$ and $U(A)$] to denote the Jacobson radical of R (i.e. the intersection of all maximal ideals of R) [resp. the Jacobson radical of A (i.e. the intersection of all maximal submodules of A)] and the set of all the *invertible elements* of R [resp. the set of all the *units* of A (Definition 1.8)], respectively. As usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$ (i.e., $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$), respectively. References for *graph theory* are [15], [16], [18], and [21]; for *commutative ring theory* and modules, see [4], [11], [7], [23], and [37].

Definition 0.1. The co-maximal submodule graph of a (unitary) module A over a ring R, denoted by $\mathbb{C}(A)$, is a graph whose vertices are the proper submodules of A which are not contained in the Jacobson radical of A, and two vertices B_1 and B_2 are adjacent if and only if $B_1 + B_2 = A$.

• Notice that, besides using many results from module theory (which are stated in Section 1 of this paper as background), the general pattern of the proofs related to characterization of $\mathbb{C}(A)$, respectively,

($*$) are exactly parallel to the ring case without any extra (major) assumptions on the module A or the ring R;

(∗) are (somewhat) parallel to the ring case by assuming that A is a (*finitely generated, cancellation, content [in particular, free, projective]*) module and the *injector ideals* (Definition 1.1) of any two maximal submodules of A are not contained in each other.

The rest of this section is devoted on a *brief historical note* on some graphs associated to some algebraic structures and concluded with a *description* on the organization of the other sections. The area of research on assigning a graph to an algebra (algebraic structure) has been very active (specially) since last two decades and There are many papers which apply combinatorial methods (using graph-theoretic properties and parameters such as *planarity, clique number, chromatic number, independence number, domination number*, and so on) to obtain algebraic results and vice versa, for instance, there are many papers on this interdisciplinary subject and for a short list of them, see for example the reference of [31].

• The co-maximal ideal graph of a ring R, denoted by $\mathbb{C}(R)$, is a graph whose vertices are the proper ideals of R which are not contained in the Jacobson radical of R, and two vertices I_1 and I_2 are adjacent if and only if $I_1 + I_2 = R$. This graph was first introduced and studied in [41]. They studied the *diameter, girth, and bipartiteness* of $\mathbb{C}(R)$ and showed that $\mathbb{C}(R)$ is a simple, connected graph with diameter less than or equal to three, and both the clique number and the chromatic number of the graph are equal to the number of maximal ideals of the ring R. They also studied the conditions under which $\mathbb{C}(R)$ is a *finite Boolean graph* if and only if R is a finite direct product of fields.

In [38], Sharma and Bhatwadekar define a graph G on a commutative ring R with vertices as elements of R, where two distinct vertices a and b are adjacent if and only if $Ra+Rb = R$. They showed that $\chi(G)$ (the chromatic number of G) is finite if and only if R is a finite ring. In this case $\chi(G) = \omega(G)$(the clique number of G) $= t + l$, where t and l, respectively, denote the number of maximal ideals and the number of units of R (see Theorem 2.3 in [38]). Further, in [26], Maimani et al. studied the graph structure defined by Sharma and Bhatwadekar and called it "comaximal graph of commutative rings". In their work, they mostly focused on the graph-theoretic and related ring-theoretic properties of the subgraph generated by nonunit elements of R (see Definition 1.20).

The *zero-divisor graph of a commutative ring* R, denoted $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero-divisors of R with two distinct vertices x and y joined by an edge if and only if $xy = 0$. Thus $\Gamma(R)$ is the *empty graph* if and only if R is an *integral domain.*

In [12] (1988), Beck introduced the concept of a *zero-divisor graph* of a commutative ring, but this work was mostly concerned with *colorings* of rings. The above definition first appeared in the work of Anderson and Livingston [5] (1999), which contains several fundamental results concerning $\Gamma(R)$. This definition, unlike the earlier work of Anderson and Naseer [6] and Beck [12], does not take zero to be a vertex of $\Gamma(R)$.

In [35], Redmond introduced the notion of an *ideal-based zero-divisor graph* of a commutative ring and his work continued and developed further in [27]. Let I be a proper ideal of R. The *zero-divisor graph* of R with respect to I, denoted by $\Gamma_I(R)$, is the graph whose vertex set is the set $\{x \in R \setminus I |\ xy \in I \text{ for some } y \in R \setminus I\}$ with distinct vertices x and y adjacent if and only if $xy \in I$. Thus, if $I = 0$ then $\Gamma_I(R) = \Gamma(R)$, and I is a *nonzero prime ideal* of R if and only if $\Gamma_I(R) = \emptyset$. In both papers [35] and [27], the authors explored the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$.

Moreover, the concept of a zero-divisor graph of a commutative ring has been generalized to a *k-zero-divisor hypergraph* by Eslahchi and Rahimi (the second author of this paper) in [20]. Besides providing many examples and introducing the notion of a k-zero-divisor, k-prime ideal and k-integral domain, there is a discussion on some of the properties and parameters of this hypergraph such as connectedness, diameter, and girth. Moreover, for other properties of this hypergraph, see [36] and [39]. Also, Rahimi in [34], in connection to the *Smarandache zero divisors* of a commutative ring, introduced the notion of a *Smarandache vertex* (S-vertex for short) of a (simple) graph (which is independent of any algebraic structure) to study the S-zero divisors of a commutative ring via its associated zero-divisor graph. Consequently, by this generalization, study of S-vertices of any simple graph can be done directly in a pure graph-theoretic sense, and specially, discussing the S-vertices of any graph associated to an algebra (algebraic structure) is possible and can lead to the study of the interplay between some graph-theoretic properties and algebraic properties of the related algebra. For instance, S. Visweswaran and Hiren D. Patel in [40] studied the S-vertices of *the complement of the annihilating-ideal graph* in connection to some ring-theoretic properties in Sections 2 (Lemma 2.5), 4 (Lemma 4.2(v)), and 5 (Proposition 5.1(iv)) of their paper.

• The concept of the *annihilating-ideal graph* of a commutative ring, denoted by $\mathbb{AG}(R)$, was first introduced by Behboodi and Rakeei in [13] and [14] (for other types of the annihilator-graph of a commutative ring R, denoted by $\mathbb{AG}(R)$ as well, see [9]). Actually, $\mathbb{AG}(R)$ is the *zero-divisor graph* of the *multiplicative semigroup* of the ideals of R (see [17]). Also in [2], Aliniaeifard, Behboodi and authors of this paper have extended and studied this notion to a more general setting as *the annihilating-ideal graph of a commutative ring R with respect to an ideal I of R*, denoted by $\mathbb{AG}_I(R)$, by replacing (nonzero) ideals whose product is zero with ideals ($\nsubseteq I$) whose product lies in I. Thus, $\mathbb{AG}_I(R) = \mathbb{AG}(R)$ for $I = (0)$. Clearly, I is a *prime ideal* of R if and only if $\mathbb{AG}_I(R) = \emptyset$. Notice that $\mathbb{AG}_I(R)$ can be regarded as an *ideal-based zero-divisor graph* of the semiring of the ideals of R and can be denoted as $\Gamma_{C_I}(\mathbb{I}(R))$, where $\mathbb{I}(R)$ is the semiring of the ideals of R and C_I is the set of all ideals of R that are contained in I.

• Furthermore, we now mention a generalization of "an annihilating-ideal graph with respect to an ideal" [2] from two different directions as follows. The authors of this paper, in [31], extended the notion of an annihilating-ideal graph with respect to an ideal to *the annihilation graphs of commutator posets and lattices with respect to an element*, which is a generalization of [2] in a very broad sense. That is, besides defining and discussing the mentioned graph on a *commutator poset* as a general model, we provide examples by applying the commutator theory to define our graph on the substructures (as vertices) of any algebra and define the edge between any two substructures to be the commutator of them that satisfies a special property. Also, in [30], from another direction, we generalized the graph in [2] (see also the last sentence in the previous paragraph) by extending the work of Redmond in [35] to *an ideal-based zero-divisor graph of a commutative semiring R*, denoted by $\Gamma_I(R)$, where I is an ideal of R, and (in contrast to the ring case [35, Theorem 3.2] and [35, Proposition 3.5]) by an example showed that $\Gamma_I(R)$ has a *cut-point* and more than one *bridge* ([30, Example 3.1]). See also [32], which is an ideal-based version of [31], namely "The annihilation graphs of commutator posets and lattices with respect to an ideal".

• The organization of this paper is as follows: In Section 1, we recall some basic properties and definitions of modules and graphs, respectively, and use them (implicitly) in the sequel. Section 2 is devoted on some fundamental properties of the graph $\mathbb{C}(A)$. In this section, besides some simple examples and trivial results, we will discuss some basic properties of the graph $\mathbb{C}(A)$ such as the *diameter* and *core* of $\mathbb{C}(A)$ (Theorems 2.3 and 2.6). Finally, we close the section by showing that $\mathbb{C}(R)$ is isomorphic to a subgraph of $\mathbb{C}(A)$ whenever A is a finitely generated *cancellation R-module* (Theorem 2.7). In Section 3, we discuss the *clique number* and the *chromatic number* of $\mathbb{C}(A)$. In this section, we will prove that the graph $\mathbb{C}(A)$ has the property that its clique number and chromatic number are equal (Theorem 3.1). Moreover, we show that a *content [in particular, free, or more generally, projective] module* A over a ring R is a finite direct sum of *simple modules* and $\mathbb{C}(A)$ is a *finite Boolean graph* whenever $\mathbb{C}(R)$ is a finite Boolean graph (Theorem 3.4). Finally, we will show that for a module A over a ring R with n maximal submodules, $\mathbb{C}(A)$ can be *retracted* to the n-Boolean graph (Corollary 3.12 and see also Proposition 3.11). In Section 4, we study the graph $\mathbb{C}(A)$ of modules A with exactly two maximal submodules. We will show that in such a situation, $\mathbb{C}(A)$ is a *complete bipartite graph* (Lemma 4.1) and consequently, $\mathrm{gr}(\mathbb{C}(A))$ is 4 or ∞ (Corollary 4.3) provided that A is finitely generated. Moreover, For a finitely generated module A, $\mathbb{C}(A)$ is a (complete) bipartite graph if and only if A has exactly two maximal submodules (Theorem 4.5). Theorem 4.7 gives a necessary and sufficient condition for $\mathrm{diam}(\mathbb{C}(A)) = 1$ when A is a finitely generated module. Finally, in Theorems 4.9 and 4.10, we study the conditions under which $\mathbb{C}(A)$ is a *complete graph* if and only if A is a direct sum of two simple R-modules.

1 Background on Modules and Graphs

In this section we recall some basic definitions and properties of (unitary) modules and (simple) graphs, respectively, and will use them (implicitly) in the sequel. The ideal generated by a subset Y of a ring R will be denoted by (Y), while the submodule generated by a subset X of a module A will be denoted by $\langle X \rangle$. We also write $B \leq A$ if B is a submodule of A, and if B is proper, by $B < A$.
Let I be an ideal of a ring R, A an R-module, and S a nonempty subset of A. Then $IS = \{\sum_{i=1}^{n} r_i a_i \mid r \in I; a \in S; n \text{ a positive integer}\}$ is a submodule of A. Similarly if $a \in A$, then $Ia = \{ra \mid r \in I\}$ is a submodule of A. A nonzero unitary R-module A is simple if its only sub- modules are 0 and A. Thus, every simple R-module is cyclic and every R-module endomorphism of a simple module A is either the zero map or an isomorphism. Note that a submodule is itself a module. Also a submodule of a unitary module over a ring with identity is necessarily unitary. In addition, when there is no confusion in the context, 0_R, 0_A, $0 \in \mathbb{Z}$ and the trivial module $\{0\}$ will all be denoted 0.

Definition 1.1. Let B be a submodule of an R-module A. Then the ideal $S_B = \{r \in R \mid rA \subseteq B\}$ of R is called the injector ideal of B. Note that if the commutative ring R has an identity, then for any submodule (ideal) B of the R-module R, S_B is precisely B.

Remark 1.2. Except in Proposition 1.19, if B is a submodule of an R-module A, we denote the annihilator ideal of A/B by S_B instead of $(B : A)$ $(= \{r \in R \mid rA \subseteq B\})$

and note that if the commutative ring R has an identity, then for any submodule (ideal) B of the R-module R, $(B : A)$ is precisely B.

Definition 1.3. Let A be an R-module. A proper submodule M of A is said to be maximal provided that for $N \leq A$ with $M \subseteq N \subseteq A$, then either $M = N$ or $N = A$.

Proposition 1.4. *Let $M < A$, where A is an R-module. Then M is maximal if and only if for each $x \in A \setminus M$, $\langle x, M \rangle = A$.*

Proof. the proof follows directly from the definition. □

The next result shows the existence of a maximal submodule in a module which is similar to the case of rings with identity (Theorem 2.18 of Chapter 3 in [22]).

Proposition 1.5. *Let A be a nonzero finitely generated R-module. Then every proper submodule of A is contained in a maximal submodule of A.*

Proof. See Theorem 2.8 in [4] or Corollary 2.1.15 of [11]. □

Definition 1.6. Let A be an R-module. The Jacobson radical of A (denoted $J(A)$) is the intersection of all maximal submodules of A. If no maximal submodules exist, then we set $J(A) = A$. Similarly the Jacobson radical of R will be denoted by $J(R)$.

The following corollary gives a necessary and sufficient condition for a finitely generated module to be trivial.

Corollary 1.7. *Let A be a finitely generated R-module. Then $J(A) = A$ if and only if $A = 0$.*

Proof. If $J(A) = A$, A has no maximal submodules. By Proposition 1.5, $\langle 0 \rangle$ is contained in some maximal submodule unless $\langle 0 \rangle = A$. Conversely, if $A = \langle 0 \rangle$, then clearly $J(A) = A$. □

Definition 1.8. An element u of an R-module A is said to be a unit provided that u is not contained in any maximal submodule of A.

Proposition 1.9. *Let A be a finitely generated R-module. Then $u \in A$ is a unit if and only if $\langle u \rangle = A$.*

Proof. "$\Leftarrow$" This is immediate. "$\Rightarrow$" Let u be a unit. Then $\langle u \rangle$ is contained in no maximal submodule of A. By Proposition 1.5, we have $\langle u \rangle = A$. □

As a consequence of Proposition 1.9, it follows that a finitely generated module A is *cyclic* if the set of all maximal submodules does not cover A.

Lemma 1.10. (cf. [28, Lemma 4.1] or see the lemma on page 2 of [29]) *Let A be an R-module and let $B, C \leq A$ be such that $RA + B = A$ and $S_B + S_C = R$. Then $B + C = A$.*

Proof. $A = RA + B = (S_B + S_C)A + B = S_B A + S_C A + B \subseteq B + C + B = B + C \subseteq A$. □

Remark 1.11. Clearly, in the above lemma, $RA = A$ and hence $RA + B = A$ for any submodule B of A if R is a ring with identity.

Proposition 1.12. [Chinese Remainder Theorem] (cf. [28, Theorem 4.2] or see the theorem on page 2 of [29]) *Let A be an R-module and let $B_1, \ldots, B_n \leq A$ be such that $RA + B_i = A$ for $i = 1, 2, \ldots, n$ and $S_{B_i} + S_{B_j} = R$ for $i \neq j$. If $a_1, a_2, \ldots, a_n \in A$, then there is $x \in A$ such that $x \equiv a_i (mod\ B_i)$ for $i = 1, 2, \ldots, n$. Furthermore, x is uniquely determined up to congruence modulo $\bigcap_{i=1}^{n} B_i$.*

Proof. See [28] or [29]. The proof is actually patterned after that given in [22]. □

Remark 1.13. On page 1 of [29], there is a counterexample that shows the comaximality condition of ideals (i.e. $S_B + S_C = R$) in the above proposition (Proposition 1.12) is not a superfluous assumption.

Definition 1.14. A proper submodule P of an R-module A is said to be prime provided that whenever $ra \in P$ with $r \in R$ and $a \in A \setminus P$, then $rA \subseteq P$.

Proposition 1.15. *If B is a prime submodule of an R-module A, then S_B is a prime ideal of R.*

Proof. Note that $1 \notin S_B$ (since B is a proper submodule of A by definition) and so S_B is strictly contained in R. Suppose B is a prime submodule of A and $rs \in S_B$ with $s \notin S_B$. Thus, there exists $a \in A$ such that $sa \notin B$ by definition. Now, by prime property of B, $r \in S_B$. See also [25]. □

Proposition 1.16. *If B is a maximal submodule of an R-module A, then B is prime and S_B is a maximal ideal of R.*

Proof. See Proposition 4 in [25]. □

We will use the next result in the proof of Theorem 3.4. For any element x of an R-module M, the content $c(x)$ of x is defined by $c(x) = \bigcap \{A \mid A \text{is an ideal of} R \text{such that} x \in AM\}$. M is called a *content R-module* if $x \in c(x)M$ for every $x \in M$. We remark that every projective module is a content module [33].

Proposition 1.17. (cf. [25, Theorem 5]) *Let F be a non-zero content [in particular, free, or more generally, projective] R-module. Then $J(R)F = J(F)$.*

Proof. See Theorem 5 in [25]. The "in particular part", follows from the fact that every free, or more generally, every projective module is content by [33]. For more properties of content modules, see [33] and Section 4 of [25]. □

The following result is an immediate consequence of the above proposition since every vector space is a free module over a field.

Corollary 1.18. *Let V be a vector space over a field. Then $J(V) = 0$.*

Proposition 1.19. [The Prime Avoidance Theorem] (cf. [24, Theorem 2.3]) *Let M be an R-module, L_1, L_2, …, L_n a finite number of submodules of M, and L a submodule of M such that $L \subseteq L_1 \cup L_2 \cdots \cup \cdots \cup L_n$. Assume that at most two of the L's are not prime, and that $(L_j : M) \nsubseteq (L_k : M)$ whenever $j \neq k$. Then $L \subseteq L_k$ for some k.*

• We close this section by recalling some definitions and notions from graph theory (for the sake of completeness) and use them throughout to keep this paper as self contained as possible.

For a graph G, by $V(G)$ and $E(G)$, we denote the set of all vertices and all edges of G, respectively. Recall that for a graph G, the *degree* of a vertex v in G is the number of edges of G incident with v. A graph G is *connected* if there is a path between any two vertices of G. The *diameter* of a connected graph G is the supremum of the distances between vertices. That is, $\operatorname{diam}(G) = \sup\{d(x,y) | x \text{ and } y \text{ are distinct vertices of } G\}$, where $d(x,y)$ is the length of a shortest path from x to y in G ($d(x,y) = \infty$ if there is no such path). The diameter is 0 if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e., each pair of distinct vertices forms an edge. The *girth* of a graph G, containing a cycle, is the smallest size of the length of the cycles of G and is denoted by $\operatorname{gr}(G)$. If G has no cycles, we define the girth of G to be infinite. An *r-partite graph* is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite graph* (2-partite graph) with parts of size m and n is denoted by $K_{m,n}$. A complete bipartite graph of the form $K_{1,n}$ is called a *star graph.* A graph in which each pair of distinct vertices is joined by an edge is called a *complete graph.* The complete graph on n vertices is denoted K_n. For a graph G, a complete subgraph of G is called a *clique*. The *clique number*, $\omega(G)$, is the greatest integer $n \geq 1$ such that $K_n \subseteq G$ and $\omega(G)$ is infinite if $K_n \subseteq G$ for all $n \geq 1$. The *chromatic number* $\chi(G)$ of a graph G is defined to be the minimum number of colors required to color the vertices of G in such a way that no two adjacent vertices have the same color. A graph G is said to be *finitely colorable* if $\chi(G)$ is finite. A graph is called *weakly perfect* if its chromatic number equals its clique number.

Definition 1.20. Let S be a nonempty set of vertices of A graph G. The *subgraph induced (= generated) by* S is the maximal subgraph of G with vertex set S and denoted by $\langle S \rangle$. That is, $\langle S \rangle$ contains precisely those edges of G joining two vertices in S.

2 Fundamental Properties of $\mathbb{C}(A)$

In this section, besides some simple examples and trivial results, we will discuss some basic properties of the graph $\mathbb{C}(A)$ such as the diameter and core of $\mathbb{C}(A)$ (Theorems 2.3 and 2.6). Finally, we close the section by showing that $\mathbb{C}(R)$ is isomorphic to a subgraph of $\mathbb{C}(A)$ whenever A is a finitely generated cancellation R-module (Theorem 2.7).

The results will show that $\mathbb{C}(A)$ has many properties similar to that of the co-maximal ideal graph $\mathbb{C}(R)$, which is defined and studied by ye and wu in [41].

We now provide a simple example of a local module before stating the next proposition. An R-module A over a ring R is called a multiplication module provided that each submodule of A is of the form IA for some ideal I of R.

Example 2.1. Clearly, a simple module A over a ring R is local with 0 its unique maximal submodule. Furthermore, a nontrivial multiplication module A over a local ring R is a local module. Suppose B is a proper submodule of A and R is a local ring with maximal ideal M. Thus, $B = IA \subseteq MA \neq A$ for some proper ideal I of R and hence MA is the unique maximal submodule of A. Note that by Nakayama's lemma, $MA = A$ implies $A = 0$ since every multiplication module over a local ring is cyclic (finitely generated) [10, Proposition 4].

Proposition 2.2. (cf. [41, Proposition 2.1]) *Let A be a module over a ring R. Then*

(a) *Let A be a nontrivial finitely generated module. Then $\mathbb{C}(A)$ is the empty graph if and only if A is a local module.*

(b) *Assume $|(A)| \geq 2$. Then for any proper submodule B such that $B \not\subseteq J(A)$, B is a vertex of $\mathbb{C}(A)$.*

Proof. (a) Clear. (b) Since $B \not\subseteq J(A)$, there is an M in (A) such that $B \not\subseteq M$. By the maximal property of M, we must have $B + M = A$ (Proposition 1.4). This gives the fact that B is a vertex. □

Throughout the rest part of this paper, all modules are assumed to be nonlocal, i.e. there are at least two maximal submodules in the module. For instance, the direct sum $A = S_1 \oplus S_2$ of two nonisomorphic simple R-modules is an R-module with exactly two maximal submodules, namely $S_1 \oplus \{0\}$ and $\{0\} \oplus S_2$. Note that a direct sum of $\mathbb{Z}_2$ and $\mathbb{Z}_2$ has three maximal submodules, i.e., the two mentioned and the diagonal $\{(0,0),(1,1)\}$.

In [41, Theorem 2.4], it is shown that for a nonlocal ring R, $\mathbb{C}(R)$ is a simple, connected graph with diameter less than or equal to 3. We now show that the co-maximal submodule graph $\mathbb{C}(A)$ has the same property.

Theorem 2.3. (cf. [41, Theorem 2.4]) *For a module A over a ring R, $\mathbb{C}(A)$ is a simple, connected graph with diameter less than or equal to three.*

Proof. Let B_1 and B_2 be any two vertices of $\mathbb{C}(A)$. If $B_1+B_2 = A$, then $\mathrm{d}(B_1, B_2) = 1$. Now we assume $B_1 + B_2 \neq A$. If there is a submodule B_3 such that $B_1 + B_3 = A$ and $B_2 + B_3 = A$, then $\mathrm{d}(B_1, B_2) = 2$. If such submodules do not exist, we can find two different maximal submodules M_1 and M_2 such that $B_1+M_1 = A$, $B_2+M_2 = A$. Since $M_1 \neq M_2$, we have $M_1 + M_2 = A$, thus $\mathrm{d}(B_1, B_2) = 3$. As the diameter of a graph is the maximum distance between any two vertices, the diameter of the graph is less than or equal to three. By the proof, we can easily see that the graph is simple and connected. □

Example 2.4. (cf. [41, Example 2.5]) Let R be a ring and $A = S \oplus S \oplus S$ the direct sum of three copies of a simple R-module S. Now it is easy to see that the diameter of $\mathbb{C}(A)$ is three. Let $a = S \oplus \{0\} \oplus \{0\}$, $b = \{0\} \oplus S \oplus S$, $c = S \oplus \{0\} \oplus S$, $d = \{0\} \oplus S \oplus \{0\}$. Clearly (a, b, c, d) is a path of length 3.

We now, similar to [41, Lemma 2.6], show that for a module A with at least three maximal submodules, the complete graph K_3 (i.e. a triangle) is an induced subgraph of $\mathbb{C}(A)$; so the girth of $\mathbb{C}(A)$ is 3.

Lemma 2.5. (cf. [41, Lemma 2.6]) *Let A be a module over a ring R and let $G = \mathbb{C}(A)$. If $|(A)| \geq 3$, then the complete graph K_3 is an induced subgraph of G. Thus G contains at least one cycle and hence its girth $gr(G) = 3$.*

Proof. If $|(A)| \geq 3$, then A has at least three different maximal submodules, say M_1, M_2, and M_3. Consider the induced subgraph on $\{M_1, M_2, M_3\}$. This graph is the complete graph K_3. As the girth of a graph with cycles is the smallest size of the length of the cycles, so $\mathrm{gr}(G) = 3$. □

• Recall that the *core* of a graph G is the subgraph induced on all vertices of cycles of G, i.e. the *union of the cycles* in G. A vertex x of G is called an *end vertex* in case the degree of x is one. In [41, Theorem 2.7], Ye and Wu proved that if $\mathbb{C}(R)$ contains a cycle, then the core of $\mathbb{C}(R)$ is always a union of triangles and squares, and a vertex in $\mathbb{C}(R)$ is either an end vertex or a vertex of the core.

We now show that the core of $\mathbb{C}(A)$ is always a union of triangles and squares for finitely generated modules.

Theorem 2.6. (cf. [41, Theorem 2.7]) *Let A be a finitely generated module over a ring R and let $G = \mathbb{C}(A)$. If G contains a cycle, then the core of G is a union of triangles and rectangles, and every vertex of G is either an end vertex or a vertex of the core.*

Proof. For the first statement, it is enough to prove that if $(B_1, B_2, \ldots, B_n, B_1)$ is a cycle in G, then each edge of this cycle is an edge of either a triangle or a rectangle. By the symmetric property of the cycle, we only have to prove that $B_1 — B_2$ is an edge of either a triangle or a rectangle. Clearly, for any cycle of size $n \geq 5$, if $B_1+B_3 = A$, or $B_2+B_{n-1} = A$, or $B_2+B_n = A$, then $B_1 — B_2$ belongs to a triangle or a rectangle. Thus we can now assume $n \geq 5$ and $B_1+B_3 \neq A$, $B_2+B_{n-1} \neq A$, and $B_2 + B_n \neq A$. In this case, either $B_1 + B_{n-1} = A$ or $B_1 + B_{n-1} \neq A$. Suppose that $B_1+B_{n-1} \neq A$. Hence, there exists a maximal submodule M of A (Proposition 1.5) such that $M \supseteq B_1 + B_{n-1}$. One can easily see that $B_2 + M = A$ and $B_n + M = A$ since $B_1 \subseteq M$ and $B_{n-1} \subseteq M$. This gives the fact that (B_1, B_2, M, B_n, B_1) is a rectangle. Now, suppose that $B_1 + B_{n-1} = A$. In this case, the result is clear when $n = 5$. Hence, by an inductive argument, we can easily see that if the edge $B_1 — B_2$ belongs to any cycle of size $5 \leq m \leq n-1$ ($n \geq 6$), must belong to a triangle or a rectangle. So the core of $\mathbb{C}(A)$ is a union of triangles and rectangles. For the second statement, we will prove that if B is not a vertex in any cycle, then B will be an end vertex. As G contains a cycle, the vertex number of G is at least 3. Let B be a vertex of G, assume B is not a vertex in any cycle. We claim that there is only one edge adjacent to B. In fact, if this is not the case, then there is a path (B_2, B, B_1, C) (note that $\mathbb{C}(A)$ is a connected graph). Let $B_3 = B_2 + C$. If $B_3 = A$, then (B_2, B, B_1, C, B_2) would be a cycle. Otherwise, we have $B_1 + B_3 = A$ and $B + B_3 = A$, which gives another cycle (B, B_1, B_3, B). Either case leads to a contradiction. □

Finally, besides a remark related to the girth and non-planarity of $\mathbb{C}(A)$, we close this section by showing that $\mathbb{C}(R)$ is isomorphic to a subgraph of $\mathbb{C}(A)$ whenever A is a finitely generated cancellation R-module. An R-module A is a *cancellation module* if for ideals I and J of R, $IA = JA$ implies $I = J$. Examples -of cancellation modules include invertible ideals and free modules (for a detailed study of cancellation modules, see [3]).

Theorem 2.7. *Let A be a finitely generated cancellation module [in particular, a finitely generated free module] over a non-local ring R. Then $\mathbb{C}(R)$ is isomorphic to a subgraph of $\mathbb{C}(A)$.*

Proof. Clearly, if R is a local ring, then $\mathbb{C}(R)$ is the empty graph and hence is a subgraph of $\mathbb{C}(A)$. Suppose I and J are two vertices of the graph $\mathbb{C}(R)$ such that $I + J = R$. Thus $IA + JA = RA = A$. Clearly, each of IA and JA is a proper submodule of A by cancellation property of A and the choice of I and J in R. On the other hand, $J(A)$ does not contain any of IA or JA since, for example, Suppose $IA \subseteq J(A)$, then there exists a maximal submodule M of A that contains JA (Proposition 1.5) and consequently contains $A = IA + JA$, which is a contradiction. Now define a map $\varphi : V(\mathbb{C}(R)) \to V(\mathbb{C}(A))$ by $I \mapsto IA$. Clearly, this function is injective and preserves edges. □

Remark 2.8. From the above result, it is clear that $\mathrm{gr}(\mathbb{C}(A)) \leq \mathrm{gr}(\mathbb{C}(R))$ and non-planarity of $\mathbb{C}(R)$ implies non-planarity of $\mathbb{C}(A)$. For example, $\mathbb{C}(A)$ is not planar if $|(R)| \geq 5$ since $\mathbb{C}(R)$ contains an isomorphic copy of K_5 and by Kuratowski's theorem [15, Theorem 10.30], it is not planar.

3 The Clique Number and the Chromatic Number of the Graph $\mathbb{C}(A)$

In this section, we will prove that the graph $\mathbb{C}(A)$ has the property that its clique number and chromatic number are equal (Theorem 3.1). Moreover, we show that a content [in particular, free, or more generally, projective] module A over a ring R is a finite direct sum of simple modules and $\mathbb{C}(A)$ is a finite Boolean graph whenever $\mathbb{C}(R)$ is a finite Boolean graph (Theorem 3.4). Finally, we will show that for a module A over a ring R with n maximal submodules, $\mathbb{C}(A)$ can be retracted to the n-Boolean graph (Corollary 3.12 and see also Proposition 3.11).

Recall that a graph G is called weakly perfect provided $\chi(G) = \omega(G)$. The following theorem shows that $\mathbb{C}(A)$ is a weakly perfect graph.

Theorem 3.1. (cf. [41, Theorem 3.1]) *Let A be a finitely generated module over a ring R and let $G = \mathbb{C}(A)$. Suppose if $(A) = \{M_1, M_2, \ldots, M_n\}$ (i.e. $|(A)|$ is finite), then $S_{M_i} \not\subseteq S_{M_j}$ for $i \neq j$. Then the following three numbers are equal.*

(a) *The number $|(A)|$ of maximal submodules of A.*

(b) *The clique number $\omega(G)$ of the graph G.*

(c) *The chromatic number* $\chi(G)$ *of the graph* G.

Proof. If $|(A)| = \infty$, it is easy to see that the other two numbers are also infinite. Thus we may assume $|(A)| = n < \infty$ and let $(A) = \{M_1, \ldots, M_n\}$ with $S_{M_i} \not\subseteq S_{M_j}$ for $i \neq j$. Consider the induced subgraph on $\{M_1, \ldots, M_n\}$. It is the complete graph K_n, so $|(A)| \leq \omega(G)$. Since $\omega(G) \leq \chi(G)$ is a well known conclusion in graph theory, we get

$|(A)| \leq \omega(G) \leq \chi(G)$. The only thing left is to prove that $|(A)| \geq \chi(G)$. Let $V_1 = \{B \in V(G) \mid B \subseteq M_1\}$, $V_2 = \{B \in V(G) \mid B \subseteq M_2, B \notin V_1\}$, $V_3 = \{B \in V(G) \mid B \subseteq M_3, B \notin V_1 \cup V_2\}$, $V_n = \{B \in V(G) \mid B \subseteq M_n, B \notin V1 \cup \cdots \cup V_{n-1}\}$.

By Prime Avoidance Theorem (Proposition 1.19), $V_i \neq \emptyset$ and hence $M_i \in V_i$ for each i. Thus this gives an n-coloring implementation on the graph G. So $|(A)| = n \geq \chi(G)$, and thus the three numbers are equal. □

The proof of the following proposition is similar to the proof of [26, Proposition 2.3] and its comaximal ideal version stated for $\mathbb{C}(R)$ in [41, Proposition 3.2] with no proof. We now prove it for $\mathbb{C}(A)$ when A is a finitely generated module.

Proposition 3.2. (cf. [41, Proposition 3.2]) *Let* A *be a finitely generated module over a ring* R *and let* $G = \mathbb{C}(A)$. *Suppose if* $(A) = \{M_1, M_2, \ldots, M_n\}$ *(i.e.* $|(A)|$ *is finite), then* $S_{M_i} \not\subseteq S_{M_j}$ *for* $i \neq j$. *Then the following hold:*

(a) *If* $|(A)| = n$, *where* $1 < n < \infty$, *then* G *is an* n*-partite graph.*

(b) *If* G *is an* n*-partite graph, then* $|(A)| \leq n$. *In this case, if* G *is not an* $(n-1)$*-partite graph, then* $|(A)| = n$.

Proof. The proof is mainly similar to the proof of Proposition 2.3 in [26]. Part (a) follows directly by constructing $V_1, V_2, \ldots, V_n$ as defined in the proof of Theorem 3.1.

For Part (b), let $V_1, V_2, \ldots, V_n$ be The n parts of vertices of $\mathbb{C}(A)$. Assume to the contrary that $|(A)| > n$ and let $M_1, \ldots, M_{n+1} \in (A)$. For any i, choose $B_i \in M_i \setminus \bigcup_{j \neq i} M_j$. Then it is easy To see That $\{B_1, B_2, \ldots, B_{n+1}\}$ is a clique in $\mathbb{C}(A)$. By The Pigeon Hole Principle, Two of B_i's should belong To one of V_i's, That is a contradiction. Therefore $|(A)| \leq n$. Now suppose That $\mathbb{C}(A)$ is not $(n-1)$-partite and $|(A)| = m < n$. By (a), the graph will be m-partite and This is a contradiction. Note that when an n-partite graph can not be reduced to an $(n-1)$-partite graph by joining two of its parts (i.e. union of two parts), then it is impossible to reduce it to an m-partite graph for any $m \leq n-1$ by joining more than two parts of it. □

The following example is a module version of [41, Example 3.3] which provides an important class of graphs, namely the n-Boolean graphs. The definition of a Boolean graph was proposed by Wu and Lu in [44] and we mention it in the following example.

Example 3.3. (cf. [41, Example 3.3]) Let R be a ring and S_i nonisomorphic simple R-modules, where $i = 1, \ldots, n$, with $n \geq 2$. Let $A = \bigoplus_{i=1}^n S_i$ be the direct sum of S_i's. Then $\mathbb{C}(A)$, which is isomorphic to the zero-divisor graph of the ring $(\mathbb{Z}_2)^n$, is called the n-Boolean graph. It is easy to see that both the clique number and the chromatic number of the n-Boolean graph is n, and the n-Boolean graph has only one maximal clique. Obviously, the 2-Boolean graph is the complete graph K_2 and for the 3-Boolean graph, see Fig. 1 in [41, Example 2.5].

The following theorem shows that if $\mathbb{C}(R)$ is a finite Boolean graph, then, similar to [41, Theorem 3.4], $\mathbb{C}(A)$ is a finite Boolean graph and A is a finite direct sum of simple modules provided that A is a content [in particular, free, or more generally, projective] R-module.

Theorem 3.4. (cf. [41, Theorem 3.4]) *Let R be a commutative ring and $\mathbb{C}(R)$ an n-Boolean graph, where $2 \leq n < \infty$. Let A be a content [in particular, free, or more generally, projective] R-module over the ring R with $|(A)| = m$, where $2 \leq m < \infty$ and for any two maximal submodules M and N of A, $S_M \nsubseteq S_N$. Then $\mathbb{C}(A)$ is isomorphic to an m-Boolean graph, $\omega(\mathbb{C}(A)) = m$, and A is isomorphic to a direct sum of m summands of simple modules. Moreover, if $m = n$, then $\mathbb{C}(A)$ is isomorphic to $\mathbb{C}(R)$.*

Proof. By Theorem 3.4(2) in [41], $R \cong \Pi_{i=1}^n F_i$, where each F_i is a field, for $i = 1, \ldots, n$. Thus $J(R) = 0$ and hence $J(A) = 0$ since A is a content module and by Proposition 1.17, $J(R)A = J(A)$. Now by Chinese Remainder Theorem (Proposition 1.12), $A = A/J(A) \cong A/M_1 \oplus A/M_2 \oplus \cdots \oplus A/M_m$. In this case, $\mathbb{C}(A)$ is isomorphic to an m-Boolean graph by Lemma 3.7 and Example 3.3. Thus $\omega(\mathbb{C}(A)) = m$ and there is only one maximal clique in $\mathbb{C}(A)$. The "in particular part", follows from the fact that every free, or more generally, every projective module is content by [33]. □

The following theorem, similar to [41, Theorem 3.4], shows that $\mathbb{C}(A)$ is a finite Boolean graph and A is a finite direct sum of simple modules provided that the Jacobson radical of A is zero.

Theorem 3.5. (cf. [41, Theorem 3.4]) *Let R be a commutative ring and let A be an R-module over the ring R with $|(A)| = m$, where $2 \leq m < \infty$ and for any two maximal submodules M and N of A, $S_M \nsubseteq S_N$. If $J(A) = 0$, then $\mathbb{C}(A)$ is isomorphic to an m-Boolean graph, $\omega(\mathbb{C}(A)) = m$, and A is isomorphic to a direct sum of m summands of simple modules.*

Proof. The proof is similar to the proof of the above theorem by using the Chinese Remainder Theorem (Proposition 1.12) since $J(A) = 0$ by hypothesis. □

• For simple graphs G and H, recall that a graph homomorphism from G to H is a map $\varphi : G \to H$ such that for distinct $u, v \in V(G)$, $\{u, v\} \in E(G)$ implies $\varphi(u) \neq \varphi(v)$ and $\varphi(u) — \varphi(v) \in E(H)$ (i.e., edge goes to edge). Moreover, if further H is a subgraph of G and the restriction of φ on H is the identity map, then H is called a *retract* of G.

Before closing this section by two results related to a retract of $\mathbb{C}(A)$, respectively, (proposition 3.11 and Corollary 3.12), we need the following (five) lemmas.

Lemma 3.6. (cf. Theorem 1.10 of Chapter 4 in [22]) *If R is a ring and B is a submodule of an R-module A, then there is a one-to-one correspondence between the set of all submodules of A containing B and the set of all submodules of A/B, given by $C \mapsto C/B$. Hence every submodule of A/B is of the form C/B, where C is a submodule of A which contains B.*

By using the above result we can prove the following lemma. Similarly, Corollary 2.10 of [4] states that A factor module M/K is simple if and only if K is a maximal submodule of M.

Lemma 3.7. *A/M is a simple module if and only if M is a maximal submodule of A.*

Lemma 3.8. (cf. Theorem 1.9 of Chapter 4 in [22]) *Let B and C be submodules of a module A over a ring R.*

(i) *There is an R-module isomorphism $B/(B \cap C) \cong (B + C)/C$;*

(ii) *if $C \subseteq B$, then B/C is a submodule of A/C, and there is an R-module isomorphism $\frac{(A/C)}{(B/C)} \cong A/B$.*

By the above result and Lemma 3.7, we have the following.

Lemma 3.9. *There is a one-to-one correspondence between the maximal submodules of A and $A/J(A)$. That is, M is a maximal submodule of A if and only if $M/J(A)$ is a maximal submodule of $A/J(A)$.*

Lemma 3.10. *Let $f : A \to A/J(A)$ be the canonical epimorphism of modules. Then $f^{-1}(J(A/J(A))) = J(A)$.*

Proof. $f^{-1}(J(A/J(A))) = f^{-1}(\bigcap M_i/J(A)) = \bigcap f^{-1}(M_i/J(A)) = \bigcap M_i = J(A)$. Note that the inverse function of any map between two sets preserves the intersection operation and inclusion relation. □

Finally, by using the preceding (five) lemmas (implicitly), we show that $\mathbb{C}(A/J(A))$ is a retract of $\mathbb{C}(A)$ and close this section by showing that for an R-module A with only finitely many maximal submodules, the graph $\mathbb{C}(A)$ can be retracted to an n-Boolean graph (Corollary 3.12).

Proposition 3.11. (cf. [41, Proposition 3.6]) *Let A be a module over a ring R. Then $\mathbb{C}(A/J(A))$ is a retract of (A).*

Proof. For the proof, we use the preceding (five) lemmas implicitly. By verifying the definition of graph retract directly, note that $B + C = A$ if and only if $[(B + J(A))/J(A)] + [(C + J(A))/J(A)] = A/J(A)$ (i.e. edge preserving), and this shows that $\mathbb{C}(A)$ contains a subgraph which is isomorphic to $\mathbb{C}(A/J(A))$. The result then follows. □

By this proposition and the Chinese Remainder Theorem (Proposition 1.12), the following corollary is easily obtained.

Corollary 3.12. (cf. [41, Corollary 3.7]) *For a module A over a ring R, let $G = \mathbb{C}(A)$. If $|(A)| = n < \infty$, then the n-Boolean graph is a retract of G.*

4 $\mathbb{C}(A)$ of Modules with Two Maximal Submodules

In this section, we study the graph $\mathbb{C}(A)$ of modules A with exactly two maximal submodules. We will show that in such a situation, $\mathbb{C}(A)$ is a complete bipartite graph (Lemma 4.1) and consequently, $\mathrm{gr}(\mathbb{C}(A))$ is 4 or ∞ (Corollary 4.3) provided that A is finitely generated. Moreover, For a finitely generated module A, $\mathbb{C}(A)$ is a (complete) bipartite graph if and only if A has exactly two maximal submodules (Theorem 4.5). Theorem 4.7 gives a necessary and sufficient condition for $\mathrm{diam}(\mathbb{C}(A)) = 1$ when A is a finitely generated module. Finally, in Theorems 4.9 and 4.10, we study the conditions under which $\mathbb{C}(A)$ is a complete graph if and only if A is a direct sum of two simple R-modules.

The following lemma shows that $\mathbb{C}(A)$ is a complete bipartite graph when A is a finitely generated module with exactly two maximal submodules.

Lemma 4.1. (cf. [41, Lemma 4.1]) *Let A be a finitely generated module over a ring R and assume $|(A)| = 2$. Then $\mathbb{C}(A)$ is one of the following:*

(a) *The complete graph $K_2 = K_{1,1}$.*

(b) *A star graph $K_{1,n}$, where $2 \leq n < \infty$ is a positive integer.*

(c) *A complete bipartite graph* $K_{m,n}$*, where* m *and* n *are positive integers with* $2 \leq m < \infty$ *and* $2 \leq n < \infty$*.*

Proof. We can assume that $(A) = \{M_1, M_2\}$. In this situation, it is enough to prove that for any two distinct submodules B_1 and B_2, neither of which is contained in $J(A)$, if they are contained in M_1 and M_2, respectively, then we must have $B_1 + B_2 = A$. In fact, if $B_1 + B_2 \neq A$, then $B_1 + B_2 \subseteq M_1$ or $B_1 + B_2 \subseteq M_2$ (Proposition 1.5). In either case, there is a contradiction. □

By Lemma 4.1 and properties of complete bipartite graphs, we can easily get the following two corollaries.

Corollary 4.2. (cf. [41, Corollary 4.2]) *Let* A *be a finitely generated module over a ring* R *with exactly two maximal submodules and let* $G = \mathbb{C}(A)$*. Then diam*$(G) = 1$ *or 2.*

Corollary 4.3. (cf. [41, Corollary 4.3]) *Let* A *be a finitely generated module over a ring* R *with exactly two maximal submodules and let* $G = \mathbb{C}(A)$*. Then gr*$(G) = 4$ *or* ∞*.*

By Lemma 2.5 and Corollary 4.3, we have the following theorem.

Theorem 4.4. (cf. [41, Theorem 4.4]) *Let* A *be a finitely generated module over a ring* R *and let* $G = \mathbb{C}(A)$*. Then gr*$(G) = 3$*, 4 or* ∞*.*

For a finitely generated module A over a ring R, the following theorem shows that $\mathbb{C}(A)$ is a (complete) bipartite graph if and only if A has exactly two maximal submodules.

Theorem 4.5. (cf. [41, Theorem 4.5]) *For a finitely generated module* A *over a ring* R*, the following statements are equivalent:*

(a) $\mathbb{C}(A)$ *is a complete bipartite graph;*

(b) $\mathbb{C}(A)$ *is a bipartite graph;*

(c) A *has only two maximal submodules, i.e.* $|(A)| = 2$*.*

Proof. (a) $\Rightarrow$ (b) is obvious and (c) $\Rightarrow$ (a) is easily obtained by Lemma 4.1. (b) $\Rightarrow$ (c) will be proved below. If $|(A)| = 1$, then the graph $\mathbb{C}(A)$ is the empty graph. If $|(A)| > 2$, by Lemma 2.5, we have that $\mathbb{C}(A)$ has a subgraph K_3. The chromatic number of K_3 is 3; so the chromatic number of $\mathbb{C}(A)$ is greater than or equal to 3. This contradicts the fact that the chromatic number of a bipartite graph is 2. □

We will use the following lemma for the proof of the next two results.

Lemma 4.6. *Let A be a module over a ring R with B a submodule of A and I an ideal of R. Suppose P is a prime submodule of A and $IB \subseteq P$. Then either $I \subseteq S_P$ or $B \subseteq P$.*

Proof. Suppose $IB \subseteq P$ and $B \nsubseteq P$. Thus, there exists $b \in B \setminus P$. Now for each $r \in I$, $rb \in P$ implies $r \in S_P$ (by definition of a prime submodule) and hence $I \subseteq S_P$. □

The following theorem gives a necessary and sufficient condition for $\text{diam}(\mathbb{C}(A)) = 1$ when A is a finitely generated module.

Theorem 4.7. (cf. [41, Theorem 4.6]) *Let A be a finitely generated module over a ring R and let $G = \mathbb{C}(A)$. Suppose for any maximal submodules M and N of A, $S_M \nsubseteq S_N$. Then G is complete if and only if diam$(G) = 1$ if and only if $G = K_2$.*

Proof. We just argue for the case when $\text{diam}(G) = 1$ implies $G = K_2$. If $|(A)| = 1$, then the graph $\mathbb{C}(A)$ is the empty graph; so its diameter cannot be 1. If $|(A)| > 2$, then there are at least three different maximal submodules in A, say M_1, M_2, and M_3. It is easy to verify that $S_{M_1}M_2 \nsubseteq J(A)$ since by primeness of M_3, either $S_{M_1} \subseteq S_{M_3}$ or $M_2 \subseteq M_3$ (Lemma 4.6). Consider $\text{d}(S_{M_1}M_2, M_1)$. As $S_{M_1}M_2 \subseteq M_1$ ($M_2 \subseteq A$ implies $S_{M_1}M_2 \subseteq S_{M_1}A \subseteq M_1$ by definition of S_{M_1}), $S_{M_1}M_2 + M_1 = M_1 \neq A$, $\text{d}(S_{M_1}M_2, M_1) \neq 1$. Also, $M_3 + M_1 = A$ and $M_3 + S_{M_1}M_2 = A$ (Proposition 1.4) shows that $\text{d}(S_{M_1}M_2, M_1) = 2$. By the definition of the diameter of a graph, we get $\text{diam}(G) \geq 2$. This contradicts the condition $\text{diam}(G) = 1$. So $|(A)| = 2$. At last, by Lemma 4.1, the only possibility for G with its diameter 1 is the complete graph K_2. □

The following proposition considers the conditions for $\text{diam}(\mathbb{C}(A)) = 2$ when $J(A)$ is a prime submodule of A.

Proposition 4.8. (cf. [42, Proposition 3.3]) *Let A be a module over a ring R and assume for each submodule $B < A$, $S_B \nsubseteq S_{J(A)}$ whenever $B \nsubseteq J(A)$. Then diam$(\mathbb{C}(A)) = 2$ provided that $\mathbb{C}(A)$ is not a complete graph and $J(A)$ is a prime submodule of A.*

Proof. Clearly, $2 \leq \text{diam}(\mathbb{C}(A)) \leq 3$ by Theorem 2.3 and hypothesis. Let $J(A)$ be a prime submodule of A. As the diameter of a graph is the supreme distance between two vertices, so if we want to prove $\text{diam}(\mathbb{C}(A)) = 2$, it suffices to prove the distance of any two vertices of $\mathbb{C}(A)$ is 1 or 2. For any two proper submodules $B, C \nsubseteq J(A)$, by the prime property of $J(A)$ and Lemma 4.6 and hypothesis, we get $S_BC \nsubseteq J(A)$. If $B + C = A$, then $\text{d}(B, C) = 1$. Now if $B + C \neq A$, thus $\text{d}(B, C) \neq 1$. Since $J(A)$ is the intersection of all maximal submodules of A, there is at least one maximal submodule M of A, such that $S_BC \nsubseteq M$. Thus $S_BC + M = A$ (Proposition 1.4). Hence we get $B+M = A$ (since $S_BC \subseteq S_BA \subseteq B$) and $C+M = A$ (since $S_BC \subseteq C$), which shows that $\mathrm{d}(B, C) = 2$. □

Finally, we close this paper with the following two theorems in which $\mathbb{C}(A)$ is a complete graph if and only if A is the direct sum of two simple modules.

Theorem 4.9. (cf. [41, Theorem 4.6]) *Let R be a commutative ring and $\mathbb{C}(R)$ an n-Boolean graph with $2 \leq n < \infty$. Let A be a finitely generated free R-module over the ring R and let $G = \mathbb{C}(A)$. Suppose for any maximal submodules M and N of A, $S_M \nsubseteq S_N$. Then the following statements are equivalent:*

(a) *G is a complete graph.*

(b) *$diam(G) = 1$.*

(c) *$G = K_2$.*

(d) *$A = S_1 \oplus S_2$, where S_1 and S_2 are simple R-modules.*

Proof. Clearly, (a), (b), and (c) are equivalent from Theorem 4.7 and (d) implies (a) is obvious. Now (c) implies (d) since by Proposition 3.2, A has only two maximal submodules and Theorem 3.4 completes the proof. □

Theorem 4.10. (cf. [41, Theorem 4.6]) *Let R be a commutative ring and let A be a finitely generated R-module over the ring R and let $G = \mathbb{C}(A)$. Suppose $J(A) = 0$ and for any maximal submodules M and N of A, $S_M \nsubseteq S_N$. Then the following statements are equivalent:*

(a) *G is a complete graph.*

(b) *$diam(G) = 1$.*

(c) *$G = K_2$.*

(d) *$A = S_1 \oplus S_2$, where S_1 and S_2 are simple R-modules.*

Proof. By using Theorem 3.5, the proof is similar to the proof of the above theorem. □

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