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To Academician Professor Radu Miron on the Occasion of his 90th Birthday

Radu Miron, member of the Romanian Academy and Emeritus Professor of the University "Al.I. Cuza" of Iaşi, Romania, celebrated his 90th birthday on October 3, 2017.

He was born in Codaesti, Vaslui county, Romania, in 1927.

In 1948 he enrolled as a student at the Faculty of Mathematics and Physics of the University "Al.I. Cuza" of Iaşi, where he graduated in 1952. In 1953 he enrolled in a Ph. D. program at the Mathematical Institute of the Romanian Academy in Iaşi and, in 1957, obtained his PhD in Mathematics and Physics, defending the thesis *The geometrization of the nonholonomic mechanical systems*", written under the direction of Academician Mendel Haimovici.

In 1973 received the *Doctor Docent* degree. In 1991 he became *Corresponding Member of the Romanian Academy*, and was named a *Full member of the Romanian Academy* in 1993.

At the Faculty of Mathematics and Physics of the University "Al.I. Cuza" of Iaşi, he was appointed *Instructor* in 1950, *Assistant Professor* in 1956, *Associate Professor* in 1963 and *Full Professor* in 1969. He has been Dean of the faculty during the period 1972-1976, and Head of the Department of Geometry in 1976.

His scientific activity began at the Mathematical Seminar "Alexandru Myller" in Iasi, founded by the Academicians Al. Myller and O. Mayer. The almost three hundred publications whose author he is, among them being 35 textbooks, books and monographs, ilustrate the coordinates of evolution of the scientific and didactic thinking of Professor Radu Miron.

He started his research with the study of the geometrization of the nonholonomic mechanical systems with scleronom links. In his Ph. D. Thesis he solved a problem that has been raised by É. Cartan. In connection with it, then he studied nonholonomic manifolds.

In 1960 Professor Radu Miron studied the so-called Myller configurations. The results were presented in a monograph that received the Gh. Tzitzeica prize of the Romanian Academy. Then he brought significant contributions to the theory of the Weyl , Norden and conformal symplectic spaces. The researche in Finslerian geometry and its generalizations that Professor Radu Miron has introduced in Romania have brought him many satisfactions and successes.

In 1974 he comes to Finsler spaces with an outstanding contribution by building a field of orthonormal frames intrinsically associated to an $n-$dimensional Finsler space. It was called "*the Miron frame*" by Prof. Dr. Makoto Matsumoto from Japan, in his monograph devoted to Finsler spaces.

In 1980 Professor Radu Miron initiates at the University of Brasov The First National Seminar on Finsler Geometry, which was held every two years ever since, and where Prof. Radu Miron presented his main discoveries : generalized Finsler metrics, Lagrange spaces, generalized Lagrange spaces, Hamilton spaces as well as a geometry of the total space of a vector bundle based on the use of a nonlinear connection.

Turning on the applications in Theoretical Physics he developed a Finslerian Theory of Relativity and published at the Romanian Academy in 1987 the monograph Vector bundles. Lagrange spaces. Applications to Relativity, written together with M. Anastasiei.

Since 1988, Professor Radu Miron concentrates much more on applications of the theory of Lagrange spaces and of generalized Lagrange spaces to Theoretical Physics. These applications were included in the book *The Geometry of Lagrange Spaces: Theory and Applications* (R. Miron, M. Anastasiei) published in 1984 by Kluwer Academic Publishers.

He continues with a deep study of the higher order Lagrange spaces that was developed in the monographs *The Geometry of Higher Order Lagrange Spaces. Applications to Mechanics and Physics* (R. Miron, 1997) published by Kluwer Academic Publishers, and *The Geometry of Higher Order Finsler Spaces*, published by Hadronic Press in 1998.

In 2001, the monograph *The geometry of Lagrange and Hamilton spaces* (jointly with D. Hrimiuc, H. Shimada and V.S. Sabau) was published by Kluwer Academic Publishers. In 2007 his book titled *Finsler-Lagrange Geometry. Applications to* *dynamic systems* (jointly with I. Bucataru) has been published by the Romanian Academy.

Professor Radu Miron received the *Doctor Honoris Causa degree* from universities from Constantza, Craiova, Bacau, Oradea, Galati, Tiraspol. Also, he received diplomas of excellence from: the Romanian Ministry of Education, the University "Al.I. Cuza" Iaşi, The University "P. Andrei" Iaşi.

He is a honorary member of the Academy of Sciences from the Republic of Moldova and Emeritus Professor of the "Al.I. Cuza" University of Iaşi.

In 2003 he received the Opera Omnia award from the Romanian National Council for Scientific Research and the V.Pogor award from City Hall of Iaşi

Professor Radu Miron was invited to lecture by well-known institutions from France, Great Britain, the former Soviet Union, Italy, Germany, Hungary, Yugoslavia, Japan.

A remarkable gifted professor, endowed with the grace of speaking, he has left an indelible mark upon numerous generation of students and collaborators. Being highly concerned with the teaching of geometry at all levels, he wrote books for pupils and students, as well as monographs having a high scientific level meant for researchers.

He supervised 30 Ph. D. students, thirteen of whom were from abroad : Japan, Italy, Hungary, Vietnam.

Although retired in 1997, he continues to work as a Consulting Professor of the Faculty of Mathematics, and as a researcher at the Mathematical Institute "O.Mayer" of the Romanian Academy in Iaşi.

This issue of Libertas Mathematica (new series) is dedicated to Professor Radu Miron in celebration of his 90th birthday, as a recognition of his outstanding contributions to mathematical research and an homage of the Romanian school of mathematics.

Vasile Staicu - Editor-in-Chief

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Selections of set-valued mappings via applications

Mitrofan M. Cioban

Abstract: Our aim is to study the problem of tightness of compact subsets of the space $M_r(X)$ of all Radon measures on the space X equipped by the topology of weak convergence. A kernel on a space Z into the space $M_r(S)$ is a continuous mapping $k : Z \longrightarrow M_r(X)$. A space X is called a uniformly Prohorov space if for each $\varepsilon > 0$, any paracompact space Z and any kernel $k : Z \longrightarrow M_r(X)$ there exists an upper semi-continuous compact-valued mapping $S_{(k,\varepsilon)} : Z \longrightarrow X$ such that $\mu_{(k,z)}(X \setminus S_{(k,\varepsilon)}(z)) \leq \varepsilon$ for each $z \in Z$. Any sieve-complete space is a uniformly Prohorov space (Corollary 3.4). Any uniformly Prohorov space is a Prohorov space. A space X is sieve-complete if and only if X is an open continuous image of a paracompact Čech-complete space. The idea of the concept of a uniformly Prohorov goes to A. Bouziad, V. Gutev and V. Valov.

Keywords: Set-valued mapping Selection, Radon measure, support of measure, Prohorov space

MSC2010: 28C15, 54C60, 54C35, 54C50

Dedicated to Academician Radu Miron on the occasion of his 90'th birthday

1 Introduction

By a space we understand a completely regular topological Hausdorff space. We use the terminology from [13]. Let βX denote the StoneÂ-Čech compactification of a space X, $cl_X A$ or clA denote the closure of a set A in a space X, $\mathbb{R}$ denote the space of reals, $\mathbb{N} = \{1, 2, ...\}$ denote the discrete space of natural numbers.

Let X be a space. By $Bo(X)$ we denote the σ-algebra of Borel subsets of X, by $C(X)$ we mean the ring of all real-valued continuous functions on X and by $C^0(X)$ we denote the Banach space of all continuous real-valued bounded functions on X

with the sup-norm $\|f\| = sup\{|f(x)| : x \in X\}$. Let $exp(X)$ be the family of all closed subsets of X and $exp_c(X)$ be the family of all compact subsets of X.

The space $exp(X)$ is endowed with the Vietoris topology. Recall that the Vietoris topology is generated by the open base of all collections of the form $e(\mathcal{V}) = \{F \in exp(X) : F \subset \cup\mathcal{V}, F \cap V \neq \emptyset \text{ for any } V \in \mathcal{V}\}$, where $\mathcal{V}$ runs over the finite families of open subsets of X. We consider $exp_c(X)$ as the subspace of the space $exp(X)$.

A *Borel measure* defined on the space X is a non-negative function $\mu : Bo(X) \longrightarrow \mathbb{R}$ with the following properties:

- $\mu(\emptyset) = 0$;

- If $A, B \in Bo(X)$ and $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$;

-$\mu(B) = sup\{\mu(F) : F \subset B, F \in exp(X)\}$ for every $B \in Bo(X)$.

Let μ be a Borel measure on X. Then for every function $f \in C^0(X)$ the integral $\int f d\mu$ is defined, and there exists a unique linear non-negative functional $u_\mu : C^0(X) \longrightarrow \mathbb{R}$ such that $u_\mu(f) = \int f d\mu$ for every $f \in C^0(X)$ (see [14, 21, 25, 18, 23, 24]). Equip the space $M(X)$ of all Borel measures with the topology of weak convergence. The weak topology on the space of all Borel measures $M(X)$ is the weakest topology for which $\mu_\alpha \to \mu$ if and only if $\int f d\mu_\alpha \to \int f d\mu$ for every $f \in C^0(X)$.

The measure μ is called:

(1) σ-additive if $\mu(\cup\{A_n : n \in \mathbb{N}\}) = \Sigma\{\mu(A_n) : n \in \mathbb{N}\}$ provided $\{A_n \in Ba(X) : n \in \mathbb{N}\}$ and $A_n \cap A_m = \emptyset$ for $n \neq m$;

(2) τ-additive or smooth if for any net of bounded continuous functions $\{f_\lambda \in C^0(X) : \lambda \in L\}$ which is monotone decreasing ($\lambda < \eta$ implies $f_\eta \leq f_\lambda$) and pointwise convergent to 0 we have $lim\{\int f_\lambda d\mu : \lambda \in L\}) = 0$;

(3) *tight* or *Radon measure* if $\mu(B) = sup\{\mu(F) : F \subset B, F \in exp_c(X)\}$ for every $B \in Bo(X)$.

If $M_\sigma(X)$ is the set of all σ-additive measures, $M_\tau(X)$ is the set of all τ-additive measures and $M_r X)$ is the set of all Radon measures on the space X, then $M_r(X) \subset M_\tau(X) \subset M_\sigma(X) \subset M(X)$. Denote by $P(X) = \{\mu \in M_r(X) : \mu(X) = 1\}$ the set of all *probabilist measures* on the space X.

For every Radon measure μ on X there exists a unique Radon measure $\beta\mu$ on βX such that $\beta\mu(H) = \mu(H \cap X)$ for every $H \in Bo(\beta X)$.

A subset Φ of the space $M(X)$ is called *tight* if for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset X$ such that $\mu(X \setminus K_\varepsilon) < \varepsilon$ for each $\mu \in \Phi$. A space X is a *Prohorov space* if any compact subset of $M_r(X)$ is tight. Our aim is to study the problem of uniform tightness of compact subsets of the space $M_r(X)$.

The class of sieve-complete spaces was defined in [26, 4, 6, 10]. Let X be a space, $\gamma = \{\gamma_n = \{U_\alpha : \alpha \in A_n\} : n \in \mathbb{N}\}$ be a sequence of families of open non-empty subsets of the space X, and let $\pi = \{\pi_n : A_{n+1} \to A_n : n \in \omega\}$ be a sequence of single-valued mappings. A sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ is called *a spectral sequence* if $\alpha_n \in A_n$ and $\pi_n(\alpha_{n+1}) = \alpha_n$ for every $n \in \mathbb{N}$. Consider the following conditions:

(SC1) $\cup\{U_\beta : \beta \in A_n\} = X$ for each $n \in \mathbb{N}$.

(SC2) $\cup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = \cup\{cl_X U_\beta : \beta \in \pi_n^{-1}(\alpha)\} = U_\alpha$ for all $\alpha \in A_n$ and $n \in \mathbb{N}$.

(SC3) For each spectral sequence $\alpha = \{\alpha_n : n \in \mathbb{N}\}$ any set $cl_X\{x_n \in U_{\alpha_n} : n \in \mathbb{N}\}$ is a compact subset of the space X and the set $H(\gamma, \alpha) = \cap\{U_{\alpha_n} : n \in \mathbb{N}\}$ is a compact subset of the space X.

(SC4) For all $n \in \mathbb{N}$, $\alpha \in A_n$ and $\beta \in q^{-1}(\alpha)$ the sets $X \setminus U_\alpha$ and U_β form a pair of completely separated subsets of the space X.

The sequences γ and π are called *an A-sieve* if they are Properties (SC1) and (SC2). The sieve with property (SC4) is called a completely separated sieve. The sequences γ and π are called an *CA-sieve* if they are Properties (SC1), (SC2) and (SC3). If on the space X there exists a CA-sieve, then on X there exists a CA-sieve with Property (SC4) [6, 10].

A space X is called *sieve-complete* if on X there exists an CA-sieve (see [6, 10, 4]). Any Čech-complete space is sieve-complete.

We mention the following characteristic of sieve-complete spaces (see [6, 9]):

Corollary 1.1. *For a space X the following assertions are equivalent:*

1. X is a sieve-complete space.

2. X is an open continuous image of a paracompact Čech-complete space.

3. $exp_c(X)$ is an open continuous image of a paracompact Čech-complete space.

4. $exp_c(X)$ is a sieve-complete space.

Proof. The equivalences $1 \to 2 \to 1$ were proved in [6, 26]. Let $\varphi : Z \longrightarrow X$ be a continuous open mapping of a sieve-complete space Z onto X. Then for any open subset U of Z and any compact subset $F \subset \varphi(U)$ there exists a compact subset $\Phi \subset U$ such that $\varphi(\Phi) = F$. Consider the mapping $\varphi_c : exp_c(Z) \longrightarrow exp_c(X)$, where $\varphi_c(\Phi) = \varphi(\Phi)$ for each $\Phi \in exp_c(Z)$. The element $\Phi \in exp_c(Z)$ is considered as a point of the space $exp_c(Z)$ and as a subset of Z. By virtue of Theorem 2 from [9], φ_c is a continuous open mapping of $exp_c(Z)$ onto $exp_c(X)$. If Z is a paracompact Čech-complete space, then $exp_c(Z)$ is a paracompact Čech-complete space too ([7], Theorems 1 and 4). This proved the implications $2 \to 3 \to 4$. Since X is a closed subspace of the space $exp_c(X)$, the proof is complete. □

2 Support of measures

A correspondence $\theta : X \longrightarrow Y$ is called a set-valued mapping if $\theta(x) \in exp(Y)$ and $\theta(x) \neq \emptyset$ for each $x \in X$. We recall that a set-valued mapping $\theta : X \longrightarrow Y$

is lower (upper) semi-continuous, or l.s.c. (u.s.c), if the set $\theta^{-1}(H) = \{x \in X : \theta(x) \cap H \neq \emptyset\}$ is open (closed) in X for for every open (closed) subset H of Y. Let $\varphi, \psi : X \longrightarrow Y$ be set-valued mappings. If $\varphi(x) \subset \psi(x)$ for each $x \in X$, then φ is called a *selection* of the mapping ψ. If $\varphi(x) \cap \psi(x) \neq \emptyset$ for each $x \in X$, then φ is called a *generalized selection* of the mapping ψ. The articles [6, 10] contain some applications of generalized selections.

Let X be a space. For every Borel measure $\mu \in M(X)$ the support $supp_X(\mu)$ is the set of all points $x \in X$ such that $\mu(U) > 0$ for each neighbourhood U of the point x in X. Obviously, the set $supp_X(\mu)$ is a closed subset of X. Every point-finite family of open subsets of the subspace $supp_X(\mu)$ is countable (see [11, 7]). Moreover, the Souslin number (cellularity) $c(supp_X(\mu))$ is countable.

In ([15], Propositions 2.1 and 2.2) for the space $P(X)$ were proved the following important assertions:

Proposition 2.1. *Let X be a space, and let $\varepsilon \in (0,1)$. Define the set-valued mappings $\Psi_\varepsilon, \Phi_\varepsilon : M_r(X) \longrightarrow exp_c(X)$ by $\Psi_\varepsilon(\eta) = \{F \in \exp_c(X) : \eta(X \setminus F) < \varepsilon\}$ and $\Phi_\varepsilon(\eta) = cl_{exp_c(X)}\Psi_\varepsilon(\eta)$. Then:*

1. Ψ_ε is a lower semi-continuous mapping.

2. Φ_ε is a lower semi-continuous mapping.

3. $\eta(X \setminus F) \leq \varepsilon$ for each $\eta \in M_r(X)$ and $F \in \Phi_\varepsilon(\eta)$.

Proof. Let $\varepsilon \in (0,1)$, $\mathcal{V} = \{V_1, V_2, ..., V_n\}$ be a finite collection of non-empty subsets of X and $V = \cup\mathcal{V}$. Fix $\eta \in M_r(X)$. Assume that $K \in \Psi_\varepsilon(\eta) \cap e(\mathcal{V})$. Since $\eta(X) = p < +\infty$, $K \subset V$ and $\eta(X \setminus K) < \epsilon$, we have $\eta(X \setminus V) < \varepsilon$. Let $3\delta = \eta(K) + \varepsilon - p$. Consider the continuous functions $f, h : X \longrightarrow [0,1]$ with properties:

- $f(x) = 1$ for each $x \in K$ and $cl_X(X \setminus f^{-1}(0)) \subset V$;
- $h(x) = 1$ for each $x \in X$.

Then the set $U = \{\xi \in M_r(X) : p-\delta < \int hd\xi < p+\delta, \eta(K)-\delta < \int fd\xi < p+\delta\}$ is open in $M_r(X)$ and $\eta \in U$. We put $H = cl_X(X \setminus f^{-1}(0))$. Let $\xi \in U$. Since $p-\delta < \int hd\xi < p+\delta$, we have $p-\delta < \xi(X) < p+\delta$. Since $\eta(K)-\delta < \int fd\xi < p+\delta$, we have $\xi(U) > \eta(K) - \delta$. Hence $p - \varepsilon + \delta < \eta(K) - \delta < \xi(H) \leq \xi(X) < p + \delta$ and $\xi(H) > \xi(X) - \varepsilon$. There exists a compact subset F of X such that $F \subset H$ and $\xi(H) \geq \xi(F) > \xi(X) - \varepsilon$. For every $i \leq n$ fix a point $a_i \in V_i$. Put $\Phi = F \cup \{a_1, a_2, ..., a_n\}$. Then $\Phi \in \Psi_\varepsilon(\xi) \cap e(\mathcal{V})$. Hence $\eta \in U \subseteq \Psi_\varepsilon^{-1}(e(\mathcal{V}))$ and the set $\Psi_\varepsilon^{-1}(e(\mathcal{V}))$ is open. Assertion 1 is proved. Assertion 2 follows from Assertion 1.

Assume that $\eta \in M_r(X)$ and $K \in \Phi_\varepsilon(\eta)$. Suppose that $\eta(X \setminus K) > \varepsilon$. There exists a compact subset $F \subset X \setminus K$ such that $\eta(F) > \varepsilon$. Let $V = X \setminus F$. The set $e(V)$ is open in $exp_c(X)$ and $K \in e(V)$. Then there exists a compact set $\Phi \in e(V) \cap \Psi_\varepsilon(\eta)$. Since $\Phi \subset V$ and $\Phi \cap F = \emptyset$, we have $\eta(X \setminus \Phi) > \varepsilon$, a contradiction. Assertion 3 is proved. This complete the proof. □

The assertions from the following proposition were proved in [25].

Proposition 2.2. *Let $\mu \in M_\tau(X)$. Then $\mu(supp_X(\mu)) = \mu(X)$ and for any family γ of open subsets of X there exists a countable subfamily ξ such that $\mu(\cup\xi) = \mu(\cup\gamma)$.*

If X is a metacompact space or a subparacompact space and $\mu \in M_\tau(X)$, then the subspace $supp_X(\mu)$ is Lindelöf ([25], Theorem 27 for a paracompact space X).

The following theorem for locally compact spaces and complete metrizable spaces was proved in [19].

Theorem 2.3. *Let X be a sieve-complete space. Then any Borel τ-additive measure on X is a Radon measure.*

Proof. Fix $\mu \in M(X)$. There exist a paracompact Čech-complete space Z and an open continuous mapping $g : Z \longrightarrow X$. Fix a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of βZ such that $Z = \cap\{U_n : n \in \mathbb{N}\}$ and $\epsilon > 0$. Let $n \in \mathbb{N}$, $\delta > 0$ and U is an open subset of Z. By virtue of Proposition 2.2, there exists an open subset $V(n, U, \delta)$ of Z such that $V(n, U, \delta) \subset U$, $cl_{\beta Z}V(n, U, \delta) \subset U_n$ and $\mu(g(U)) \setminus \mu(V(n, U, \delta)) < \delta$. Therefore, there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of open subsets of βZ such that $V_{n+1} \subset cl_{\beta Z}V_{n+1} \subset V_n \subset cl_{\beta Z}V_n \subset U_n$ and $\mu(g(Z \cap V_n)) > \mu(X) - \epsilon$ for each $n \in \mathbb{N}$. Then $F = \cap\{V_n : n \in \mathbb{N}\}$ is a compact subset of Z and $g(F) = \cap\{g(Z \cap V_n) : n \in \mathbb{N}\}$. Hence $\mu(X \setminus g(F)) \leq \epsilon$. □

3 Uniformly Prohorov spaces

Let X and Z be topological spaces. A *measurable kernel* with source Z and target X is a mapping $\lambda : Z \times Bo(X) \longrightarrow \mathbb{R}$ with the following properties:

(1) $\mu_{(\lambda,z)} : Bo(X) \longrightarrow \mathbb{R}$, where $\mu_{(\lambda,z)}(A) = \lambda(z, A)$ for each $A \in Bo(X)$, is a Radon measure on X for each $z \in Z$;

(2) $\chi_{(\lambda,z)} : Z \longrightarrow \mathbb{R}$, where $\chi_{(\lambda,z)}(A) = \lambda(z, A)$ for each $z \in Z$, is a Borel measurable function on Z for each $A \in Bo(X)$;

(3) The mapping $k_\lambda : Z \longrightarrow M_r(X)$, where $\kappa_\lambda(z) = \mu_{(\lambda,z)}$ for each $z \in Z$, is continuous.

We say that a *kernel* on Z into the space $M_r(X)$ of Radon measures on X is a continuous mapping $k : Z \longrightarrow M_r(X)$. If $k : Z \longrightarrow M_r(X)$ is a kernel, then $\mu_(k, z)$ $= k(z) \in M_r(X)$ [22].

A space X is called a *uniformly Prohorov space* if for each $\varepsilon > 0$ any paracompact space Z and any kernel $k : Z \longrightarrow M_r(X)$ there exists an upper semi-continuous compact-valued mapping $S_{(k,\varepsilon)} : Z \longrightarrow X$ such that $\mu_{(k,z)}(X \setminus S_{(k,\varepsilon)}(z)) \leq \varepsilon$ for each $z \in Z$.

The idea of the concept of a uniformly Prohorov goes to the articles of A. Bouziad [3] V. Gutev and V. Valov [15]. That is a "continuous version" of the notion of Prohorov space [21, 15].

Proposition 3.1. *Let X be a uniformly Prohorov space. Then X is a Prohorov space.*

Proof. Let Φ be a compact subset of the space $M_r(X)$ and $\varepsilon > 0$. Consider the identical mapping $k : \Phi \longrightarrow M_r(X)$, where $k(\mu) = \mu$ for each $\mu \in \Phi$. The space Φ is compact and there exists an upper semi-continuous compact-valued mapping $S_\varepsilon : \Phi \longrightarrow X$ such that $\mu(X \setminus S_\varepsilon(\mu)) \leq \varepsilon$ for each $\mu \in \Phi$. Then $F = S_\epsilon(\Phi)$ is a compact subset of X and $\mu(X \setminus F) \leq \mu(X \setminus S_\varepsilon(\mu)) \leq \varepsilon$ for each $\mu \in \Phi$. The proof is complete. □

For the mapping $S_\varepsilon : P(X) \longrightarrow X$ the following theorem was proved in ([15].

Theorem 3.2. *Let X be a sieve-complete space. Then for any paracompact space Z, any kernel $k : Z \longrightarrow M_r(X)$ and any $\varepsilon > 0$ there exists an upper semi-continuous compact-valued mapping $S_{(k,\varepsilon)} : Z \longrightarrow X$ such that $\mu_{(k,z)}(X \setminus S_{(k,\varepsilon)}(z) \leq \varepsilon$ and $S_{(k,\varepsilon)}(F)$ is a compact of countable character in X for each compact subset F of Z.*

Proof. Consider the lower semi-continuous set-valued mappings $\Phi_\varepsilon : M_r(X) \longrightarrow exp_c(X)$ defined in Proposition 2.1. We have $\eta(X \setminus F) \leq \varepsilon$ for each $\eta \in M_r(X)$ and $F \in \Phi_\varepsilon(\eta)$. By virtue of Corollary 1.1, the space $exp_c(X)$ is sieve-complete. Consider now the lower semi-continuous set-valued mapping $\theta : Z \longrightarrow exp_c(X)$, where $\theta(z) = \Phi_\varepsilon(k(z)$ for each $z \in Z$.

As was proved in [6], Theorem 8.4, there exists an upper semi-continuous compact-valued mapping $G_\varepsilon : Z \longrightarrow exp_c X$ such that $G_\varepsilon(z) \cap \theta(z) \neq \emptyset$ for each $z \in Z$. From the construction of the mapping S_ε, it follows that the set and $S_\varepsilon(\eta)$ is a compact of countable character in X for each $\eta \in M_r(X)$. We put $S_{(k,\varepsilon)}(z) = \cup\{F \subset X : F \in G_\varepsilon(z)\}$ for each $z \in Z$. The mapping $S_{(k,\varepsilon)}$ is compact-valued (see [8], Lemma 4). Let U be an open subset of X. Then the set $V = \{F \in exp_c(X) : F \subset U\}$ is open in $exp_c(X)$. Hence the mapping $S_{(k,\varepsilon)}$ is upper semi-continuous. If $z \in Z$ and $F \in G_\varepsilon(z) \cap \Phi_\varepsilon(z)$, then $F \subset S_\varepsilon(\mu_{(k,z)})$ and $\mu_{(k,z)}(X \setminus S_{(k,\varepsilon)}(z)) \leq \mu_{(k,z)}(X \setminus F) \leq \varepsilon$. The proof is complete. □

The space X is a paracompact Čech-complete space if and only if $M_r(X)$ is a paracompact Čech-complete space (see [2, 23, 7]). Hence, from Theorem 3.2 it follows:

Corollary 3.3. *Let X be a paracompact Čech-complete space. Then for any $\epsilon > 0$ there exists a upper semi-continuous compact-valued mapping $S_\varepsilon : M_r(X) \longrightarrow X$*

such that $\mu(X \setminus S_\varepsilon(\mu) \leq \varepsilon$ and $S_\varepsilon(F)$ is a compact of countable character in X for each compact subset F of $M_r(X)$.

Corollary 3.4. *Any sieve-complete space is a uniformly Prohorov space.*

Corollary 3.5. *[7] Any sieve-complete space is a Prohorov space.*

Example 3.6. Let X be a space. A typical distance between measures is of the form

$$\rho_{\mathcal{D}}(\mu,\nu) = sup\{|\int f d\mu - \int f d\nu| : f \in \mathcal{D}\},$$

where $\mathcal{D}$ is some class of Borel measurable functions on the space X.

The total variation distance between two measures μ and ν on a sigma-algebra $Bo(X)$ of subsets of the space X is defined via

$$\delta(\mu,\nu) = sup\{|\mu(A)–\nu(A)| : A \in Bo(X)\}.$$

If $\mathcal{D}_1 = \{1_A : A \in Bo(X)\}$, then $\delta(\mu,\nu) = sup\{|\int f d\mu - \int f d\nu| : f \in \mathcal{D}_1\}$ and $\delta = \rho_{\mathcal{D}_1}$.

In probability theory, the total variation distance is a distance measure for probability distributions. It is an example of a statistical distance metric, and is sometimes just called "the" statistical distance. If μ and ν are both probability measures, then the total variation distance is also given by

$$\| \mu - \nu \|_T V = 2 \cdot sup\{|\mu(A)–\nu(A)| : A \in Bo(X)\}.$$

Let $\mathcal{D}_2$ be the family of all Borel measurable functions $f : X \to [-1,1]$. Then the distance $\rho_{\mathcal{D}_2}(\mu,\nu)$ also is called the total variation distance between two measures μ and ν.

Radon distance between two measures μ and ν to be

$$d_r(\mu,\nu) = sup\{|\int f d\mu - \int f d\nu| : f \in C(X,[-1,1])\}.$$

For Radon measures μ and ν we have

$$\delta(\mu,\nu) = d_r(\mu,\nu) = \rho_{\mathcal{D}_2}(\mu,\nu).$$

If μ and $\{\mu_n : n \in \omega\}$ are Radon measures and $lim_{n\to\infty} d_r(\mu,\mu_n) = 0$, then $\mu_n \longrightarrow \mu$ in the weak topology too. Hence the space $(M_r(X), d_r)$ with the identical mapping k of $(M_r(X), d_r)$ onto $M_r(X)$ is a kernel on $(M_r(X), d_r)$ and the space $(M_r(X), d_r)$ is paracompact as a metric space.

4 On scattered spaces

A point $x \in X$ is called *a point of countable type* if there exists a compact subset F with a countable base of open neighborhoods $\{U_n : n \in \mathbb{N}\}$ in X such that $x \in F$. A space X is called a space of *pointwise countable type* if each $x \in X$ is a point of countable type [1, 13].

A space X is called a space of *countable type* if for each compact subset F of X there exists a compact subset H with a countable base of open neighborhoods $\{U_n : n \in \mathbb{N}\}$ in X such that $F \subseteq H$ [1, 13, 16]. Any sieve-complete space is of countable type [6].

A space X is called *a scattered* space if any non-empty subspace Y of X contains some isolated point in Y.

Proposition 4.1. *Let X be a space, $X \subset Y \subset \mu X$, Y be a space of pointwise countable type and a paracompact space Z is scattered or a union of countable family of closed discrete subspaces. Then for each lower semi-continuous mapping $\theta : Z \longrightarrow X$ there exists an upper semi-continuous compact-valued mapping $\varphi : Z \longrightarrow Y$ with the following properties:*

1. $\psi(z) = \varphi(z) \cap \theta(z) \neq \emptyset$ for any $z \in Z$.

2. If F is a compact subset of Z, then $\Phi = \varphi(F)$ is a compact of countable character in Y, $\Phi \cap X$ is a bounded subset of X and $\Phi = cl_Y(\Phi \cap X)$.

3. If U is a functionally open subset of Y and $V = U \cap X$, then $\{z \in Z : \varphi(z) \subset U\} = \{z \in Z : \psi(z) \subset V\} = \{z \in Z : \psi(z) \subset U\}$.

Proof. The case when Z is a union of countable family of closed discrete subspaces follows from results in [5, 6].

Assume that Z is a paracompact scattered space. Then $dimZ = 0$. It is sufficient to prove that for any point $b \in Z$ and any open subset H of Y for which $\theta(z) \cap H \neq \emptyset$, there exist an open-and-closed subspace $Z(b, H)$ and such an upper semi-continuous compact-valued mapping $\varphi_{(b,H)} : Z(b, H) \longrightarrow Y$ with the following properties:

1. $\varphi_{(b,H)}(z) \cap \theta(z) \neq \emptyset$ and $\varphi_{(b,H)}(z) \subset H$ for any $z \in Z$.

2. If F is a compact subset of $Z(b, H)$, then $\Phi = \varphi_{(b,H)}(F)$ is a compact of countable character in Y, $\Phi \cap X$ is a bounded subset of X and $\Phi = cl_Y(\Phi \cap X)$.

Fix a point $b \in Z$ and an open subset H of Y for which $\theta(z) \cap H \neq \emptyset$. If $b \in Z_(1)$, then we put $Z(b, H) = \{b\}$. Then we fix a point $x(b) \in X \cap \theta(z) \cap H$ and a compact Φ of countable character in Y such that $x(b) \in \Phi$, and put $\varphi_{(b,H)}(b) = \Phi$.

Assume that $\alpha \geq 2$ and for any point $b \in \cup\{Z_{(\beta)} : \beta < \alpha\}$ the objects $Z(b, H)$ and $\varphi_{(b,H)}$ there exist provided H is open in Y and $\theta(z) \cap H \neq \emptyset$.

Fix $b \in Z_{(\alpha)}$ and an open in Y subset for which $\theta(z) \cap H \neq \emptyset$. Then we fix a point $x(b) \in X \cap \theta(z) \cap H$ and a compact Φ of countable character in Y such that

$x(b) \in \Phi$. Let $\{U_n : n \in \mathbb{N}\}$ be a base of Φ in Y and $\{V_n : n \in \mathbb{N}\}$ be a sequence of open-and-closed subsets of Z such that:

- $b \in Z(b, H) = V_1 \subset \theta^{-1}(U_1)$;

- $U_{n+1} \subset U_n \subset H$ and $b \in V_{n+1} \subset V_n \subset \theta^{-1}(U_n$ for each $n \in \mathbb{N}$.

We put $B = \cap\{V_n : n \in \mathbb{N}\}$. Obviously that $\theta(z) \cap \Phi \neq \emptyset$ for any $z \in B$. Denote $B_n = V_n \setminus V_{n+1}$. Then $Z(b, H = V_1 = B \cup (\cup\{B_n : n \in \mathbb{N}\}$. Assume that $z \in Z(b, H)$. If $z \in B$, then $\varphi_{(b,H)}(z) = \Phi$. Suppose that $z \in B_n$ and $n \in \mathbb{N}$. Since B_n is a paracompact space and $dim Z_n = 0$, there exist a subset $A_n \subset B_n$ and a discrete cover $\{Wx : x \in A_n\}$ of the subspace B_n such that $Wx \subset Z(x, U_n)$ for each $x \in A_n$. We put $\varphi_{(b,H)}(z) = \varphi_{(x,U_n)}(z)$ for all $z \in Wx$ and $x \in A_n$. The objects $Z(b, H)$ and $\varphi_{(b,H)}$ are the desired set and mapping. The proof is complete. □

Corollary 4.2. *Let X be a space of countable type. Then:*

1. Any scattered compact subset of the space $M_r(X)$ is uniformly tight.

2. If Z is a paracompact scattered or a paracompact F_σ-discrete subspace of $M_r(X)$, then for any $\varepsilon > 0$ there exists an upper semi-continuous compact-valued mapping $S_\varepsilon : Z \longrightarrow X$ such that:

- $\eta(X \setminus S_\varepsilon(\eta) \leq \varepsilon$ and $S_\varepsilon(\eta)$ is a compact of countable character in X for each $\eta \in M_r(X)$;

- the set $S_\varepsilon(\Phi)$ is compact for any compact subset Φ of Y.

The assertion 1 of Corollary 4.2 was proved in [17, 23] and for metric spaces - in [12].

In [20] was proved that the space of rational numbers $\mathbb{Q}$ is not a Prohorov space.

Let $S = \{0\} \cup \{n{-}1 : n \in \mathbb{N}\}$ be the convergent sequence as a subspace of the closed interval $[0, 1]$. In [11, 7] was proved that there exist a non-Prohorov space Y, a Prohorov space X, an open continuous compact mapping $\varphi : Y \longrightarrow S$ of Y onto S and an open continuous compact mapping $\psi : X \longrightarrow Y$ of X onto Y such that:

(1) There exist $a \in X$ and $b \in Y$ such that $\varphi^{-1}(0) = \{b\}$ and $\psi^{-1}(b) = \{a\}$;

(2) Every compact subset of X is finite;

(3) Every compact subset of Y is finite;

(4) The subspaces $X \setminus \{a\}$ and $Y \setminus \{b\}$ are discrete and countable;

(5) The spaces S, X, Y are scattered.

Similar example was constructed in [3].

In [23] F. Topsoe raised the problem: Is a continuous open image of a Prohorov space a Prohorov space? Hence, in general, the answer is not "yes". Nevertheless the next problems which were formulated in [11, 7] remain open.

Question 1. Is a continuous open image of a metric Prohorov space a Prohorov space?

Question 2. Is a continuous open compact image of a first countable Prohorov space a Prohorov space?

Question 3. Is a continuous closed image of a metric Prohorov space a Prohorov space?

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Conjugate covariant derivatives on vector bundles and duality

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Abstract: The notion of *conjugate connections*, discussed in [2] for a given manifold M and its tangent bundle, is extended here to covariant derivatives on an arbitrary vector bundle E endowed with quadratic endomorphisms. A main property of pairs of such covariant derivatives, namely the duality, is pointed out. As generalization, the case of anchored (particularly Lie algebroid) covariant derivatives on E is considered. As applications we study the Finsler bundle of M as well as the Finsler connections on the slit tangent bundle of a Finsler geometry.

Keywords: (Finsler) vector bundle; quadratic endomorphism; (conjugate) covariant derivatives; duality; mean covariant derivative; anchored bundle.

MSC2010: 53C05, 53C07, 53C60

Introduction

In a series of papers, namely [7]-[9], we have studied the geometry of pairs of linear connections on a manifold M which are conjugated with respect to a tensor field of $(1,1)$-type $T \in \mathcal{T}^1_1(M)$ satisfying the reduced quadratic equation: $T^2 = \varepsilon I$. Here I is the usual Kronecker tensor field $I = (\delta^i_j)$ and $\varepsilon \in \{0, -1, +1\}$. Recently, an unified approach for the non-degenerate case $\varepsilon = \pm 1$ was presented in [2]. Related to the almost product case is the study [6] concerned with golden structures and [10] dealing with metallic structures.

Let us remark that all the previous studies involve only the tangent bundle TM of M. The aim of the present work is to extend these objects to an arbitrary vector bundle E endowed with a non-degenerate quadratic endomorphism λ and a given covariant derivative ∇. An important remark is that the main property mentioned above, namely the duality of conjugate linear connections (∇, ∇^λ), continues to hold in this general setting. Two new features of the present paper are: i) the use of local expressions for all the involved objects, which leads to a better picture; for example in Section 2, for the structural and virtual tensor fields of a pair (linear connection, endomorphism), ii) a special attention is given to the mean covariant derivative ∇^0 which parallelizes the given λ. More generally, we use ∇^0 and the first associated Obata operator in order to determine the whole class $C(\lambda)$ of covariant derivatives with respect to which λ is parallel. The second Obata operator is used to express the variation of the curvature tensor field. We finish the first section with the study of anchored, particularly Lie algebroid, covariant derivatives on E.

The above computations are applied in the cases of two particular geometries involving again the tangent bundle. The first concerns the so-called Finsler bundle $TM \times_M TM$ while the second is a proper Finsler geometry provided by a 1-homogeneous function $F : TM \to \mathbb{R}_+$ with non-degenerate square. Let us remark that the class of almost complex and almost product connections in vector bundles endowed with such endomorphisms are discussed also in [19] but our study follows a different path: we unify the treatment of these geometries and in this way we firstly determine the mean covariant derivative from an arbitrary pair (∇, ∇^λ) and secondly we derive the set $C(\lambda)$. A strong motivation for an unified treatment of almost complex and almost product geometries comes from the relationship (1.20) of curvatures of the conjugate covariant derivatives; the same relation holds for both geometries i.e. it does not depends on ε. For the usual case of the tangent bundle we associate to ∇ two tensor fields of $(1,2)$-type called *the structural* and *the virtual tensor field* of the pair (λ, ∇) respectively. In the last section we compute these tensor fields in the case of a Finsler connection.

1 Conjugate covariant derivatives for ε-endomorphisms

Let $\pi : E \to N$ be a vector bundle of (paracompact) base N^n and fibre $\mathbb{R}^k$. As usual, $\mathcal{X}(N) = \Gamma(TN)$ and $\Gamma(\pi)$ denote the $C^\infty(N)$-module of sections for the vector bundle $\tau_N : TN \to N$ and π, respectively; in fact we mainly follow the notations of [21]. A classical notion for this setting is:

Definition 1.1. ([21, p. 277]) A *covariant derivative operator* on π is an $\mathbb{R}$-bilinear map $\nabla : \mathcal{X}(N) \times \Gamma(\pi) \to \Gamma(\pi)$, $(X, s) \to \nabla_X s$, such that:
i) ∇ is tensorial in its first variable: $\nabla_{fX} s = f\nabla_X s$,
ii) ∇ is a derivation in its second variable: $\nabla_X(fs) = X(f) \cdot s + f\nabla_X s$,
for all $f \in C^\infty(N)$, $X \in \mathcal{X}(N)$ and $s \in \Gamma(\pi)$.

Let now $\varepsilon \in \{\pm 1\}$ and $\lambda \in \Gamma(End(\pi)) \cong End(\Gamma(\pi))$. Recall from [21, p. 284] that ∇ induces a covariant derivative operator $\hat{\nabla}$ on the bundle $End(E)$ through the relation:

$$(\hat{\nabla}_X \lambda)s := \nabla_X(\lambda(s)) - \lambda(\nabla_X s), \tag{1.1}$$

for all $X \in \mathcal{X}(N)$ and $s \in \Gamma(\pi)$.

For the given λ the natural problem is to obtain the class of all ∇ such that $\hat{\nabla}\lambda = 0$; let us denote by $\mathcal{C}(\lambda)$ the set of these covariant derivatives. In the following we restrict to a particular remarkable type of such endomorphisms:

Definition 1.2. λ is an ε-*endomorphism* if: $\lambda^2 = \varepsilon 1_{\Gamma(E)}$.

From now on we suppose that the fixed λ is an ε-endomorphism. In order to study the above problem we follow the method of [2] and [7]-[9] by introducing:

Definition 1.3. The λ-*conjugate* of ∇ is $\nabla^\lambda : \mathcal{X}(N) \times \Gamma(\pi) \to \Gamma(\pi)$ given by:

$$\nabla^\lambda_X s := \varepsilon\lambda(\nabla_X(\lambda s)). \tag{1.2}$$

More generally, the λ-*conjugate* of ∇ is $\nabla^\lambda : \mathcal{X}(N) \times \Gamma(\pi) \to \Gamma(\pi)$ given by:

$$\nabla^\lambda := \lambda^{-1} \circ \nabla \circ (1_{\mathcal{X}(N)}, \lambda), \tag{1.2gen}$$

for any invertible endomorphism λ.

It follows immediately that ∇^λ is also a covariant derivative operator on π. Let us remark that defining the *mean covariant derivative* of ∇ and ∇^λ:

$$\nabla^0 = \frac{1}{2}\left(\nabla + \nabla^\lambda\right) \tag{1.3}$$

we obtain the desired solution of the problem, namely: $\hat{\nabla}^0\lambda = 0$; see also the Proposition 1.5 below. It follows that a study of ∇^λ is necessary and in the following we provide some properties of this conjugate covariant derivative. A first important property is the duality announced in the title:

Proposition 1.4. $\left(\nabla^\lambda\right)^\lambda = \nabla$, *hence* $\left(\nabla^\lambda\right)^0 = \nabla^0$.

Proof. A direct consequence of the definition is:

$$\nabla^\lambda_X(\lambda s) = \lambda(\nabla_X s) \tag{1.4}$$

and hence:

$$\left(\nabla^\lambda\right)^\lambda_X s = \varepsilon\lambda(\nabla^\lambda_X(\lambda s)) = \varepsilon\lambda\left[\lambda(\nabla_X s)\right] = \varepsilon^2\nabla_X s$$

which implies the conclusion. □

The next property concerns with the hat-versions of the involved ∇'s:

Proposition 1.5. $\hat{\nabla}^\lambda\lambda = -\hat{\nabla}\lambda$ *and then* $\nabla \in \mathcal{C}(\lambda)$ *if and only if* $\nabla^\lambda \in \mathcal{C}(\lambda)$, *that is* $\nabla^\lambda = \nabla = \nabla^0$.

Proof. We have directly from the definitions:

$$\left(\hat{\nabla}^{\lambda}_{X}\lambda\right)(s) = \nabla^{\lambda}_{X}(\lambda s) - \lambda(\nabla^{\lambda}_{X}s) = \lambda(\nabla_{X}s) - \varepsilon^{2}(\nabla_{X}\lambda s) = -(\hat{\nabla}_{X}\lambda)(s)$$

which is the claimed equation. □

In the following we express locally the above objects. Let $h = (U, u^{\alpha}; \alpha = 1, ..., n)$ be a local chart on N and suppose that $E|_{U} := \pi^{-1}(U)$ has a trivialization. By the same arguments as in [21, p. 279] it follows that there exists a local frame field $S_{U} = \{s_{i}; i = 1, ..., k\}$ of π over U. So, any local section of $\Gamma_{U}(\pi)$ can be uniquely written as a $C^{\infty}(U)$-linear combination of elements of S_{U}. In particular:

$$\nabla^{U}_{\frac{\partial}{\partial u^{\alpha}}} s_{i} = \Gamma^{j}_{\alpha i} s_{j} \tag{1.5}$$

for some smooth functions $\Gamma^{j}_{\alpha i} \in C^{\infty}(U)$. Following the cited book we call these functions *the Christoffel symbols* of ∇ with respect to the chart h and the local frame field S_{U}. Also we have the local expression of λ:

$$\lambda(s_{i}) = \lambda^{j}_{i} s_{j} \tag{1.6}$$

with $\lambda^{j}_{i} \in C^{\infty}(U)$. Let us denote by $\overset{\lambda}{\Gamma}{}^{j}{}_{\alpha i}$ ($\overset{0}{\Gamma}{}^{j}{}_{\alpha i}$) the Christoffel symbols of ∇^{λ} (∇^{0}) with respect to the same pair (h, S_{U}). A straightforward computation yields the change $\Gamma \rightarrow \overset{\lambda}{\Gamma}$:

$$\overset{\lambda}{\Gamma}{}^{j}{}_{\alpha i} = \varepsilon \lambda^{j}_{k}\left(\frac{\partial \lambda^{k}_{i}}{\partial u^{\alpha}} + \Gamma^{k}_{\alpha l}\lambda^{l}_{i}\right). \tag{1.7}$$

Recall the usual local expression of the ∇-covariant derivative of λ:

$$\lambda^{k}_{i|\alpha} := \frac{\partial \lambda^{k}_{i}}{\partial u^{\alpha}} + \Gamma^{k}_{\alpha l}\lambda^{l}_{i} - \Gamma^{l}_{\alpha i}\lambda^{k}_{l}. \tag{1.8}$$

Then the conjugate and the mean covariant derivatives can be expressed in the following way:

$$\begin{cases} \overset{\lambda}{\Gamma}{}^{j}{}_{\alpha i} = \varepsilon \lambda^{j}_{k}\left(\lambda^{k}_{i|\alpha} + \Gamma^{l}_{\alpha i}\lambda^{k}_{l}\right) = \varepsilon \lambda^{j}_{k}\lambda^{k}_{i|\alpha} + \Gamma^{j}_{\alpha i}, \\ \overset{0}{\Gamma}{}^{j}{}_{\alpha i} = \frac{\varepsilon}{2}\lambda^{j}_{k}\lambda^{k}_{i|\alpha} + \Gamma^{j}_{\alpha i} = -\frac{\varepsilon}{2}\lambda^{k}_{i}\lambda^{j}_{k|\alpha} + \Gamma^{j}_{\alpha i}. \end{cases} \tag{1.9}$$

The relationship in formulae (1.3) and (1.9) can be represented as follows:

∇	$\xrightarrow{+\frac{\varepsilon}{2}\lambda^{j}_{k}\lambda^{k}_{i\vert\alpha}}$	∇^{0}	$\xleftarrow{-\frac{\varepsilon}{2}\lambda^{j}_{k}\lambda^{k}_{i\vert\alpha}}$	∇^{λ}

. $(1.9fig)$

If $\omega = (\omega^j_i)$ is the connection 1-form of ∇ and $\overset{\lambda}{\omega}$ is the connection 1-form of $\overset{\lambda}{\nabla}$ then, from $\omega^j_i = \Gamma^j_{\alpha i} du^\alpha$ it follows that:

$$\overset{\lambda}{\omega}{}^j_i - \omega^j_i = \varepsilon \lambda^j_k \lambda^k_{i|\alpha} du^\alpha = -\varepsilon \lambda^k_i \lambda^j_{k|\alpha} du^\alpha. \tag{1.10}$$

Recall also the curvature of ∇:

$$R\left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}\right) s_i = R^j_{\alpha\beta i} s_j, \tag{1.11}$$

where:

$$R^j_{\alpha\beta i} = \frac{\partial \Gamma^j_{\beta i}}{\partial u^\alpha} - \frac{\partial \Gamma^j_{\alpha i}}{\partial u^\beta} + \Gamma^k_{\beta i}\Gamma^j_{\alpha k} - \Gamma^k_{\alpha i}\Gamma^j_{\beta k}. \tag{1.12}$$

Example 1.6. A) Suppose that E is the tangent bundle TN, hence λ is a tensor field of $(1,1)$-type on N and $n = k$ while $\alpha \in \{1, ..., n\}$.
i) The case $\varepsilon = -1$, corresponding to almost complex geometry, was studied in [7]; n must be an even positive integer.
ii) The case $\varepsilon = +1$, corresponding to almost product geometry, was studied in [8].
iii) From the expression (1.9) we derive also a relation between the torsions of ∇ and ∇^λ, ∇^0:

$$\overset{\lambda}{T}{}^j{}_{\alpha i} = T^j_{\alpha i} + \varepsilon \lambda^j_k (\lambda^k_{i|\alpha} - \lambda^k_{\alpha|i}), \quad \overset{0}{T}{}^j{}_{\alpha i} = T^j_{\alpha i} + \frac{\varepsilon}{2} \lambda^j_k (\lambda^k_{i|\alpha} - \lambda^k_{\alpha|i}). \tag{1.13}$$

iv) In the setting of G-structures suppose that λ is integrable. Then there exists an atlas on N such that the components of λ are constant and hence:

$$\overset{\lambda}{\Gamma}{}^j{}_{\alpha i} = \varepsilon \lambda^j_k \Gamma^k_{\alpha l} \lambda^l_i, \quad \overset{0}{\Gamma}{}^j{}_{\alpha i} = \frac{\varepsilon}{2} \lambda^j_k \Gamma^k_{\alpha l} \lambda^l_i + \frac{1}{2}\Gamma^j_{\alpha i} \tag{1.14}$$

which yields:

$$\overset{\lambda}{R}{}^j{}_{\alpha\beta i} = \varepsilon \lambda^j_k R^k_{\alpha\beta l} \lambda^l_i, \quad \overset{0}{R}{}^j{}_{\alpha\beta i} = \frac{\varepsilon}{2} \lambda^j_k R^k_{\alpha\beta l} \lambda^l_i + \frac{1}{2} R^j_{\alpha\beta i}. \tag{1.15}$$

We recover the result of Proposition 2.1 of [2], namely ∇ is flat if and only if ∇^λ is also flat; this result holds even if λ is not-integrable as well as for ∇^0. The relationships between the torsions of ∇ and ∇^λ, ∇^0 are:

$$\overset{\lambda}{T}{}^j{}_{\alpha i} = \varepsilon \lambda^j_k T^k_{\alpha l} \lambda^l_i, \quad \overset{0}{T}{}^j{}_{\alpha i} = \frac{1}{2} T^j_{\alpha i} + \frac{\varepsilon}{2} \lambda^j_k T^k_{\alpha l} \lambda^l_i. \tag{1.16}$$

B) Almost complex structures on the vertical bundle associated to the tangent bundle are studied in [4], while almost product structures on the same bundle are studied in [17]. □

Returning to the general case we describe now the $C^\infty(N)$-affine module $C(\lambda)$. The Kronecker endomorphism $I \in \Gamma(End(\pi)) \cong End(\Gamma(\pi))$ acts locally as:

$$I(s_i) = \delta_i^j s_j \tag{1.17}$$

and λ has two associated $(2,2)$-tensor fields, called *Obata operators*:

$$\Omega_{ij}^{hk} = \frac{1}{2}\left(\delta_i^h \delta_j^k + \varepsilon \lambda_i^h \lambda_j^k\right), \quad \Psi_{ij}^{hk} = \frac{1}{2}\left(\delta_i^h \delta_j^k - \varepsilon \lambda_i^h \lambda_j^k\right). \tag{1.18}$$

A straightforward computation yields:

Proposition 1.7. *1. The generic element ∇^g of $C(\lambda)$ has the expression:*

$$\overset{g}{\Gamma^j}_{\alpha i} = \overset{0}{\Gamma^j}_{\alpha i} + \Omega_{ia}^{lj} X_{\alpha l}^a, \tag{1.19}$$

with arbitrary $X = (X_{\alpha l}^a)$.
2. If $\nabla \in C(\lambda)$ then Ω and Ψ are also covariant constant with respect to ∇.

The second Obata operator is useful to express globally the first equation in (1.15) through:

$$\overset{\lambda}{R} = R - 2\Psi(R), \tag{1.20}$$

a relation obtained in [11] for $E = TN$ and conjugate connections with respect to non-degenerate $(0,2)$-tensor fields; see also [12]. In conclusion $\overset{0}{R} = R - \Psi(R)$. Let us remark that from $\nabla^g \in \mathcal{C}(\lambda)$ it results that $\overset{g}{R}$ commutes with λ:

$$\overset{g}{R}(\cdot,\cdot) \circ \lambda = \lambda \circ \overset{g}{R}(\cdot,\cdot). \tag{1.21}$$

Another approach for the pair $(\nabla^\lambda, \nabla^0)$ is expressed in terms of *quasi-covariant derivatives*, more precisely λ-*covariant derivatives*, which are maps D as in Definition 1.1 with the second condition replaced by:
iiλ) $D_X(fs) = fD_X s + X(f) \cdot \lambda(s)$.

It is easy to see that any covariant derivative ∇ yields a λ-covariant derivative D^∇ through:

$$D_X^\nabla := \nabla_X \circ \lambda \tag{1.22}$$

and then:

$$\nabla^\lambda = \varepsilon\lambda \circ D^\nabla, \quad \nabla_X^0 = \frac{\varepsilon}{2}(\lambda \circ D_X^\lambda + D^\lambda \circ \lambda) \tag{1.23}$$

for every $X \in \mathcal{X}(N)$. Also, it follows that: $\nabla_X = \varepsilon D_X^\nabla \circ \lambda$.

We finish this section with a slight generalization concerning algebroid covariant derivatives. Namely, let $(\mathcal{A}, N, \tau : \mathcal{A} \to N, \rho)$ be an *anchored vector bundle* over N of rank r, i.e. $\rho : \mathcal{A} \to TN$ is a morphism of vector bundles over the identity of N, called *anchor*, [13, p. 7]. Following the cited book we consider:

Definition 1.8. An $\mathcal{A}$-*covariant derivative* on π is an $\mathbb{R}$-bilinear map $D : \Gamma(\tau) \times \Gamma(\pi) \to \Gamma(\pi)$, $(\xi, s) \to D_\xi s$ satisfying:
i) D is tensorial in the first variable: $D_(f\xi)s = fD_\xi s$,
ii) D is a derivation in the second variable: $D_\xi(fs) = fD_\xi s + (\rho \circ \xi)(f)s$, for all $f \in C^\infty(N)$, $\xi \in \Gamma(\tau)$ and $s \in \Gamma(\pi)$.

We note that this notion also occurs in the infinite dimensional setting of Banach vector bundles, but under another name, in [3]. We can consider the same problem for λ as above in the setting of anchored covariant derivatives and all the definitions and results 1.3-1.5 hold with X replaced by ξ. Let us express locally these objects. Let $h = (U, u^\alpha), \alpha = 1, ..., n$ be a local chart on N and the trivialization of τ and π respectively:
i) $\mathcal{A}|_U := \tau^{-1}(U)$ has a trivialization $s_U^{\mathcal{A}} = \{e_A, A = 1, ..., r\}$,
ii) $E|_U$ has a trivialization as above.
The trivialization $s_U^{\mathcal{A}}$ yields:
1) the linear fibre coordinates z^A i.e. $\mathcal{A}|_U$ has the local coordinates (u^α, z^A),
2) the smooth functions ([13, p. 7]) $\rho_A^\alpha := \dot{u}^\alpha \circ \rho \circ e_A \in C^\infty(U)$. Hence: $\rho(e_A) = \rho_A^\alpha \frac{\partial}{\partial u^\alpha}$.

A fixed $\mathcal{A}$-covariant derivative D has the local expression: $D^U_{e_A} s_i := \Gamma^j_{Ai} s_j$ with $\Gamma^j_{Ai} \in C^\infty(U)$. Then its conjugate $\mathcal{A}$-covariant derivative D^λ is the generalization of (1.7):

$$\overset{\lambda}{\Gamma^j}_{Ai} = \varepsilon\lambda_k^j \left(\rho_A^\alpha \frac{\partial \lambda_i^k}{\partial u^\alpha} + \Gamma^k_{Al}\lambda_i^l\right). \tag{1.24}$$

Also, we consider the generalization of (1.8):

$$\lambda^k_{i|A} := \rho_A^\alpha \frac{\partial \lambda_i^k}{\partial u^\alpha} + \Gamma^k_{Al}\lambda_i^l - \Gamma^l_{Ai}\lambda_l^k \tag{1.25}$$

and hence (1.9) generalizes to:

$$\begin{cases} \overset{\lambda}{\Gamma^j}_{Ai} = \varepsilon\lambda_k^j \left(\lambda^k_{i|A} + \Gamma^l_{Ai}\lambda_l^k\right) = \varepsilon\lambda_k^j\lambda^k_{i|A} + \Gamma^j_{Ai}, \\ \overset{0}{\Gamma^j}_{Ai} = \frac{\varepsilon}{2}\lambda_k^j\lambda^k_{i|A} + \Gamma^j_{Ai} = -\frac{\varepsilon}{2}\lambda_i^k\lambda^j_{k|A} + \Gamma^j_{Ai}. \end{cases} \tag{1.26}$$

We remark that if $\mathcal{A}$ is the usual tangent bundle of N then (1.24) – (1.26) reduce to the formulae (1.7) – (1.9) since ρ is the Kronecker endomorphism (1.17).

In order to introduce the curvature we suppose that, in addition, $\mathcal{A}$ is a *Lie algebroid* i.e. $\Gamma(\tau)$ is endowed with a Lie bracket $[\cdot, \cdot]_{\mathcal{A}}$ following [13, p. 7]. Then the curvature tensor field of D is ([13, p. 48]):

$$R(\xi, \eta)s := D_\xi D_\eta s - D_\eta D_\xi s - D_{[\xi,\eta]_{\mathcal{A}}} s \tag{1.27}$$

with local coefficients:

$$R(e_A, e_B)s_i := R^j_{ABi}s_j, \quad R^j_{ABi} \in C^\infty(U). \tag{1.28}$$

A straightforward computation yields:

$$R^j_{ABi} := \rho^\alpha_A \frac{\partial \Gamma^j_{Bi}}{\partial u^\alpha} - \rho^\alpha_B \frac{\partial \Gamma^j_{Ai}}{\partial u^\alpha} + \Gamma^k_{Bi}\Gamma^j_{Ak} - \Gamma^k_{Ai}\Gamma^j_{Bk} - \Theta^C_{AB}\Gamma^j_{Ci} \tag{1.29}$$

where the coefficients Θ are provided by the $\mathcal{A}$-Lie bracket:

$$\Theta^C_{AB} := z^C \circ [e_A, e_B]_{\mathcal{A}}. \tag{1.30}$$

In the particular case $\tau = \pi$ we can also define *the torsion* of D, ([13, p. 48]):

$$T(\xi, \eta) := D_\xi \eta - D_\eta \xi - [\xi, \eta]_{\mathcal{A}} \tag{1.31}$$

and hence:

$$T(e_A, e_B) := T^C_{AB}e_C, \quad T^C_{AB} := \Gamma^C_{AB} - \Gamma^C_{BA} - \Theta^C_{AB}. \tag{1.32}$$

2 The case of a Finsler bundle endowed with an ε-endomorphism

Let now M be a smooth manifold of dimension m and $\tau = \tau_M : TM \to M$ be its tangent bundle. A local chart $h_M = (U, x^i; i = 1, ..., m)$ on M induces a local chart $(\tau^{-1}(U), x^i, y^i)$ on $TM|_U$, where $u \in TM|_U$ is expressed as $u = y^i \frac{\partial}{\partial x^i}$.

The vector bundle in the preceding section is $\pi : E = TM \times_M TM \to N = TM$ with [21, p. 179]:

$$TM \times_M TM = \{(u, v) \in TM \times TM; \tau(u) = \tau(v)\} \tag{2.1}$$

and $\pi(u, v) = u$. The fiber of π over $u \in TM$ is $\pi^{-1}(u) = \{u\} \times T_{\tau(u)}M$ and hence $k = m$ and $n = 2m$. Usually, π is called *the Finsler bundle of M* and its sections are called *Finsler vector fields*. The $C^\infty(TM)$-module $\Gamma(\pi)$ of Finsler vector fields is canonically isomorphic with the $C^\infty(TM)$-module of sections of τ_M along τ_M:

$$\Gamma_{\tau_M}(TM) := \{\underline{X} \in C^\infty(TM, TM); \tau_M \circ \underline{X} = \tau_M\}. \tag{2.2}$$

Hence the Finsler vector fields are of the form:

$$\tilde{X} : u \in TM \to \tilde{X}(u) = (u, \underline{X}(u)) \in TM \times_M TM \tag{2.3}$$

or briefly $\tilde{X} = (1_{TM}, \underline{X})$. A remarkable Finsler vector field $\mathbb{C}$ corresponds to $\underline{X} = 1_{TM} \in \Gamma_{\tau_M}(TM)$ and then:

$$\mathbb{C} = (1_{TM}, 1_{TM}) : u \in TM \to (u, u) \in TM \times_M TM. \tag{2.4}$$

In a compact form we have the *Liouville vector field* $\mathbb{C} = y^i \frac{\partial}{\partial y^i} \in \mathcal{X}(TM)$.

The local chart h_M of M determines for $u \in \tau^{-1}(U) \subset TM|_U$ the basis $\{\bar{\partial}_i(u)\}$ of the fibre $\pi^{-1}(u)$ given by:

$$\bar{\partial}_i(u) := \left(u, \frac{\partial}{\partial x^i}|_{\tau(u)}\right). \tag{2.5}$$

Hence $S|_U = \{s_i = \bar{\partial}_i; i = 1, ..., n\}$ is a local frame field of π over U.

Fix now an ε-endomorphism λ of this π; hence λ is a particular case of a *Finsler tensor field* of $(1,1)$-type. The equations:

$$\lambda(\bar{\partial}_i) = \lambda_i^j \bar{\partial}_j \tag{2.6}$$

give its components $\lambda_i^j \in C^\infty(\tau^{-1}(U))$:

$$\lambda_i^j = \lambda_i^j(u^\alpha) = \lambda_i^j(x^1, ..., x^m, y^1, ..., y^m). \tag{2.7}$$

Fix now the covariant differential operator ∇ for π. The local expression of ∇:

$$\nabla^U_{\frac{\partial}{\partial x^i}} \bar{\partial}_j = \Gamma^k_{ij} \bar{\partial}_k, \quad \nabla^U_{\frac{\partial}{\partial y^i}} \bar{\partial}_j = C^k_{ij} \bar{\partial}_k \tag{2.8}$$

generates the pair of Christoffel symbols: i) horizontal: $\Gamma^k_{ij} = \Gamma^k_{ij}(x, y)$, ii) vertical: $C^k_{ij} = C^k_{ij}(x, y)$. The conjugate ∇^λ has the pair $(\overset{\lambda}{\Gamma}{}^k{}_{ij}, \overset{\lambda}{C}{}^k{}_{ij})$:

$$\overset{\lambda}{\Gamma}{}^k{}_{ij} = \varepsilon \lambda^k_a \left(\frac{\partial \lambda^a_j}{\partial x^i} + \Gamma^a_{il} \lambda^l_j\right), \quad \overset{\lambda}{C}{}^k{}_{ij} = \varepsilon \lambda^k_a \left(\frac{\partial \lambda^a_j}{\partial y^i} + C^a_{il} \lambda^l_j\right). \tag{2.9}$$

Then ∇ yields two covariant derivatives: one horizontal $|$ and one vertical ${}_{|}$. For λ we have:

$$\lambda^k_{j|i} := \frac{\partial \lambda^k_j}{\partial x^i} + \Gamma^k_{il} \lambda^l_j - \Gamma^l_{ij} \lambda^k_l, \quad \lambda^k_{j_| i} := \frac{\partial \lambda^k_j}{\partial y^i} + C^k_{il} \lambda^l_j - C^l_{ij} \lambda^k_l \tag{2.10}$$

and then:

$$\overset{\lambda}{\Gamma}{}^k{}_{ij} = \varepsilon \lambda^k_a \left(\lambda^a_{j|i} + \Gamma^l_{ij} \lambda^a_l\right), \quad \overset{\lambda}{C}{}^k{}_{ij} = \varepsilon \lambda^k_a \left(\lambda^a_{j_| i} + C^l_{ij} \lambda^a_l\right). \tag{2.11}$$

Also, the mean covariant derivative ∇^0 has the pair $(\overset{0}{\Gamma}{}^k{}_{ij}, \overset{0}{C}{}^k{}_{ij})$:

$$\overset{0}{\Gamma}{}^k{}_{ij} = \frac{\varepsilon}{2} \lambda^k_a \lambda^a_{j|i} + \Gamma^k_{ij}, \quad \overset{0}{C}{}^k{}_{ij} = \frac{\varepsilon}{2} \lambda^k_a \lambda^a_{j_| i} + C^k_{ij}. \tag{2.12}$$

Example 2.1. Suppose that λ is a tensor field on the base M; then $\lambda = \lambda(x)$. It follows that:

$$\overset{\lambda}{C}{}^k{}_{ij} = \varepsilon \lambda^k_a C^a_{il} \lambda^l_j, \quad \overset{0}{C}{}^k{}_{ij} = \frac{\varepsilon}{2} \lambda^k_a C^a_{il} \lambda^l_j + \frac{1}{2} C^k_{ij} \tag{2.13}$$

and then, the vertical part of both ∇^0 and ∇^λ follows the path of formula (1.14). In particular, if the vertical part of ∇ vanishes then also the vertical parts of ∇^λ and ∇^0 vanish; in particular this is the case discussed in [2] when Γ lives also on the base M i.e. $\Gamma = \Gamma(x)$.

Now, with $m = 2$ and following the example 1.6 part iv) we consider:

$$\lambda = \overset{\varepsilon}{\lambda} := \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}. \tag{2.14}$$

The first equations (2.13) become:

$$\overset{\lambda}{C}{}^1{}_{ij} = \varepsilon C^2_{il} \lambda^l_j, \quad \overset{\lambda}{C}{}^2{}_{ij} = C^1_{il} \lambda^l_j \tag{2.15}$$

or, in more details:

$$\overset{\lambda}{C}{}^1{}_{i1} = C^2_{i2}, \quad \overset{\lambda}{C}{}^1{}_{i2} = \varepsilon C^2_{i1}, \quad \overset{\lambda}{C}{}^2{}_{i1} = \varepsilon C^1_{i2}, \quad \overset{\lambda}{C}{}^2{}_{i2} = C^1_{i1}. \tag{2.16}$$

□

The corresponding result of Proposition 1.7 for this setting is:

Proposition 2.2. *The generic element of $C(\lambda)$ has the expression:*

$$\overset{g}{\Gamma}{}^k{}_{ij} = \overset{0}{\Gamma}{}^k{}_{ij} + \Omega^{lk}_{ja} X^a_{il}, \quad \overset{g}{C}{}^k{}_{ij} = \overset{0}{C}{}^k{}_{ij} + \Omega^{lk}_{ja} Y^a_{il}, \tag{2.17}$$

with arbitrary $X = (X^a_{il})$ and $Y = (Y^a_{il})$.

Let us finish this section with the simple case of the tangent bundle τ_M; then $\lambda \in \mathcal{T}^1_1(M)$. Two tensor fields of $(1,2)$-type are associated in [2] to a given linear connection ∇: *the structural* and *the virtual* tensor field, denoted respectively by C^λ_∇ and B^λ_∇. After a short computation of their initial expression we write them in more useful form as:
1) the structural tensor field:

$$C^\lambda_\nabla(X, Y) := \frac{1}{2}\{(\nabla_{\lambda X}\lambda)Y - \lambda[(\nabla_X \lambda)(Y)]\}, \tag{2.18}$$

2) the virtual tensor field:

$$B^{\lambda}_{\nabla}(X,Y) := \frac{1}{2}\{(\nabla_{\lambda X}\lambda)Y + \lambda[(\nabla_X\lambda)(Y)]\}. \tag{2.19}$$

Their utility is provided by the relation (30) in [2]:

$$\nabla^{\lambda} = \nabla + \varepsilon(B^{\lambda}_{\nabla} - C^{\lambda}_{\nabla}) \Rightarrow \nabla^0 = \nabla + \frac{\varepsilon}{2}(B^{\lambda}_{\nabla} - C^{\lambda}_{\nabla}). \tag{2.20}$$

Suppose that locally we write:

$$C^{\lambda}_{\nabla}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = (C^{\lambda}_{\nabla})^k_{ij}\frac{\partial}{\partial x^k}, \quad B^{\lambda}_{\nabla}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = (B^{\lambda}_{\nabla})^k_{ij}\frac{\partial}{\partial x^k}. \tag{2.21}$$

Since the equation (1.8) is expressed by:

$$(\nabla_{\frac{\partial}{\partial x^i}}\lambda)(\frac{\partial}{\partial x^j}) = \lambda^k_{j|i}\frac{\partial}{\partial x^k}, \tag{2.22}$$

we derive:

$$(C^{\lambda}_{\nabla})^k_{ij} = \frac{1}{2}(\lambda^l_i\lambda^k_{j|l} - \lambda^k_l\lambda^l_{j|i}), \quad (B^{\lambda}_{\nabla})^k_{ij} = \frac{1}{2}(\lambda^l_i\lambda^k_{j|l} + \lambda^k_l\lambda^l_{j|i}). \tag{2.23}$$

3 Finsler geometry endowed with an ε-endomorphism

Recall from [5] that a *Finsler fundamental function* on M is a map $F : TM \to \mathbb{R}_+$ with the following properties:
F1) F is smooth on the slit tangent bundle $T_0M := TM \setminus O$ and continuous on the null section O of τ_M,
F2) F is positive homogeneous of degree 1: $F(x, \lambda y) = \lambda F(x, y)$ for every $\lambda > 0$,
F3) the matrix $(g_{ij}) = \left(\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}\right)$ is invertible and its associated quadratic form is positive definite. The tensor field $g = \{g_{ij}(x, y); 1 \leq i, j \leq m\}$ is called *the Finsler metric* and the homogeneity of F implies:

$$F^2(x, y) = g_{ij}y^iy^j = y_iy^i, \tag{3.1}$$

where $y_i = g_{ij}y^j$. The pair (M, F) is called *Finsler manifold*. In particular, if g does not depend on y, we recover the Riemannian geometry.

On $N := T_0M$ we have two distributions:
i) $V(T_0M) := \ker \pi_*$, called *the vertical distribution*; does not depends of F. It is integrable and has the basis $\{\frac{\partial}{\partial y^i}; 1 \leq i \leq m\}$. A remarkable section of it is *the*

Liouville vector field $\mathbb{C} = y^i \frac{\partial}{\partial y^i}$.
ii) $H(T_0M)$ with the basis $\{\frac{\delta}{\delta x^i} := \frac{\partial}{\partial y^i} - N_i^j \frac{\partial}{\partial y^j}\}$, where:

$$N_j^i = \frac{1}{2} \frac{\partial \gamma_{00}^i}{\partial y^j} \tag{3.2}$$

and $\gamma_{00}^i = \gamma_{jk}^i y^j y^k$ is built from the usual Christoffel symbols:

$$\gamma_{jk}^i = \frac{1}{2} g^{ia} \left(\frac{\partial g_{ak}}{\partial x^j} + \frac{\partial g_{ja}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^a} \right). \tag{3.3}$$

$H(T_0M)$ is often called the *Cartan* (or canonical) *nonlinear connection* of the geometry (M, F) and a remarkable section of it is *the geodesic spray*:

$$S_F = y^i \frac{\delta}{\delta x^i}. \tag{3.4}$$

The Finslerian connections are triples $\Gamma = (N_i^k, F_{ij}^k(x,y), C_{ij}^k(x,y))$ where $F_{\cdot\cdot}^{\cdot}$ behave like the coefficients of a linear connection on M and C is a d-tensor field on T_0M. Such a Finslerian connection yields the covariant derivative $\Gamma\Delta$ on T_0M given by:

$$\begin{cases} \Gamma\Delta_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta x^i} := F_{ij}^k \frac{\delta}{\delta x^k}, & \Gamma\Delta_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial y^i} := F_{ij}^k \frac{\partial}{\partial y^k} \\ \Gamma\Delta_{\frac{\partial}{\partial y^j}} \frac{\delta}{\delta x^i} := C_{ij}^k \frac{\delta}{\delta x^k}, & \Gamma\Delta_{\frac{\partial}{\partial y^j}} \frac{\partial}{\partial y^i} := C_{ij}^k \frac{\partial}{\partial y^k}. \end{cases} \tag{3.5}$$

There are four remarkable Finslerian connections, [5, p. 227]:
-Cartan $Ca = (N_i^k, F_{ij}^k, C_{ij}^k)$,
-Chern-Rund $CR = (N_i^k, F_{ij}^k, 0)$,
-Berwald $B = (N_i^k, G_{ij}^k, 0)$,
-Hashiguchi $H = (N_i^k, G_{ij}^k, C_{ij}^k)$,
where $G^k = N_j^i y^j$ and:

$$F_{ij}^k = \frac{1}{2} g^{ka} \left(\frac{\delta g_{aj}}{\delta x^i} + \frac{\delta g_{ia}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^a} \right), C_{ij}^k = \frac{1}{2} g^{ka} \frac{\partial g_{ij}}{\partial y^a}, G_{ij}^k = \frac{\partial^2 G^k}{\partial y^i \partial y^j} = \gamma_{00}^i. \tag{3.6}$$

Let now λ be an ε-endomorphism given locally by $\lambda = (\lambda_i^j(x,y))$ as in (2.7). Γ, or equivalently $\Gamma\Delta$, yields two covariant derivatives: one horizontal $|$ and one vertical $_|$ which for λ are:

$$\lambda_{j|i}^k := \frac{\delta \lambda_j^k}{\delta x^i} + F_{il}^k \lambda_j^l - F_{ij}^l \lambda_l^k, \quad \lambda_{j_| i}^k := \frac{\partial \lambda_j^k}{\partial y^i} + C_{il}^k \lambda_j^l - C_{ij}^l \lambda_l^k. \tag{3.7}$$

Then Γ has:

1) a conjugate Finsler connection $\Gamma^{\lambda} = (N^k_i, \overset{\lambda}{F}{}^k_{ij}(x,y), \overset{\lambda}{C}{}^k_{ij}(x,y))$ with coefficients as in (2.11),

2) a mean Finsler connection $\Gamma^0 = (N^k_i, \overset{0}{F}{}^k_{ij}(x,y), \overset{0}{C}{}^k_{ij}(x,y))$ with coefficients as in (2.12). λ is covariant constant with respect to this Finsler connection:

$$\lambda^k_{\underset{0}{j|i}} = \lambda^k_{\underset{0}{j|i}} = 0. \tag{3.8}$$

Example 3.1. The almost complex case ($\varepsilon = -1$) of Γ^0 appears in [20, p. 15] while the almost product case ($\varepsilon = +1$) in [18, p. 61], see also [1]. An important remark of these works is that the set $C(\lambda)$ derived from (2.12) is a commutative group. □

Returning to the general case let us remark that the Finsler connection Γ defines the pairs: $(\overset{h}{C^{\lambda}}\Gamma, \overset{h}{B^{\lambda}}\Gamma)$, $(\overset{v}{C^{\lambda}}\Gamma, \overset{v}{B^{\lambda}}\Gamma)$ with:

$$2(\overset{h}{C^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\delta\lambda^k_j}{\delta x^l} - \lambda^k_l \frac{\delta\lambda^l_j}{\delta x^i} + \varepsilon F^k_{ij} + \lambda^l_i F^k_{ls}\lambda^s_j - \lambda^k_l[\lambda^s_i F^l_{sj} + \lambda^s_j F^l_{si}] \tag{3.9}$$

$$2(\overset{h}{B^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\delta\lambda^k_j}{\delta x^l} + \lambda^k_l \frac{\delta\lambda^l_j}{\delta x^i} - \varepsilon F^k_{ij} + \lambda^l_i F^k_{ls}\lambda^s_j - \lambda^k_l[\lambda^s_i F^l_{sj} - \lambda^s_j F^l_{si}] \tag{3.10}$$

$$2(\overset{v}{C^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\partial\lambda^k_j}{\partial y^l} - \lambda^k_l \frac{\partial\lambda^l_j}{\partial y^i} + \varepsilon C^k_{ij} + \lambda^l_i C^k_{ls}\lambda^s_j - \lambda^k_l[\lambda^s_i C^l_{sj} + \lambda^s_j C^l_{si}] \tag{3.11}$$

$$2(\overset{v}{B^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\partial\lambda^k_j}{\partial y^l} + \lambda^k_l \frac{\partial\lambda^l_j}{\partial y^i} - \varepsilon C^k_{ij} + \lambda^l_i C^k_{ls}\lambda^s_j - \lambda^k_l[\lambda^s_i C^l_{sj} - \lambda^s_j C^l_{si}]. \tag{3.12}$$

Hence, for the case $\Gamma \in \{Chern - Rund,\ Berwald\}$ we have:

$$2(\overset{v}{C^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\partial\lambda^k_j}{\partial y^l} - \lambda^k_l \frac{\partial\lambda^l_j}{\partial y^i}, \quad 2(\overset{v}{B^{\lambda}}\Gamma)^k_{ij} = \lambda^l_i \frac{\partial\lambda^k_j}{\partial y^l} + \lambda^k_l \frac{\partial\lambda^l_j}{\partial y^i} \tag{3.13}$$

and these tensor fields are zero if λ is a basic endomorphism, i.e. $\lambda \in \mathcal{T}^1_1(M)$, or an integrable one, i.e. with constant components in a preferential atlas on M.

Let $(dx^i, \delta y^i := dy^i + N^i_j dx^j)$ be the dual of the Berwald basis $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$. A final remark is that the given λ yields four endomorphisms of N:

$$A_{\pm}(\lambda) := \lambda^j_i \frac{\delta}{\delta x^j} \otimes dx^i \pm \lambda^j_i \frac{\partial}{\partial y^j} \otimes \delta y^i, \quad B_{\pm}(\lambda) := \lambda^j_i \frac{\delta}{\delta x^j} \otimes \delta y^i \pm \lambda^j_i \frac{\partial}{\partial y^j} \otimes dx^i. \tag{3.14}$$

We note that $A_{\pm}(\lambda)$ and $B_{+}(\lambda)$ are exactly ε-endomorphisms on N while $B_{-}(\lambda)$ is an $(-\varepsilon)$-endomorphism on N.

Example 3.2. Suppose that $\varepsilon = +1$ and $\lambda = \delta = (\delta^j_i)$. Then: $A_+(\delta) = 1_{TN}$ is the Kronecker tensor field of tangent bundle TN; $A_-(\delta)$ is (together with the Sasaki-type metric G_F on TN induced by (g_{ij})) the almost para-Kähler structure P_F from [15, p. 1880] while $B_-(\delta)$ is (again together with G_F) the almost Kähler structure Ψ_F from [16, p. 243]. For the general setting of tangent manifolds, particularly tangent bundles, endowed with nonlinear connections, the almost product structure $A_-(\delta)$ appears in [14, p. 14] and it is well-known that $A_-(\delta)$ is integrable if and only if the corresponding nonlinear connection is without curvature i.e. flat. Also, for any Finsler connection Γ we have $\overset{v}{C}{}^{\delta}\Gamma = \overset{v}{B}{}^{\delta}\Gamma = \overset{h}{C}{}^{\delta}\Gamma = \overset{h}{B}{}^{\delta}\Gamma = 0$. □

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A note on submanifolds of generalized Kähler manifolds

Izu Vaisman

Abstract: In this note, we consider submanifolds of a generalized Kähler manifold that are CR-submanifolds for the two associated Hermitian structures. Then, we establish the conditions for the induced, generalized F structure to be a CRFK structure. The results extend similar conditions which we obtained for hypersurfaces in an earlier paper.

Keywords: Generalized F structure, generalized CRF structure, generalized CRFK structure, CR-submanifold, bi-CR-submanifold.

MSC2010: 53C15

Dedicated to Academicians Radu Miron and Constantin Corduneanu on their 90-th anniversary.

1 Introduction

This note is a complement to our previous paper [8]. All manifolds and mappings are of class C^∞ and the terminology and notation are classical [3]. An exception is the use of Cartan's conventions for exterior products and differentials, e.g.,

$$\alpha \wedge \beta(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X),$$
$$d\alpha(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]).$$

Furthermore, we shall assume that the reader is familiar with the basic notions and facts of generalized geometry in the sense of Hitchin as they already appeared in many papers. In particular, we shall refer to [2, 4, 5, 6].

In this note we consider a class of submanifolds of a generalized Kähler manifold, which bear a naturally induced generalized metric F structure and we study the conditions for the induced structure to be a CRFK structure[1] in the sense of [5]. In [8] we studied this problem in the case of hypersurfaces.

[1]CR stands for Cauchy-Riemann, F stands for Yano's F structure and K comes from Kähler.

First, we shall deduce a result in the classical framework. Namely, we consider a CR-submanifold of a Hermitian manifold and its induced F structure [1] and we establish the conditions for the latter to be classical CRF in the sense of [5]. As a corollary, it follows that these conditions hold for totally geodesic and totally umbilical CR-submanifolds. Then, we shall consider bi-CR-submanifolds of a generalized Kähler manifold, i.e. submanifolds that have the CR property for the two associated Hermitian structures. Bi-CR-submanifolds have an induced, generalized metric F structure and we establish the conditions for the induced structure to be CRFK. As a corollary, it follows that, if the bi-CR-submanifold is totally geodesic, the induced structure is a generalized CRFK structure.

2 Bi-CR-submanifolds

Let M^{2n} be a generalized almost Hermitian manifold, with the generalized Riemannian metric G and the compatible generalized almost complex structure $\mathcal{J}$. Then, the following results hold [2].

G is equivalent to $\mathcal{G} \in End(\mathbf{T}M)$ ($\mathbf{T}M = TM \oplus T^*M$) defined by

$$G(\mathcal{G}\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, \mathcal{Y}) = \frac{1}{2}(\alpha(Y) + \beta(X)),$$

where $\mathcal{X} = (X, \alpha), \mathcal{Y} = (Y, \beta) \in \mathbf{T}M$ and

$$\mathcal{G}^2 = Id,\ g(\mathcal{G}\mathcal{X}, \mathcal{G}\mathcal{Y}) = g(\mathcal{X}, \mathcal{Y}).$$

G is also equivalent to a pair (γ, ψ) where γ is a Riemannian metric and ψ is a 2-form on M. The equivalence is via the ± 1-eigenbundles of $\mathcal{G}$

$$V_\pm = \{(X, \flat_{\psi\pm\gamma}X), X \in TM\}\ (\flat_{\psi\pm\gamma}X = i(X)(\psi \pm \gamma))$$

and the projections $\tau_\pm = pr_{TM} : V_\pm \to TM$ are *transfer isomorphisms.*

For the structure $\mathcal{J}$ one has

$$\mathcal{J}^2 = -Id,\ g(\mathcal{J}\mathcal{X}, \mathcal{Y}) + g(\mathcal{X}, \mathcal{J}\mathcal{Y}) = 0,\ G(\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}) = G(\mathcal{X}, \mathcal{Y}).$$

The bundles $V_\pm$ are $\mathcal{J}$-invariant and the transfer by $\tau_\pm$ produces two γ-compatible almost complex structures $J_\pm$ of M such that

$$\mathcal{J}(X, \flat_{\psi\pm\gamma}X) = (J_\pm X, \flat_{\psi\pm\gamma}(J_\pm X)).$$

Thus, $(G, \mathcal{J})$ is equivalent to the quadruple $(\gamma, \psi, J_\pm)$.

Furthermore, a complementary, G-compatible, generalized almost complex structure is defined by $\mathcal{J}' = \mathcal{G} \circ \mathcal{J} = \mathcal{J} \circ \mathcal{G}$ and $\mathcal{J} \circ \mathcal{J}' = \mathcal{J}' \circ \mathcal{J}$, $\mathcal{G} = -\mathcal{J} \circ \mathcal{J}'$. The complementary structure corresponds to $(\gamma, \psi, J_+, -J_-)$.

On an arbitrary manifold M, a *generalized F structure* $\mathcal{F} \in End\, \mathbf{T}M$ [5] is defined by the conditions

$$\mathcal{F}^3 + \mathcal{F} = 0,\ g(\mathcal{F}\mathcal{X}, \mathcal{Y}) + g(\mathcal{X}, \mathcal{F}\mathcal{Y}) = 0$$

and the structure is *metric* with respect to a generalized Riemannian metric G if

$$G(\mathcal{F}\mathcal{X}, \mathcal{Y}) + G(\mathcal{X}, \mathcal{F}\mathcal{Y}) = 0.$$

Then, like in the almost Hermitian case, there exists a complementary generalized metric F structure $\mathcal{F}' = \mathcal{G} \circ \mathcal{F}$.

By Proposition 4.2 of [5], $(\mathcal{F}, G)$ is a generalized metric F structure iff there exists two classical metric F structures $(F_\pm, \gamma)$ on M, i.e.,

$$F_\pm^3 + F_\pm = 0,\ \gamma(F_\pm X, Y) + \gamma(X, F_\pm Y) = 0\ (X, Y \in TM) \tag{2.1}$$

and the generalized F structure is given by

$$\mathcal{F}(X, \flat_{\psi\pm\gamma}X) = (F_\pm X, \flat_{\psi\pm\gamma}(F_\pm X)),\ \forall X \in TM.$$

Now, let $\iota : N^k \hookrightarrow M$ be a submanifold of M and let $\nu N = T^{\perp_\gamma} N$ be the normal bundle of N. Then, $T_N M = TN \oplus \nu N$ and we shall identify $T^*N = ann\, \nu N, \nu^* N = ann\, TN, \mathbf{T}N = TN \oplus ann\, \nu N$. It follows easily that

$$\mathbf{T}^{\perp_g} N = \nu N \oplus ann\, TN,$$

hence, the restriction $g|_{\mathbf{T}N}$ coincides with the pairing metric on the manifold N, thus, it is non degenerate, and

$$\mathbf{T}_N M = \mathbf{T}N \oplus \mathbf{T}^{\perp_g} N. \tag{2.2}$$

The metric G induces a generalized Riemannian metric G' on N that corresponds to the induced pair $(\gamma' = \iota^*\gamma, \psi' = \iota^*\psi)$ and has the ±1-eigenbundles

$$V'_\pm = \{(X, pr_{ann\, \nu N}(\flat_{\psi\pm\gamma}X)) \,/\, X \in TN\} = pr_{\mathbf{T}N} V_\pm,$$

where the projection is defined by (2.2) (e.g., [7]). In the particular case $\psi = 0$, we get $V'_\pm = V_\pm \cap \mathbf{T}N$, we have

$$\mathbf{T}N = (V_+ \cap \mathbf{T}N) \oplus (V_- \cap \mathbf{T}N) \subseteq \mathbf{T}M \tag{2.3}$$

and G' is induced by G via the inclusion (2.3).

Now, we define the class of submanifolds that we want to study.

Definition 2.1. 1. If (M, γ, J) is a classical almost Hermitian manifold, a submanifold $\iota : N \hookrightarrow M$ is called a *CR-submanifold* if the equality

$$TN = (TN \cap J(TN)) \oplus (TN \cap J(\nu N)) \tag{2.4}$$

holds at every point of N and the rank of the terms is constant.

2. If $(M, \gamma, \psi, J_{\pm})$ is a generalized almost Hermitian manifold, a submanifold $\iota : N \hookrightarrow M$ is called a *bi-CR-submanifold* if it is a CR-submanifold with respect to the two almost Hermitian structures $(\gamma, J_{\pm})$.

Part 1 of Definition 2.1 is equivalent to Bejancu's original definition [1], the distributions $D, D^{\perp}$ of [1] being the terms of the direct sum (2.4). Among the examples of CR-submanifolds we notice the hypersurfaces and the Ω-coisotropic submanifolds ($\Omega(X, Y) = \gamma(JX, Y)$ is the Kähler form). The CR terminology is justified by the fact that, if J is integrable with i-eigenbundle $L \subseteq T^cM$, then, $L \cap T^cN$ is a CR structure (the index c denotes complexification). In the particular case $\psi = 0$, if we apply the transfer $\tau_{\pm}^{-1}$ to the equalities (2.4) for $J_{\pm}$ and use (2.3), we can see that the bi-CR-submanifolds are characterized by

$$\mathbf{T}N = (\mathbf{T}N \cap \mathcal{J}(\mathbf{T}N)) \oplus (\mathbf{T}N \cap \mathcal{J}(\nu N \oplus \nu^* N)),$$

hence, if $\psi = 0$, a bi-CR-submanifold is an F submanifold in the sense of [5].

If (2.4) holds, N has the induced metric F structure

$$F|_{TN \cap J_{\pm}(TN)} = J|_{TN \cap J_{\pm}(TN)}, \; F|_{TN \cap J(\nu N)} = 0. \tag{2.5}$$

Notice that F of (2.5) coincides with the tensor ϕ of [1].

In the generalized case, the use of $J_{\pm}$ in (2.5) yields two structures $F_{\pm}$ and we get an *induced* generalized metric F structure $\mathcal{F}$ defined by the quadruple $(\gamma, \psi, F_{\pm})$.

3 Submanifolds of generalized Kähler manifolds

The generalized almost complex structure $\mathcal{J}$ may be identified with its $\pm i$-eigenbundles $\mathcal{L}, \bar{\mathcal{L}}$ (the bar denotes complex conjugation) and in the generalized almost Hermitian case of $(G, \mathcal{J}, \mathcal{J}')$ one has [2]

$$\mathcal{L} = (\mathcal{L} \cap V_+) \oplus (\mathcal{L} \cap V_-), \; \mathcal{L}' = (\mathcal{L} \cap V_+) \oplus (\bar{\mathcal{L}} \cap V_+).$$

The structure $\mathcal{J}$ is *integrable* (*generalized complex*), respectively $(G, \mathcal{J}, \mathcal{J}')$ is *generalized Hermitian*, if $\mathcal{L}$ is closed under Courant brackets. Furthermore, $(G, \mathcal{J}, \mathcal{J}')$ is *generalized Kähler* if $\mathcal{J}, \mathcal{J}'$ are integrable, which turns out to be equivalent to the following pair of properties: (i) the pairs $(\gamma, J_\pm)$ are Hermitian structures, (ii) for the Hermitian structures $(\gamma, J_\pm)$ one has

$$\gamma(\nabla^\gamma_X J_\pm(Y), Z) = \mp\frac{1}{2}[d\psi(X, J_\pm Y, Z) + d\psi(X, Y, J_\pm Z)], \tag{3.1}$$

where ∇^γ is the Levi-Civita connection of γ [2, 6]. If the 2-form ψ is closed, M is a *bi-Kählerian manifold*, i.e., a manifold with two Kähler structures with the same Riemannian metric γ.

The generalized F structure $\mathcal{F}$ may be identified with its $(\pm i, 0)$-eigenbundles $\mathcal{E}, \bar{\mathcal{E}}, \mathcal{S}$ and $\mathcal{F}$ is *integrable* or *CRF* if $\mathcal{E}$ is closed under Courant brackets [5]. Furthermore [5], the generalized metric F structure $(G, \mathcal{F}, \mathcal{F}')$ is a *generalized CRFK structure* if $\mathcal{F}, \mathcal{F}'$ are integrable and the eigenbundles of $\mathcal{G}, \mathcal{F}$ satisfy the Courant bracket condition

$$[V_+ \cap \mathcal{S}, V_- \cap \mathcal{S}] \subseteq \mathcal{S}.$$

These properties are equivalent to the pair of properties [5] (a) the corresponding structures $F_\pm$ are classical CRF structures, i.e., $\mathcal{F}_\pm(X, \alpha) = (F_\pm X, -\alpha \circ F_\pm)$ are generalized CRF structures, (b) one has the equalities

$$\gamma(F_\pm(\nabla^\gamma_X F_\pm)Y, Z) = \pm\frac{1}{2}[d\psi(X, Y, F^2_\pm Z) + d\psi(X, F_\pm Y, F_\pm Z)]. \tag{3.2}$$

If the form ψ is closed, M is a *partially bi-Kählerian submanifold*, i.e., a Riemannian manifold such that its metric γ has two de Rham decompositions that have one Kählerian term [5].

Hereafter, we shall assume that $(M, G, \mathcal{J}, \mathcal{J}')$ is a generalized Káhler manifold, N is a bi-CR-submanifold and $(G', \mathcal{F})$ is the induced structure. Then, we will look for the conditions that characterize the case where the induced structure is a CRFK structure and we begin by the following preparations.

Riemannian geometry gives us the Gauss-Weingarten equations along the submanifold N of the Riemannian manifold (M, γ),

$$\nabla^\gamma_X Y = \nabla^{\gamma'}_X Y + b(X, Y),\ \nabla^\gamma_X U = -W_U X + \nabla^\nu_X U, \tag{3.3}$$

where $X, Y \in TN, U \in \nu N$, $\nabla^\gamma, \nabla^{\gamma'}$ are the Levi-Civita connections of the metrics $\gamma, \gamma' = \iota^*\gamma$, ∇^ν is the induced connection of the normal bundle of N and $b(X, Y) = b(Y, X) \in \nu N, W_\nu X \in TN$ are the νN-valued second fundamental form and the Weingarten operator, respectively. The latter are related by the formula $\gamma(W_U X, Y) = \gamma(b(X, Y), U)$.

Using these equations, we can extend the proof of Proposition 3.2 of [8] and get the following result.

Theorem 3.1. *Let $\iota : N \hookrightarrow M$ be a CR-submanifold of the Hermitian manifold (M, γ, J). Then, the induced structure F of N is a classical CRF structure iff*

$$d\Omega(X, Y, Z) = d\Omega(JX, JY, Z),\ \gamma(b(FX, FY) - b(X, Y), JZ) = 0, \tag{3.4}$$

for all $X, Y \in TN \cap (JTN)$ and $Z \in TN \cap (J\nu N)$.

Proof. As earlier, $\Omega(X, Y) = \gamma(JX, Y)$ is the Kähler form. Following [5], the structure F is classical CRF iff

$$[H, H] \subseteq H,\ [H, Q^c] \subseteq H \oplus Q^c, \tag{3.5}$$

where $H, \bar{H}, Q$ are the $\pm i, 0$-eigenbundles of F and the brackets are Lie brackets.

The definition (2.5) of F shows that $Q = TN \cap (J\nu N)$, $H \oplus \bar{H}$ is the complexification of $P = im\, F = TN \cap (JTN)$ and $H = L \cap T^cN$, where L is the i-eigenbundle of J. In particular, the integrability of J (M is Hermitian) implies the first condition (3.5) and we have to take care of the second condition only.

The second condition (3.5) is equivalent to [5]

$$F[FX, Z] - F^2[X, Z] = 0,\ \forall X \in P, Z \in Q. \tag{3.6}$$

Because of the second condition (2.1) and since, $F|_P = J_P, F^2|_P = -Id$, (3.6) is equivalent to

$$\gamma([JX, Z], JY) = \gamma([X, Z], Y),\ \forall X, Y \in P, Z \in Q. \tag{3.7}$$

Indeed, (3.7) means that the left hand side of (3.6) is orthogonal to P and it is also orthogonal to Q because $F|_Q = 0$.

We shall express (3.7) using the equalities

$$[X, Z] = \nabla^\gamma_X Z - \nabla^\gamma_Z X,\ [JX, Z] = \nabla^\gamma_{JX} Z - \nabla^\gamma_Z (JX),$$
$$\nabla^\gamma_Z (JY) = (\nabla^\gamma_Z J)Y + J\nabla^\gamma_Z Y$$

and the γ-compatibility if J. The result is

$$\gamma((\nabla^\gamma_Z J)X, JY) - \gamma(\nabla^\gamma_{JX} Z, JY) + \gamma(\nabla^\gamma_X Z, Y) = 0. \tag{3.8}$$

On the other hand, we have

$$-\gamma(\nabla^\gamma_{JX} Z, JY) = \gamma(J\nabla^\gamma_{JX} Z, Y) = \gamma(\nabla^\gamma_{JX}(JZ), Y) - \gamma((\nabla^\gamma_{JX} J)Z, Y),$$
$$\gamma(\nabla^\gamma_X Z, Y) = -\gamma(\nabla^\gamma_X (J^2 Z), Y) = -\gamma((\nabla^\gamma_X J)(JZ), Y) + \gamma(\nabla^\gamma_X (JZ), JY),$$

and it follows that (3.8) is equivalent to

$$\begin{aligned}\gamma((\nabla^{\gamma}_{Z}J)X, JY) - \gamma((\nabla^{\gamma}_{JX}J)Z, Y) + \gamma(\nabla^{\gamma}_{JX}(JZ), Y)\\ -\gamma((\nabla^{\gamma}_{X}J)(JZ), Y) + \gamma(\nabla^{\gamma}_{X}(JZ), JY) = 0.\end{aligned}$$

Then, since $Z \in Q$ implies $JZ \in \nu N$, the third and fifth term of the previous equality may be expressed using the Gauss-Weingarten equations and the relation between the Weingarten operator and the second fundamental form. As a result, we get the following equivalent form of the condition (3.6)

$$\begin{aligned}\gamma((\nabla^{\gamma}_{Z}J)X, JY) - \gamma((\nabla^{\gamma}_{JX}J)Z, Y) - \gamma((\nabla^{\gamma}_{X}J)(JZ), Y)\\ = \gamma(b(JX, Y) + b(X, JY), JZ).\end{aligned} \tag{3.9}$$

To continue, we recall that the integrability of J is equivalent to the following equality

$$2\gamma(\nabla^{\gamma}_{X}J(Y), Z) = d\Omega(X, Y, Z) - d\Omega(X, JY, JZ),\ \forall X, Y, Z \in TM \tag{3.10}$$

(this result is given by Proposition IX.4.2 of [3] with our conventions for the sign of Ω and the evaluation of the exterior differential). We also recall the equality

$$d\Omega(JZ, JX, JY) = d\Omega(JZ, X, Y) + d\Omega(Z, JX, Y) + d\Omega(Z, X, JY) \tag{3.11}$$

(check for arguments of complex type $(1, 0), (0, 1)$).

Modulo (3.10) and (3.11) condition (3.9) becomes

$$d\Omega(Z, X, JY) + d\Omega(Z, JX, Y) = 2[\gamma(b(JX, Y) + b(X, JY), JZ)]. \tag{3.12}$$

In (3.12) the left hand side is skew symmetric in X, Y and the right hand side is symmetric. Therefore, the equality holds iff both of its sides vanish and the replacement of Y by JY shows that the result is exactly (3.4). □

Corollary 3.2. *If N is a totally umbilical (in particular, totally geodesic) submanifold of a Kähler manifold M, the induced F structure of N is a classical CRF structure.*

Proof. Under the hypotheses of the corollary, conditions (3.4) obviously hold. □

Corollary 3.3. *If N is an Ω-coisotropic submanifold of the Hermitian manifold (M, γ, J) it is a CR-submanifold and the induced F structure is classical CRF iff the first condition (3.4) holds and the second fundamental form satisfies the equality*

$$b(FX, FY) = b(X, Y),\ \ \forall X, Y \in im\, F.$$

Proof. It is well known that N is a CR-submanifold and, in this case, (2.4) takes the form

$$TN = (TN \cap J(TN)) \oplus J(\nu N).$$

Indeed, we have $J(\nu N) = T^{\perp_\Omega} N \subseteq TN$ and the second term of the right hand side of (2.4) is $J(\nu N)$. On the other hand, it follows easily that $(J\nu N)^{\perp_{\gamma'}} = TN \cap J(TN)$. Then, the assertion of the corollary follows from the fact that, in the second condition (3.4), JZ runs through the whole normal bundle νN. □

With the preparations done, we now give the answer to the motivating question of the note. It turns out to be a straightforward extension of Proposition 3.3 of [8].

Theorem 3.4. *Let $\iota : N \hookrightarrow M$ be a bi-CR-submanifold of the generalized Kähler manifold $(M, \gamma, \psi, J_\pm)$. Then, the induced generalized metric F structure $\mathcal{F}$ of N is a generalized CRFK structure iff, $\forall Z \in Q_\pm = ker\, F_\pm$, one has*

$$\begin{aligned} &d\psi(X, Y, J_\pm Z) = d\psi(J_\pm X, J_\pm Y, J_\pm Z), \;\; \forall X, Y \in P_\pm = im\, F_\pm, \\ &d\psi(X, J_\pm Y, J_\pm Z) = \mp 2\gamma(b(X, F_\pm Y), J_\pm Z), \;\; \forall X \in TN, Y \in P_\pm. \end{aligned} \tag{3.13}$$

Proof. Since M is generalized Kähler, $(\gamma, J_\pm)$ are Hermitian structures, hence, (3.10) holds for these two structures, and it implies that condition (3.1) is equivalent to [2]

$$d\psi(X, Y, Z) = \mp d\Omega_\pm(J_\pm X, J_\pm Y, J_\pm Z), \; \forall X, Y, Z \in TM. \tag{3.14}$$

For $\mathcal{F}$ to be CRFK, the first required condition, condition (a), is that $(\gamma', F_\pm)$ are classical metric CRF structures, i.e., that conditions (3.4) hold for both structures. Modulo (3.14) the first condition (3.4) becomes the first condition (3.13) and the second condition (3.4) is

$$\gamma(b(F_\pm X, F_\pm Y) - b(X, Y), J_\pm Z) = 0, \;\; \forall X, Y \in P_\pm, Z \in Q_\pm. \tag{3.15}$$

The second required condition, condition (b), is (3.2) on (N, γ'), where we may assume $Z \in P_\pm$ since the condition holds trivially if $F_\pm Z = 0$. Then, $F_\pm^2 Z = -Z$ and, using (2.1), (3.2) becomes

$$\gamma'((\nabla^{\gamma'}_X F_\pm)Y, F_\pm Z) = \mp\frac{1}{2}[d\psi(X, F_\pm Y, F_\pm Z) - d\psi(X, Y, Z)], \tag{3.16}$$

with $X, Y \in TN, Z \in P_\pm$.

We consider the cases (i) $Y \in P_\pm$, (ii) $Y \in Q_\pm$ separately. In case (i), since $F_\pm|_{P_\pm} = J_\pm|_{P_\pm}$, the Gauss equation yields

$$\gamma'((\nabla^{\gamma'}_X F_\pm)Y, F_\pm Z) = \gamma((\nabla^{\gamma}_X J_\pm)Y, J_\pm Z),$$

which makes (3.16) take the form (3.1) with Z replaced by $J_\pm Z$. Therefore it holds because M is generalized Kähler.

In case (ii), we have $F_\pm Y = 0$ and we get

$$\gamma'(F_\pm(\nabla^{\gamma'}_X F_\pm)Y, Z) = -\gamma'(F^2_\pm \nabla^{\gamma'}_X Y, Z) = -\gamma'(\nabla^{\gamma'}_X Y, F^2_\pm Z) = \gamma'(\nabla^{\gamma'}_X Y, Z),$$

which, together with the Gauss equation, changes (3.16) into

$$\gamma(\nabla^\gamma_X Y, Z) = \pm\frac{1}{2} d\psi(X, Y, Z). \tag{3.17}$$

Furthermore, we shall take into account that $Y \in Q_\pm$ implies $J_\pm Y \in \nu N$ and use the Weingarten equation. We get

$$\begin{aligned}\gamma(\nabla^\gamma_X Y, Z) &= \gamma(J_\pm \nabla^\gamma_X Y, J_\pm Z) = \gamma(\nabla^\gamma_X (J_\pm Y), J_\pm Z) - \gamma((\nabla^\gamma_X J_\pm)Y, J_\pm Z) \\ &\overset{(3.3),(3.1)}{=} -\gamma(b(X, J_\pm Z), J_\pm Y) \mp \tfrac{1}{2}[d\psi(X, J_\pm Y, J_\pm Z) - d\psi(X, Y, Z)].\end{aligned}$$

Accordingly, condition (3.17) becomes

$$\gamma(b(X, J_\pm Z), J_\pm Y) = \pm\frac{1}{2} d\psi(X, J_\pm Y, J_\pm Z),$$

which is the second condition (3.13) with the replacements $Y \mapsto Z, Z \mapsto Y$.

To end the proof we only have to remark that conditions (3.13) imply (3.15). This follows by noticing that, if we replace $X \in P_\pm$ by $F_\pm X$, $X \in P_\pm$, the second condition (3.13) becomes

$$\gamma(b(X, Y), J_\pm Z) = \mp\frac{1}{2} d\psi(X, Y, J_\pm Z), \ , \ \forall X, Y \in P_\pm, Z \in Q_\pm$$

and by using the first condition (3.13). □

Corollary 3.5. *If M is a generalized Kähler manifold with a closed associated form ψ, then, any totally geodesic, bi-CR-submanifold of M has an induced CRFK structure.*

Proof. The assertion is an obvious consequence of conditions (3.13). □

Corollary 3.6. *If M is a generalized Kähler manifold with a closed associated form ψ and N is a bi-coisotropic submanifold, then, the induced generalized F structure is CRFK iff $b(X, Y) = 0$, $\forall X \in TN, Y \in P_\pm$.*

Proof. By bi-coisotropic we understand that N is coisotropic with respect to the two Kähler forms $\Omega_\pm$. The assertion follows because $J_\pm Z$ of the right hand side of (3.13) runs through the whole bundle νN (see the proof of Corollary 3.3). □

Because of the symmetry of the second fundamental form, the CRFK condition of Corollary 3.6 may also be seen as $b(X,Y) = 0$, $\forall X \in P_{\pm}, Y \in TN$. Thus, it follows that, if the induced structure F of the corollary is CRFK, and if $b(Z,Z') = 0$ for either $Z,Z' \in Q_+$ or $Z,Z' \in Q_-$, then, N is a totally geodesic submanifold of M.

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Golden warped product Riemannian manifolds

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Abstract: The aim of our paper is to introduce the Golden warped product Riemannian manifold and study its properties with a special view towards its curvature. We obtain a characterization of the Golden structure on the product of two Golden manifolds in terms of Golden maps and provide a necessary and sufficient condition for the warped product of two locally Golden Riemannian manifolds to be locally Golden. The particular case of product manifolds is discussed and an example of Golden warped product Riemannian manifold is also given.

Keywords: Warped product manifold, Golden Riemannian structure.

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Dedicated to Academician Radu Miron on the occasion of his 90'th birthday

1 Introduction

The notion of *Golden number* is provided by the positive solution of the equation $x^2 - x - 1 = 0$ and it has the form:

$$\sigma = \frac{1+\sqrt{5}}{2}. \tag{1.1}$$

Starting from a polynomial structure, which was generally defined by S. I. Goldberg, K. Yano and N. C. Petridis in ([7] and [8]), we consider a polynomial structure on an m-dimensional Riemannian manifold (M, g), called by us a *Golden structure* ([5], [10], [6] and [11]), determined by a $(1,1)$-tensor field J which satisfies the equation:

$$J^2 = J + I, \tag{1.2}$$

where I is the identity operator on the Lie algebra of vector fields on M identified with the set of smooth sections $\Gamma(T(M))$ (and we'll simply denote $X \in T(M)$).

Remark that a Golden structure J verifies the recurrence relation:

$$J^{n+1} = f_{n+1} \cdot J + f_n \cdot I, \tag{1.3}$$

where $(f_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence defined by $f_{n+2} = f_{n+1} + f_n$, $f_1 = f_2 = 1$.

We say that the metric g is J-compatible if the following equality holds:

$$g(JX, Y) = g(X, JY). \tag{1.4}$$

If (M, g) is a Riemannian manifold endowed with a Golden structure J such that the Riemannian metric g is J-compatible, then (M, g, J) is called a *Golden Riemannian manifold.*

Also, we can remark that:

$$g(JX, JY) = g(X, JY) + g(X, Y), \tag{1.5}$$

for any $X, Y \in T(M)$.

We proved that an almost product structure F on M induces two Golden structures on M ([12]):

$$J_1 = \frac{1}{2} \cdot I + \frac{2\sigma - 1}{2} \cdot F, \quad J_2 = \frac{1}{2} \cdot I - \frac{2\sigma - 1}{2} \cdot F. \tag{1.6}$$

Conversely, every Golden structure J on M induces two almost product structures on M:

$$F_\pm = \pm \left(\frac{2}{2\sigma - 1} \cdot J - \frac{1}{2\sigma - 1} \cdot I \right). \tag{1.7}$$

In particular, if the almost product structure F is compatible with the Riemannian metric, then J_1 and J_2 are Golden Riemannian structures.

On a Golden manifold (M, J) there are two complementary distributions $\mathcal{D}_l$ and $\mathcal{D}_m$ corresponding to the projection operators l and m ([12]) given by:

$$l = \frac{\sigma}{2\sigma - 1} \cdot I - \frac{1}{2\sigma - 1} \cdot J, \quad m = \frac{\sigma - 1}{2\sigma - 1} \cdot I + \frac{1}{2\sigma - 1} \cdot J. \tag{1.8}$$

Moreover, the operators l and m verify the following equalities:

$$l + m = I, \quad l^2 = l, \quad m^2 = m, \quad lm = ml = 0, \tag{1.9}$$

$$Jl = lJ = (1 - \sigma)l, \quad Jm = mJ = \sigma m. \tag{1.10}$$

The analogue concept of locally product manifold is considered in the context of Golden geometry. We say that the Golden Riemannian manifold (M, g, J) is *locally Golden* if J is parallel with respect to the Levi-Civita connection associated to g.

2 Golden warped product Riemannian manifolds

2.1 Warped product manifolds

Consider (M_1, g_1) and (M_2, g_2) two Riemannian manifolds of dimensions n and m, respectively. Denote by p_1 and p_2 the projection maps from the product manifold $M_1 \times M_2$ to M_1 and M_2 and by $\widetilde{\varphi} := \varphi \circ p_1$ the lift to $M_1 \times M_2$ of a smooth function φ on M_1.

In this context, we shall call M_1 *the base* and M_2 *the fiber* of $M_1 \times M_2$, the unique element $\widetilde{X}$ of $T(M_1 \times M_2)$ that is p_1-related to $X \in T(M_1)$ and to the zero vector field on M_2, the *horizontal lift of* X and the unique element $\widetilde{V}$ of $T(M_1 \times M_2)$ that is p_2-related to $V \in T(M_2)$ and to the zero vector field on M_1, the *vertical lift of* V. Also denote by $\mathcal{L}(M_1)$ the set of all horizontal lifts of vector fields on M_1 and by $\mathcal{L}(M_2)$ the set of all vertical lifts of vector fields on M_2.

Let $f > 0$ be a smooth function on M_1 and

$$\widetilde{g} := p_1^* g_1 + (f \circ p_1)^2 p_2^* g_2 \tag{2.1}$$

be a Riemannian metric on $M_1 \times M_2$.

Definition 2.1. ([4]) The product manifold of M_1 and M_2 together with the Riemannian metric $\widetilde{g}$ defined by (2.1) is called *the warped product* of M_1 and M_2 by the warping function f [and it is denoted by $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$].

Remark that if f is constant (equal to 1), the warped product becomes the usual product of the Riemannian manifolds.

For $(x, y) \in \widetilde{M}$, we shall identify $X \in T(M_1)$ with $(X_x, 0_y) \in T_{(x,y)}(\widetilde{M})$ and $Y \in T(M_2)$ with $(0_x, Y_y) \in T_{(x,y)}(\widetilde{M})$ ([3]).

Let $\pi_1 =: Tp_1$ and $\pi_2 =: Tp_2$ be the projection mappings of $T(M_1 \times M_2)$ into $T(M_1)$ and $T(M_2)$, respectively. They verify:

$$\pi_1 + \pi_2 = I, \quad \pi_1^2 = \pi_1, \quad \pi_2^2 = \pi_2, \quad \pi_1 \circ \pi_2 = \pi_2 \circ \pi_1 = 0. \tag{2.2}$$

The Riemannian metric of the warped product manifold $\widetilde{M} = M_1 \times_f M_2$ equals to $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(\pi_1 \widetilde{X}, \pi_1 \widetilde{Y}) + (f \circ p_1)^2 g_2(\pi_2 \widetilde{X}, \pi_2 \widetilde{Y})$, thus:

$$\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2), \tag{2.3}$$

for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$. From the definition of $\widetilde{g}$, one can verify that the leaves $M_1 \times \{y\}$, for $y \in M_2$, are totally geodesic submanifolds of $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g})$.

If we denote by $\widetilde{\nabla}$, ${}^{M_1}\nabla$, ${}^{M_2}\nabla$ the Levi-Civita connections on $\widetilde{M}$, M_1 and M_2, we know that for any $X_1, Y_1 \in T(M_1)$ and $X_2, Y_2 \in T(M_2)$ ([13]):

$$\widetilde{\nabla}_{(X_1,X_2)}(Y_1,Y_2) = ({}^{M_1}\nabla_{X_1}Y_1 - \frac{1}{2}g_2(X_2,Y_2)\cdot grad(f^2), \tag{2.4}$$

$${}^{M_2}\nabla_{X_2}Y_2 + \frac{1}{2f^2}X_1(f^2)Y_2 + \frac{1}{2f^2}Y_1(f^2)X_2).$$

Remark 2.2. For the case of product Riemannian manifolds:

i) the Riemannian curvature tensors verify ([2]):

$$R(\widetilde{X},\widetilde{Y})\widetilde{Z} = (R_1(X_1,Y_1)Z_1, R_2(X_2,Y_2)Z_2), \tag{2.5}$$

for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2), \widetilde{Z} = (Z_1, Z_2) \in T(M_1 \times M_2)$, where R, R_1 and R_2 are respectively the Riemannian curvature tensors of the Riemannian manifolds $(M_1 \times M_2, \widetilde{g})$, (M_1, g_1) and (M_2, g_2);

ii) the Ricci curvature tensors verify ([2]):

$$S(\widetilde{X},\widetilde{Y}) = S_1(X_1,Y_1) + S_2(X_2,Y_2), \tag{2.6}$$

for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(M_1 \times M_2)$, where S, S_1 and S_2 are respectively the Ricci curvature tensors of the Riemannian manifolds $(M_1 \times M_2, \widetilde{g})$, (M_1, g_1) and (M_2, g_2).

Remark that the Riemannian curvature tensor of a locally Golden Riemannian manifold has the following properties:

Proposition 2.3. *If (M, g, J) is a locally Golden Riemannian manifold, then for any $X, Y, Z \in T(M)$:*

$$R(X,Y)JZ = J(R(X,Y)Z), \tag{2.7}$$

$$R(JX,Y) = R(X,JY), \tag{2.8}$$

$$R(JX,JY) = R(JX,Y) + R(X,Y), \tag{2.9}$$

$$R(J^{n+1}X,Y) = f_{n+1}\cdot R(JX,Y) + f_n \cdot R(X,Y), \tag{2.10}$$

where $(f_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence defined by $f_{n+2} = f_{n+1} + f_n$, $f_1 = f_2 = 1$.

Proof. The locally Golden condition $\nabla J = 0$ is equivalent to $\nabla_X JY = J(\nabla_X Y)$, for any $X, Y \in T(M)$ and (2.7) follows from the definition of R. The relations (2.8), (2.9) and (2.10) follows from the symmetries of R and from the recurrence relation $J^{n+1} = f_{n+1}\cdot J + f_n \cdot I$. □

Let S, S_{M_1}, S_{M_2} be the Ricci curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{S_{M_1}}$, $\widetilde{S_{M_2}}$ the lift on $\widetilde{M}$ of S_{M_1} and S_{M_2}. Then:

Lemma 2.4. *([4]) If $(\widetilde{M} := M_1 \times_f M_2, \widetilde{g})$ is the warped product of M_1 and M_2 by the warping function f and $m > 1$, then for any X, $Y \in \mathcal{L}(M_1)$ and any V, $W \in \mathcal{L}(M_2)$, we have:*

1. $S(X,Y) = \widetilde{S_{M_1}}(X,Y) - \frac{m}{f} H^f(X,Y)$, *where H^f is the lift on $\widetilde{M}$ of $Hess(f)$;*
2. $S(X,V) = 0$;
3. $S(V,W) = \widetilde{S_{M_2}}(V,W) - \left[\frac{\Delta(f)}{f} + (m-1)\frac{|grad(f)|^2}{f^2}\right] g(V,W)$.

2.2 Golden warped product Riemannian manifolds

i) Golden Riemannian structure on $(\widetilde{M}, \widetilde{g})$ induced by the projection operators

The endomorphism

$$F = \pi_1 - \pi_2 \tag{2.11}$$

verifies $F^2 = I$ and $\widetilde{g}(F\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, F\widetilde{Y})$, thus F is an almost product structure on $M_1 \times M_2$.

By using relations (1.6) we can construct on $M_1 \times M_2$ two Golden structures, given by:

$$\widetilde{J}_{\pm} = \frac{I \pm \sqrt{5}F}{2}. \tag{2.12}$$

Also from $\widetilde{g}(F\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, F\widetilde{Y})$ follows $\widetilde{g}(\widetilde{J}_{\pm}\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{J}_{\pm}\widetilde{Y})$. Therefore, we can state the following result:

Theorem 2.5. *There exists two Golden Riemannian structure $\widetilde{J}_{\pm}$ on $(\widetilde{M}, \widetilde{g})$ given by:*

$$\widetilde{J}_{\pm} = \frac{I \pm \sqrt{5}F}{2}, \tag{2.13}$$

where $\widetilde{M} = M_1 \times_f M_2$ and $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2)$, for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$.

We remark that for $\widetilde{J}_+ = \frac{I+\sqrt{5}F}{2}$, the projection operators are $\pi_1 = m$, $\pi_2 = l$ and for $\widetilde{J}_- = \frac{I-\sqrt{5}F}{2}$ we have $\pi_1 = l$, $\pi_2 = m$, where m and l were given in relations (1.8).

Remark 2.6. If we denote by $\widetilde{\nabla}$ the Levi-Civita connection on $\widetilde{M}$ with respect to $\widetilde{g}$, we can check that $\widetilde{\nabla}F = 0$ [hence $\widetilde{\nabla}\widetilde{J}_{\pm} = 0$ and so $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J}_{\pm})$ is a locally Golden Riemannian manifold].

For the case of product Riemannian manifolds, from (2.5) and Proposition 2.3 we have:

Proposition 2.7. *Let $(\widetilde{M} = M_1 \times M_2, \widetilde{g}, \widetilde{J}_{\pm})$ (with $\widetilde{g}$ given by* (2.1) *for $f = 1$ and $\widetilde{J}_{\pm}$ given by* (2.12)*) be the product of the Riemannian manifolds (M_1, g_1) and (M_2, g_2) with the Golden structure $\widetilde{J}_{\pm}$. Then, for any $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in T(\widetilde{M}) = T(M_1 \times M_2)$, the Riemannian curvature tensor verifies the relations:*

$$R(\widetilde{X}, \widetilde{Y})\widetilde{J}_{\pm}\widetilde{Z} = \widetilde{J}_{\pm}(R(\widetilde{X}, \widetilde{Y})\widetilde{Z}), \tag{2.14}$$

$$R(\widetilde{J}_{\pm}\widetilde{X}, \widetilde{Y}) = R(\widetilde{X}, \widetilde{J}_{\pm}\widetilde{Y}), \tag{2.15}$$

$$R(\widetilde{J}_{\pm}\widetilde{X}, \widetilde{J}_{\pm}\widetilde{Y}) = R(\widetilde{J}_{\pm}\widetilde{X}, \widetilde{Y}) + R(\widetilde{X}, \widetilde{Y}), \tag{2.16}$$

$$R(\widetilde{J}_{\pm}^{n+1}\widetilde{X}, \widetilde{Y}) = f_{n+1} \cdot R(\widetilde{J}_{\pm}\widetilde{X}, \widetilde{Y}) + f_n \cdot R(\widetilde{X}, \widetilde{Y}), \tag{2.17}$$

where $(f_n)_{n \in \mathbb{N}}$ is the Fibonacci sequence defined by $f_{n+2} = f_{n+1} + f_n$, $f_1 = f_2 = 1$.

ii) Golden Riemannian structure on $(\widetilde{M}, \widetilde{g})$ induced by two Golden structures on M_1 and M_2

For any vector field $\widetilde{X} = (X, Y) \in T(M_1 \times M_2)$ we define a linear map $\widetilde{J}$ of tangent space $T(M_1 \times M_2)$ into itself as follows:

$$\widetilde{J}\widetilde{X} = (J_1 X, J_2 Y), \tag{2.18}$$

where J_1 and J_2 are two Golden structures defined on M_1 and M_2, respectively. It follows that:

$$\begin{aligned} \widetilde{J}^2\widetilde{X} = \widetilde{J}(J_1 X, J_2 Y) = (J_1^2 X, J_2^2 Y) = \\ = (J_1 X + X, J_2 Y + Y) = (J_1 X, J_2 Y) + (X, Y). \end{aligned} \tag{2.19}$$

Also from $g_i(J_i X_i, Y_i) = g_i(X_i, J_i Y_i)$, $i \in \{1, 2\}$, we get $\widetilde{g}(\widetilde{J}\widetilde{X}, \widetilde{Y}) = \widetilde{g}(\widetilde{X}, \widetilde{J}\widetilde{Y})$. Therefore, we can state the following result:

Theorem 2.8. *If (M_1, g_1, J_1) and (M_2, g_2, J_2) are Golden Riemannian manifolds, then there exists a Golden Riemannian structure $\widetilde{J}$ on $(\widetilde{M}, \widetilde{g})$ given by:*

$$\widetilde{J}\widetilde{X} = (J_1 X, J_2 Y), \tag{2.20}$$

for any $\widetilde{X} = (X, Y) \in T(\widetilde{M})$, where $\widetilde{M} = M_1 \times_f M_2$ and $\widetilde{g}(\widetilde{X}, \widetilde{Y}) = g_1(X_1, Y_1) + (f \circ p_1)^2 g_2(X_2, Y_2)$, for any $\widetilde{X} = (X_1, X_2), \widetilde{Y} = (Y_1, Y_2) \in T(\widetilde{M}) = T(M_1 \times_f M_2)$.

For the case of product Riemannian manifolds, from (2.5) we have:

Proposition 2.9. *Let* $(\widetilde{M} = M_1 \times M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by* (2.1) *for* $f = 1$ *and* $\widetilde{J}$ *given by* (2.18)*) be the product of the locally Golden Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. Then for any* $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in T(\widetilde{M}) = T(M_1 \times M_2)$*, the Riemannian curvature tensor verifies the relations:*

$$R(\widetilde{X}, \widetilde{Y})\widetilde{J}\widetilde{Z} = \widetilde{J}(R(\widetilde{X}, \widetilde{Y})\widetilde{Z}), \tag{2.21}$$

$$R(\widetilde{J}\widetilde{X}, \widetilde{Y}) = R(\widetilde{X}, \widetilde{J}\widetilde{Y}), \tag{2.22}$$

$$R(\widetilde{J}\widetilde{X}, \widetilde{J}\widetilde{Y}) = R(\widetilde{J}\widetilde{X}, \widetilde{Y}) + R(\widetilde{X}, \widetilde{Y}), \tag{2.23}$$

$$R(\widetilde{J}^{n+1}\widetilde{X}, \widetilde{Y}) = f_{n+1} \cdot R(\widetilde{J}\widetilde{X}, \widetilde{Y}) + f_n \cdot R(\widetilde{X}, \widetilde{Y}), \tag{2.24}$$

where $(f_n)_{n\in\mathbb{N}}$ *is the Fibonacci sequence defined by* $f_{n+2} = f_{n+1} + f_n$*,* $f_1 = f_2 = 1$*.*

Now we shall obtain a characterization of the Golden structure on the product of two Golden manifolds (M_1, J_1) and (M_2, J_2) in terms of *Golden maps*, that are smooth maps $\Phi : M_1 \to M_2$ satisfying:

$$T\Phi \circ J_1 = J_2 \circ T\Phi.$$

Remark that for the Golden structure $\widetilde{J} := (J_1, J_2)$ given by (2.18), the projections p_1 and p_2 on the two factors M_1 and M_2 are Golden maps. Indeed:

$$(Tp_i \circ \widetilde{J})(X_1, X_2) = Tp_i(J_1X_1, J_2X_2) = J_iX_i = J_i(Tp_i(X_1, X_2)),$$

$i \in \{1, 2\}$, for any $X_1 \in T(M_1)$ and $X_2 \in T(M_2)$.

Conversely, if we assume that the two projections p_1 and p_2 are Golden maps, then $\widetilde{J} = (J_1, J_2)$, since:

$$(Tp_i \circ \widetilde{J})(X_1, X_2) = J_i(Tp_i(X_1, X_2)) = J_iX_i,$$

$i \in \{1, 2\}$, for any $X_1 \in T(M_1)$ and $X_2 \in T(M_2)$, if we denote by $(Y_1, Y_2) =: \widetilde{J}(X_1, X_2)$, then $Y_i = J_iX_i$, $i \in \{1, 2\}$.

Therefore:

Proposition 2.10. *The Golden structure* $\widetilde{J} := (J_1, J_2)$ *given by* (2.18) *is the only Golden structure on the product manifold* $\widetilde{M} = M_1 \times M_2$ *such that the projections* p_1 *and* p_2 *on the two factors* M_1 *and* M_2 *are Golden maps.*

A necessary and sufficient condition for the warped product of two locally Golden Riemannian manifolds to be locally Golden will be further provided:

Theorem 2.11. *Let* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by* (2.1) *and* $\widetilde{J}$ *given by* (2.18)*) be the warped product of the locally Golden Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. Then* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *is locally Golden if and only if:*

$$\begin{cases} (df^2 \circ J_1) \otimes I = df^2 \otimes J_2 \\ g_2(J_1\cdot, \cdot) \cdot grad(f^2) = g_2(\cdot, \cdot) \cdot J_1(grad(f^2)) \end{cases}.$$

Proof. Replacing the expression of $\widetilde{\nabla}$ from (2.4), for any $X_1, Y_1 \in T(M_1)$ and $X_2, Y_2 \in T(M_2)$, we have:

$$(\widetilde{\nabla}_{(X_1,X_2)}\widetilde{J})(Y_1,Y_2) := \widetilde{\nabla}_{(X_1,X_2)}\widetilde{J}(Y_1,Y_2) - \widetilde{J}(\widetilde{\nabla}_{(X_1,X_2)}(Y_1,Y_2)) =$$

$$= ((^{M_1}\nabla_{X_1}J_1)Y_1 - \frac{1}{2}g_2(J_2X_2,Y_2)\cdot grad(f^2) + \frac{1}{2}g_2(X_2,Y_2)\cdot J_1(grad(f^2)),$$

$$(^{M_2}\nabla_{X_2}J_2)Y_2 + \frac{1}{2f^2}(J_1Y_1)(f^2)X_2 - \frac{1}{2f^2}Y_1(f^2)J_2X_2).$$

Under the assumptions $^{M_1}\nabla J_1 = 0$ and $^{M_2}\nabla J_2 = 0$ we get:

$$(\widetilde{\nabla}_{(X_1,X_2)}\widetilde{J})(Y_1,Y_2) = (-\frac{1}{2}g_2(J_2X_2,Y_2)\cdot grad(f^2) + \frac{1}{2}g_2(X_2,Y_2)\cdot J_1(grad(f^2)),$$

$$\frac{1}{2f^2}(J_1Y_1)(f^2)X_2 - \frac{1}{2f^2}Y_1(f^2)J_2X_2) =$$

$$= (-\frac{1}{2}[g_2(J_2X_2,Y_2)\cdot grad(f^2) - g_2(X_2,Y_2)\cdot J_1(grad(f^2))],$$

$$\frac{1}{2f^2}[df^2(J_1Y_1)X_2 - df^2(Y_1)J_2X_2]),$$

from where we obtain the conclusion. □

Theorem 2.12. *Let* $(\widetilde{M} = M_1 \times_f M_2, \widetilde{g}, \widetilde{J})$ *(with* $\widetilde{g}$ *given by* (2.1) *and* $\widetilde{J}$ *given by* (2.18)*) be the warped product of the Golden Riemannian manifolds* (M_1, g_1, J_1) *and* (M_2, g_2, J_2)*. If* M_1 *and* M_2 *have* J_1*- and* J_2*-invariant Ricci tensors, respectively (i.e.* $Q_{M_i} \circ J_i = J_i \circ Q_{M_i}$, $i \in \{1,2\}$*), then* $\widetilde{M}$ *has* $\widetilde{J}$*-invariant Ricci tensor if and only if*

$$Hess(f)(J_1\cdot, \cdot) - Hess(f)(\cdot, J_1\cdot) \in \{0\} \times T(M_2).$$

Proof. If we denote by S, S_{M_1}, S_{M_2} the Ricci curvature tensors on $\widetilde{M}$, M_1 and M_2 and $\widetilde{S_{M_1}}$, $\widetilde{S_{M_2}}$ the lift on $\widetilde{M}$ of S_{M_1} and S_{M_2}, remark that the J_i-invariance of the Ricci tensor Q_{M_i}, $i \in \{1,2\}$, is equivalent to $S_{M_i}(J_iX,Y) = S_{M_i}(X,J_iY)$, $i \in \{1,2\}$, which implies $\widetilde{S_{M_i}}(\widetilde{J}X,Y) = \widetilde{S_{M_i}}(X,\widetilde{J}Y)$, $i \in \{1,2\}$.

Now using Lemma 2.4, we have for any X, $Y \in \mathcal{L}(M_1)$:

$$S(\widetilde{J}X,Y)=\widetilde{S_{M_1}}(\widetilde{J}X,Y)-\frac{m}{f}H^f(\widetilde{J}X,Y)=\widetilde{S_{M_1}}(X,\widetilde{J}Y)-\frac{m}{f}H^f(\widetilde{J}X,Y)=$$

$$=S(X,\widetilde{J}Y)+\frac{m}{f}H^f(X,\widetilde{J}Y)-\frac{m}{f}H^f(\widetilde{J}X,Y),$$

where H^f is the lift on $\widetilde{M}$ of $Hess(f)$.

Also from Lemma 2.4, for any V, $W \in \mathcal{L}(M_2)$ we similarly obtain:

$$S(\widetilde{J}V,W)=\widetilde{S_{M_2}}(\widetilde{J}V,W)-[f\Delta(f)+(m-1)|grad(f)|^2]g_2(J_2V,W)=$$

$$=\widetilde{S_{M_2}}(V,\widetilde{J}W)-[f\Delta(f)+(m-1)|grad(f)|^2]g_2(V,J_2W)=S(V,\widetilde{J}W).$$

□

3 Example of Golden warped product Riemannian manifold

Let M be a submanifold in $\mathbb{R}^{2n}$ with the local coordinates $(x_1,y_1,x_2,y_2,...,x_n,y_n)$ given by:

$$x_i=u\cos\alpha_i,\qquad y_i=u\sin\alpha_i, \tag{3.1}$$

for $i\in\{1,...,n\}$, where $u>0$ and α_i denote arbitrary parameters.

Let $X_i:=\frac{\partial}{\partial x_i}$ and $Y_i:=\frac{\partial}{\partial y_i}$, $i\in\{1,2,...,n\}$. Denote by:

$$(X^1,Y^1,...,X^k,Y^k,X^{k+1},Y^{k+1},...,X^n,Y^n):=(X^i,Y^i,X^j,Y^j),$$

for $i\in\{1,...,k\}$, $j\in\{k+1,...,n\}$, $k\in\{2,...,n-1\}$, and consider the $(1,1)$-the tensor field $J:\Gamma(T\mathbb{R}^{2n})\to\Gamma(T\mathbb{R}^{2n})$ defined by:

$$J(X^i,Y^i,X^j,Y^j):=(\sigma X^i,\sigma Y^i,\overline{\sigma}X^j,\overline{\sigma}Y^j), \tag{3.2}$$

where $\sigma=\frac{1+\sqrt{5}}{2}$ is the Golden number and $\overline{\sigma}=\frac{1-\sqrt{5}}{2}=1-\sigma$.

We can verify that J is a Golden structure on $\mathbb{R}^{2n}$ (i.e. $J^2=J+I$) and for any (X^i,Y^i,X^j,Y^j), $(X'^i,Y'^i,X'^j,Y'^j)\in\Gamma(T\mathbb{R}^{2n})$, the scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{2n}$ is J-compatible:

$$\langle J(X^i,Y^i,X^j,Y^j),(X'^i,Y'^i,X'^j,Y'^j)\rangle=\langle(X^i,Y^i,X^j,Y^j),J(X'^i,Y'^i,X'^j,Y'^j)\rangle.$$

Therefore, $(\mathbb{R}^{2n},\langle\cdot,\cdot\rangle,J)$ is a Golden Riemannian manifold.

Using a similar construction as in ([1]), we can check that the tangent bundle of M is spanned by the vectors:

$$Z_0 = \sum_{i=1}^{n} \left(\cos\alpha_i \frac{\partial}{\partial x_i} + \sin\alpha_i \frac{\partial}{\partial y_i} \right) \tag{3.3}$$

and

$$Z_i = -u \sin\alpha_i \frac{\partial}{\partial x_i} + u \cos\alpha_i \frac{\partial}{\partial y_i}, \tag{3.4}$$

for $i \in \{1, ..., n\}$. Then:

$$\|Z_0\|^2 = n, \quad \|Z_i\|^2 = u^2,\ i \in \{1, ..., n\} \tag{3.5}$$

and

$$Z_0 \perp Z_i, \quad Z_i \perp Z_j,\ i \neq j, \quad i, j \in \{1, ..., n\}. \tag{3.6}$$

From (3.2) and (3.3) we obtain:

$$JZ_0 = \sigma \sum_{i=1}^{k} \left(\cos\alpha_i \frac{\partial}{\partial x_i} + \sin\alpha_i \frac{\partial}{\partial y_i} \right) + \overline{\sigma} \sum_{j=k+1}^{n} \left(\cos\alpha_j \frac{\partial}{\partial x_j} + \sin\alpha_j \frac{\partial}{\partial y_j} \right) \tag{3.7}$$

and

$$JZ_i = \sigma Z_i, \quad JZ_j = \bar{\sigma} Z_j, \tag{3.8}$$

for $i \in \{1, ..., k\}$, $j \in \{k+1, ..., n\}$, $k \in \{2, ..., n-1\}$.

The Riemannian metric on M is given by:

$$g = ndu^2 + u^2 \sum_{i=1}^{n} d\alpha_i^2 =: g_1 \times_u g_2. \tag{3.9}$$

Thus M is an $(n+1)$-dimensional warped product submanifold of the Golden Riemannian manifold $(\mathbb{R}^{2n}, \langle\cdot,\cdot\rangle, J)$ with the warping function u.

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Equivalent definitions for connections in higher order tangent bundles

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Abstract: We establish a one to one correspondence between the connections $C^{(k-1)}$ (in the bundle $T^kM \to M$, used by R. Miron in his work on higher order spaces) and $C^{(0)}$ (in the affine bundle $T^kM \to T^{k-1}M$, used for example in [3]).

Keywords: Higher order bundles, connections, fibered manifolds.

MSC2010: 70H03, 70H50

The authors dedicate the paper to Academician Radu Miron on the occasion of his 90'th birthday

1 Introduction

We establish a correspondence one to one between the connections $C^{(k-1)}$ (in the bundle $T^kM \to M$, used by R. Miron) and $C^{(0)}$ (in the affine bundle $T^kM \to T^{k-1}M$, used for example in [3]). The coefficients of $C^{(0)}$ are called, following Miron, as dual coefficients, but not related as the coefficients of a connection. In this case we use notations more adequate to our setting, different from [1] or [4]. A sinthetic review of some problems involving higher order connections is performed in [2].

2 Some notions regarding higher order spaces

The higher order space T^kM of a manifold M can be viewed as a fibered manifold in k ways, over the bases T^iM, $0 \leq i \leq k-1$. Firstly, let us consider the extreme cases $i = 0$ and $i = k-1$, i.e. the fibered manifolds $T^kM \overset{\pi_0}{\to} M$ and $T^kM \overset{\pi_{k-1}}{\to} T^{k-1}M$.

The fibered manifold $T^kM \overset{\pi_0}{\to} M$ was systematically used by R. Miron in the study of non-linear connections, semi-sprays and Lagrangians of order k (for example in [4]).

The affine bundle $T^kM \overset{\pi_{k-1}}{\to} T^{k-1}M$ was used in the papers [3] and [8], in the study of affine Hamiltonians of order k, a natural dual counterpart of Lagrangians of order k.

The higher order space T^kM is defined according by the change rule of the local coordinates given by $ky^{(k)i'} = k\frac{\partial x^{i'}}{\partial x^i}y^{(k)i} + \Gamma_U^{(k-1)}(y^{(k-1)i'})$, where $\Gamma_U^{(k-1)} = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + (k-1)y^{(k-1)i}\frac{\partial}{\partial y^{(k-2)i}}$ is only a local vector field. The Liouville vector field has the local form $\overset{k}{\Gamma} = y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + y^{(k)i}\frac{\partial}{\partial y^{(k)i}}$ and it is a global vector field.

Proposition 2.1. *The fibered manifold $(T^kM, \pi_k, T^{k-1}M)$ is an affine bundle, for $k \geq 2$.*

Notice that the coordinates $y^{(p)i}$ are in accordance with [4]-[7].

In the sequel we use the dual k-structures $J : TT^kM \to TT^kM$ and $J^* : T^*T^kM \to T^*T^kM$, given by:

$$J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \cdots + \frac{\partial}{\partial y^{(k)i}} \otimes dy^{(k-1)i}, \tag{J}$$

$$J^* = dy^{(k-1)i} \otimes \frac{\partial}{\partial y^{(k)i}} + dy^{(k-2)i} \otimes \frac{\partial}{\partial y^{(k-1)i}} + \cdots + dx^i \otimes \frac{\partial}{\partial y^{(1)i}}, \tag{J*}$$

or

$$\frac{\partial}{\partial x^i} \overset{J}{\to} \frac{\partial}{\partial y^{(1)i}} \overset{J}{\to} \cdots \overset{J}{\to} \frac{\partial}{\partial y^{(k)i}} \overset{J}{\to} 0,$$
$$dy^{(k)i} \overset{J^*}{\to} dy^{(k-1)i} \overset{J^*}{\to} \cdots \overset{J^*}{\to} dx^i \overset{J^*}{\to} 0.$$

It is easy to see that J and J^* are (globally defined) endomorphisms on T^kM, $rank\ J = rank\ J^* = km$.

The vector bundle canonically associated with the affine bundle $(T^kM, \pi_k, T^{k-1}M)$ is the vector bundle q_{k-1}^*TM, where $q_{k-1} : T^{k-1}M \to M$ is $q_{k-1} = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_{k-1}$. The fibered manifold (T^kM, q_k, M) is used in [4], [7] in the study of the geometrical objects of order k on M, in particular in the study of Lagrangians of order k on M.

The tensors defined on the fibers of the vertical vector bundle $V_{k-1}^kM \to T^kM$ of the affine bundle $(T^kM,\ \pi_k, T^{k-1}M)$, or on the fibers of an open fibered submanifold of $V_{k-1}^kM \to T^kM$, are called *d-tensors of order* k on M. We denote by $V_j^kM \to T^kM$ the vertical bundle of the bundle $T^kM \to T^jM$, where $T^{(0)}M = M$. Obviously $TT^kM \supset V_0^kM \supset V_1^kM \supset \cdots \supset V_{k-1}^kM$. Their *polar bundles* are $W_j^kM = (V_j^kM)^\circ$, thus $W_0^kM \subset W_1^kM \subset \cdots \subset W_{k-1}^kM \subset T^*T^kM$.

It is easy to see that $\operatorname{Im} J = V_0^k$, $\ker J = V_{k-1}^k$, $\operatorname{Im} J^* = W_{k-1}^k$, $\ker J^* = W_0^k$.

Notice that the local coordinates $(x^i,\ y^{(1)i},\ldots,\ y^{(k)i})$ on T^kM change according to the rules:

$$\begin{aligned}
y^{(1)i'} &= y^{(1)i}\frac{\partial x^{i'}}{\partial x^i},\\
2y^{(2)i'} &= y^{(1)i}\frac{\partial y^{(1)i'}}{\partial x^i} + 2y^{(2)i}\frac{\partial y^{(1)i'}}{\partial y^{(1)i}},\\
&\vdots\\
ky^{(k)i} &= y^{(1)i}\frac{\partial y^{(k-1)i'}}{\partial x^i} + \cdots + ky^{(k)i}\frac{\partial y^{(k-1)i'}}{\partial y^{(k-1)i}}.
\end{aligned} \tag{2.1}$$

Local bases of modules of sections in $\Gamma(V_j^kM)$ and $\Gamma(W_j^kM)$ are $\left\{\dfrac{\partial}{\partial y^{(j)i}},\ldots,\dfrac{\partial}{\partial y^{(k-1)i}}\right\}$ and $\left\{dx^i, dy^{(j)i},\ldots,dy^{(j-1)i}\right\}$ respectively.

3 Connections

A non-linear connection in the affine bundle $(T^kM,\ \pi_k, T^{k-1}M)$ is a left splitting $C^{(k-1)}$ of the inclusion $V_{k-1}^kM \overset{I_{k-1}}{\to} TT^kM$, i.e. a vector bundle map $TT^kM \overset{C^{(k-1)}}{\to} V_{k-1}^kM$ such that $C^{(k-1)}\circ I_{k-1} = 1_{V_{k-1}^kM}$. Using local coordinates, $C^{(k-1)}$ has the local form:

$$\begin{aligned}
&(x^i, y^{(1)i},\ldots,y^{(k)i}, Y^{(0)i}, Y^{(1)i},\ldots,Y^{(k)i}) \overset{C^{(k-1)}}{\to}\\
&(x^i, y^{(1)i},\ldots,y^{(k)i}, Y^{(k)i} + M_{(0)j}^iY^{(0)j} + \cdots + M_{(k-1)j}^iY^{(k-1)j}).
\end{aligned} \tag{3.1}$$

A non-linear connection in the bundle $(T^kM,\ \pi_k, M)$ is a left splitting $C^{(0)}$ of the inclusion $V_0^kM \overset{I_0}{\to} TT^kM$, i.e. a vector bundle map $TT^kM \overset{C^{(0)}}{\to} V_0^kM$ such that $C^{(0)}\circ I_0 = 1_{V_0^kM}$. Using local coordinates, $C^{(0)}$ has the local form:

$$\begin{aligned}
&(x^i, y^{(1)i},\ldots,y^{(k)i}, Y^{(0)i}, Y^{(1)i},\ldots,Y^{(k)i})) \overset{C^{(0)}}{\to}\\
&(x^i, y^{(1)i},\ldots,y^{(k)i}, Y^{(1)i} + N_j^{(1)i}Y^{(0)j},\ldots,Y^{(k)i} + N_j^{(k)i}Y^{(0)j}).
\end{aligned} \tag{3.2}$$

We are going to establish a one to one correspondence between connections $C^{(0)}$ and $C^{(k-1)}$.

Using constructions performed in [4] (see also [5]), we can perform the following constructions using $C^{(0)}$.

If one denotes by $N_0 \subset TT^kM$ the horizontal distribution of $C^{(0)}$, then the successive distributions:

$$N_1 = J(N_0), \ldots, N_{k-1} = J^{k-1}(N_0), V^k_{k-1}M = J^k(N_0) \tag{3.3}$$

give the decomposition:

$$TT^kM = N_0 \oplus N_1 \oplus \cdots \oplus N_{k-1} \oplus V^k_{k-1} \tag{3.4}$$

and the corresponding projectors denoted by h, $v_1, \ldots, v_k$. A local base $\mathcal{B}_0 = \left\{ \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^{(1)j} \frac{\partial}{\partial y^{(1)j}} - \cdots - N_i^{(k-1)j} \frac{\partial}{\partial y^{(k-1)j}} \right\}$ for $\Gamma(N_0)$ generate the bases $\mathcal{B}_1 = J(\mathcal{B}_0) \subset \Gamma(N_1)$,..., $\mathcal{B}_{k-1} = J^{k-1}(\mathcal{B}_0) \subset \Gamma(N_{k-1})$, $\mathcal{B}_k = J^k(\mathcal{B}_0) \subset \Gamma(V^k_{k-1})$ and $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ is a local base for $\mathcal{X}(T^kM)$, called a *Berwald base* (see [1]).

Let us consider a dual base of $\mathcal{B}$, $\mathcal{B}^* = \{\delta x^i = dx^i, \delta y^{(1)i}, \ldots, \delta y^{(k)i}\}$, where

$$\delta y^{(k)i} = M^i_{(k)j} dx^j + M^i_{(k-1)j} dy^{(1)j} + \cdots + M^i_{(1)j} dy^{(k-1)j} + dy^{(k)i}$$

and $\delta y^{(k-1)i} = J^* \delta y^{(k)i}$,..., $\delta y^{(1)i} = (J^*)^{k-1} \delta y^{(k)i}$, $\delta x^i = dx^i = (J^*)^k \delta y^{(k)i}$. The *dual coefficients* of the connection $C^{(0)}$ are the above coefficients of $\delta y^{(k)i}$. The duality conditions give a link between the coefficients $N_j^{(\alpha)i}$ and $M^i_{(\beta)j}$, given by:

$$\begin{aligned}
&N_j^{(1)i} = M^i_{(1)j}, \\
&N_j^{(2)i} = M^i_{(2)j} - M^i_{(1)p} N_j^{(1)p}, \\
&\vdots \\
&N_j^{(k)i} = M^i_{(k)j} - M^i_{(k-1)p} N_j^{(1)p} - \cdots - M^i_{(1)p} N_j^{(k-1)p},
\end{aligned} \tag{3.5}$$

and conversely:

$$\begin{aligned}
&M^i_{(1)j} = N_j^{(1)i}, \\
&M^i_{(2)j} = N_j^{(2)i} + M^i_{(1)p} N_j^{(1)p} \\
&\vdots \\
&M^i_{(k)j} = N_j^{(k)i} + M^i_{(k-1)p} N_j^{(1)p} + \cdots + M^i_{(1)p} N_j^{(k-1)p}.
\end{aligned} \tag{3.6}$$

The coefficients M change according to the rule:

$$\begin{cases}
M^j_{(1)i} \frac{\partial x^{i'}}{\partial x^j} = M^{i'}_{(1)j'} \frac{\partial x^{j'}}{\partial x^i} + \frac{\partial y^{(1)i'}}{\partial x^i}, \\
\vdots \\
M^j_{(k)i} \frac{\partial x^{i'}}{\partial x^j} = M^{i'}_{(k)j'} \frac{\partial x^{j'}}{\partial x^i} + M^{i'}_{(k-1)j'} \frac{\partial y^{(1)j'}}{\partial x^i} + \cdots + M^{i'}_{(1)j'} \frac{\partial y^{(k-1)j'}}{\partial x^i} + \frac{\partial y^{(k)i'}}{\partial x^i}.
\end{cases} \tag{3.7}$$

The change rule of $\delta y^{(k)i}$ is $\delta y^{(k)i'} = \dfrac{\partial x^{i'}}{\partial x^i}\delta y^{(k)i}$. By a straightforward computation one can prove that a connection $C^{(k-1)}$ that has the local form (3.1) can be defined. Its horizontal projector has the form:

$$\begin{aligned} & h\left(Y^{(0)i}\frac{\partial}{\partial x^i} + Y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + Y^{(k-1)i}\frac{\partial}{\partial y^{(k-1)i}}\right) \\ = \; & Y^{(0)i}\left(\frac{\partial}{\partial x^i} - M^j_{(k)i}\frac{\partial}{\partial y^{(k)i}}\right) + Y^{(1)i}\left(\frac{\partial}{\partial y^{(k-1)i}} - M^j_{(1)i}\frac{\partial}{\partial y^{(k)i}}\right) + \cdots \\ & + Y^{(k-1)i}\left(\frac{\partial}{\partial y^{(k-1)i}} - M^j_{(1)i}\frac{\partial}{\partial y^{(k)i}}\right). \end{aligned}$$

It can be also proved that conversely, given a connection $C^{(k-1)}$ having the coefficients $(M^i_{(\alpha)j})$, then a (non-linear) connection $C^{(0)}$ with coefficients $(N^{(\alpha)i}_j)$ given by relations (3.5) is obtained, having $(M^i_{(\alpha)j})$ as dual coefficients.

A link between the connections $C^{(0)}$ and $C^{(k-1)}$ can be obtained without involving coordinates, as follows. Let us consider a given connection $C^{(0)}$, thus the distribution N_0 is given. We can construct the distributions (3.3) and the distribution (3.4). We have $TT^kM = N_k \oplus V^k_{k-1}M$, where $N_k = N_0 \oplus N_1 \oplus \cdots \oplus N_{k-1}$. The polar subbundle $N^\circ_k \subset T^*T^kM$ (given by forms in T^*T^kM that are null on N_k) is canonically isomorphic with the dual of the vertical bundle $V^k_{k-1}M$, thus the inclusion $N^\circ_k \subset T^*T^kM$ reverses by duality to a vector bundle epimorphism $TT^kM \overset{C^{(k-1)}}{\rightarrow} V^k_{k-1}M$ that defines the connection $C^{(k-1)}$.

In order to obtain $C^{(0)}$ from $C^{(k-1)}$ we proceed analogously, following a converse way: The vector bundle epimorphism $TT^kM \overset{C^{(k-1)}}{\rightarrow} V^k_{k-1}M$ that defines the connection $C^{(k-1)}$ reverses by duality to an inclusion $W^k = \left(V^k_{k-1}M\right)^* \subset T^*T^kM$. We consider the subbundles of T^*T^kM: $W^{k-1} = J^*(W^k)$,..., $W^1 = (J^*)^{k-1}(W^k)$, $W^0 = (J^*)^k(W^k)$. We have that $T^*T^kM = W^0 \oplus \cdots \oplus W^k$ and $W^0 = (V^k_0M)^\circ$. Then considering $\mathcal{U} = W^1 \oplus \cdots \oplus W^k$ we have $T^*T^kM = \mathcal{U} \oplus (V^k_0M)^\circ$ and $\mathcal{U}$ is canonically isomorphic with $(V^k_0M)^*$. Thus the inclusion $\mathcal{U} \subset T^*T^kM$ reverses by duality to a vector bundle epimorphism $TT^kM \overset{C^{(0)}}{\rightarrow} V^k_0M$ that defines the connection $C^{(0)}$.

All these prove the following interpretation of the dual coefficients of a non-linear connection of order k, which is the main result of the paper.

Theorem 3.1. *There is a one to one correspondence between the connections $C^{(0)}$ in the bundle $T^kM \overset{\pi_0}{\rightarrow} M$ and connections $C^{(k-1)}$ in the bundle $T^kM \overset{\pi_{k-1}}{\rightarrow} T^{k-1}M$.*

As we have seen, the correspondence given by the above Proposition associates, in local coordinates, but also in an invariant form, free of coordinates, a connection

$C^{(k-1)}$ with a connection $C^{(0)}$ that has the dual coefficients exactly the coefficients of $C^{(k-1)}$.

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On Some Inequalities with Operators in Hilbert Spaces

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Abstract: Operators defined on Hilbert spaces represent a major subfield of (or base for) the Functional Analysis. Several types of inequalities among such operators were established and studied in the last decades, mainly by the early '50s and then from the '80s. In this paper there are reviewed some of the most important types of inequalities, introduced and studied by H. Bohr, E. Heinz - T. Kato, T. Ando, etc. They were extended and/or sharpened by other authors, mentioned in the Introduction. Some of the definitions and proofs, found in several references, are completed (by the author) with specific formulas involving H-space operators, several details are also added to certain proofs and definitions as well. The main ways for establishing inequalities with operators are pointed out: scalar inequalities like the Cauchy-Schwarz and Bohr's inequalities over the complex field or on an H-space, certain identities with H-space operators, etc.

Keywords: Operators on Hilbert spaces, Bohr's inequality, Heinz's and Kato's inequalities.

MSC2010: 46C05, 46L05, 47A05, 47A63, 47B65

1 Introduction

Many and various types of inequalities, involving operators defined on Hilbert spaces, were introduced and studied in the latest 6-7 decades. The main class of H-space operators taken into account in such studies is the one of (bounded) selfadjoint operators on complex Hilbert spaces. A large number of papers approaching this subject of inequalities with operators were published, mainly in the '80s, and the names of a few most significant authors deserve mention: F. Kittaneh (1986, 1988), F. Kittaneh (1988), O. Hirzallah (2003), several Croatian authors headed by Academician J. Pečarić of Zagreb, F. Zhang (2007). But – as mentioned in the Abstract – several forerunning papers appeared around 1950. In the next Section 2, there are recalled basic definitions and notations from the theory of Hilbert space operators. The subsequent Section 3 is dedicated to operator inequalities induced by scalar inequalities over Hilbert spaces like the mixed Schwarz inequality, H. Bohr inequality, F. Heinz − T. Kato inequality, inequalities induced by operator positivity. The author of this paper identifies some ways to derive inequalities with H-Space operators, some relations between specific types of inequalities are discussed and several details are added to certain definitions and proofs in ten Remarks and four Propositions.

2 Preliminaries on Hilbert space operators

2.1 Basic definitions and notations

A Hilbert space over the complex field is denoted, in [6], as $\boldsymbol{H}$.The standard definition of a *norm* on an inner product space, including on a Hilbert space, is

$$(\forall\, x \in \boldsymbol{H}\,)\ \ ||\, x\, || \ = \sqrt{\langle\, x\,,\ x\,\rangle}\,. \tag{2.1}$$

A linear operator $T : \boldsymbol{H} \longrightarrow \boldsymbol{H}$ obeys the natural property of linearity,

$$(\,\forall\, \alpha_1,\, \alpha_2 \in \mathbb{C}\ )\,(\,\forall\ x_1,\, x_2 \in \boldsymbol{H}\,)\quad T\,(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T\, x_1 + \ \alpha_2 T x_2\,. \tag{2.2}$$

Definition 2.1. *Domain* and *range* of an operator $T : \boldsymbol{H} \to \boldsymbol{H}$. This notation means that T is defined everywhere on $\boldsymbol{H}$, but it is possible that the domanin of T is only a subset (subspace) of $H : \mathfrak{D}_T \subseteq \boldsymbol{H}$. The range of T is defined by

$$\mathfrak{R}_T = \{\ y \in \boldsymbol{H} : (\ \exists\ \ x \in \boldsymbol{H}\ )\ \ y = T\, x\}\,. \tag{2.3}$$

The *kernel* and the *image* of T are (respectively) defined by

$$\text{Ker}\ T = \{x \in \boldsymbol{H} : T\, x = \mathbf{0}\} \quad \text{Im}\ T = \{y \in \boldsymbol{H} : (\,\exists\, x \in \mathfrak{D}_T)\ \ y = T\, x\}. \tag{2.4}$$

Remark 2.1. The image of T, as defined in (2.5), is slightly more general than its range of (2.4) since a posibly smaller domain (than the whole space $\boldsymbol{H}$) is considered. They coincide if $\mathfrak{D}_T = \boldsymbol{H}$. The zero vector is denoted as $\mathbf{0}$ for avoiding possible confusions with the zero scalar and/or the zero operator as well, which is denoted as 0 in most references. The two subspaces that occur in (2.5) can be also defined in an equivalent (may be simpler) way as in our textbook [3] :

$$\text{Ker}\ T = T_{-1}\ (\mathbf{0})\,, \quad \text{Im}\ T = T\,\boldsymbol{H} \quad \text{or} \quad \text{Im}\ T = T\,\mathfrak{D}_T. \tag{2.5}$$

In (2.6) , T_{-1} denotes the counterimage mapping, from $\mathfrak{R}_T$ to $\mathcal{P}(\mathfrak{D}_T)$: if $y \in \mathfrak{R}_T$ then its counterimage is $T_{-1}(y) = \{x \in \boldsymbol{H} : Tx = y\}$. It is well known that T is injective $\Leftrightarrow$ Ker $T = \{\mathbf{0}\}$ and it is surjective (or onto) $\Leftrightarrow$ Im $T = \boldsymbol{H}$. The identity operator $I_{\boldsymbol{H}}$ is (most often) denoted as $\mathbf{1}$. The connection between this $\mathbf{1}$, the zero operator 0 and the two subspaces that occur in (2.5) is

$$\text{Ker } \mathbf{1} = \{\mathbf{0}\}\,, \quad \text{Im } \mathbf{1} \ = \boldsymbol{H}\ , \qquad \text{Ker } 0 = \boldsymbol{H}\ , \quad \text{Im } 0 \ = \{\mathbf{0}\}.$$

The linear operations with operators are defined as usually : the sum $T_1 + T_2$ and $\alpha\, T =$ the multiplication by a scalar. The composition (or product) of two operators T_1, T_2 from $\boldsymbol{H}$ to $\boldsymbol{H}$ is defined, as usually for the composition of two maps on the same set / space :

$$(\ \forall\ x \in \boldsymbol{H}\ )\ \ (T_1\ T_2)\, x = T_2\ (T_1)\, x \underset{\text{not}}{=} (T_1 \circ T_2)\, x\,. \tag{2.6}$$

Endowed with these three operations, the set of operators defined on the Hilbert space $\boldsymbol{H}$ has the structure of a $C^{\star}-$ algebra, that is an involutive Banach algebra.

2.2 Classes of H-space operators and characterizations

Several inequalities involving operators defined on a Hilbert space $\boldsymbol{H}$ were formulated and studied for operators of certain (particular) types. Certain classes of H-space operators need a few preliminary definitions for being introduced. The first of them follows, but it implies the notion of a bounded operator, which follows to be presented together with other types after this one. According to [6], an operator $T : \boldsymbol{H} \longrightarrow \boldsymbol{H}$ is said to be bounded if there exists a positive number α such that $(\forall\ x \in \boldsymbol{H}\ )\ \ ||\, T\, x\, ||\ \leq \alpha\, ||\, x\, ||$. This class of bounded operators is denoted (in almost all references) as $\mathcal{B}(\boldsymbol{H})$. A large variety of classes (or types) of H-space operators can be found in [6]. A selection of five classes is presented in Jan Hamhalter's extended LN [7], together with characterizations for three of them. But some of them involve the notion of the adjoint of an operator $T : \boldsymbol{H} \longrightarrow \boldsymbol{H}$, and also the notion of norm. Two definitions of the *norm* of an H-space operator T are :

$$||\, T\, ||\ = \sup\{||\, T\, x\, ||\, /\, ||\, x\, ||\ : x \neq \mathbf{0}\} \text{ or } ||\, T\, || \ = \sup_{||\, x\, || = ||\, y\, || = 1} |\, \langle\, y\,,\ T\, x\, \rangle\, |\,. \tag{2.7}$$

The above definition of (2.8) occurs in E. Heinz's paper [9] while the latter was given by Ch. Remling in [17]. According to [5], these two definitions are equivalent if T is selfadjoint.

Definition 2.2. ([6] or [17])The *adjoint* of an operator $T : \boldsymbol{H} \longrightarrow \boldsymbol{H}$, $T \in \mathcal{B}(\boldsymbol{H})$, denoted T^*, is defined by

$$(\forall\ x\ ,\ y \in \boldsymbol{H}\ )\ \ \langle\, T^* y\,,\ x\,\rangle = \langle\, y\,, T\, x\,\rangle. \tag{2.8}$$

The main properties of the adjoint operator were stated in Ch. Remling's just quoted lecture note :

Theorem 2.1 (Th. 6.1 in [20]). *Let* $S, T \in \mathcal{B}(\boldsymbol{H})$ *and* $c \in \mathbb{C}$. *Then* (a) $T^* \in \mathcal{B}(\boldsymbol{H})$; (b) $(S+T)^* = S^* + T^*$, $(cT)^* = \bar{c}\,T^*$; (c) $(S\,T)^* = T^*S^*$; (d) $T^{**} = T$; (e) *If* T *is invertible, then* T^* *is also invertible and* $(T^*)^{-1} = (T^{-1})^*$; (f) $\|T\| = \|T^*\|$, $\|T\,T^*\| = \|T^*T\| = \|T\|^2$ *(the* $C^{\star}-$ *property).*

Definition 2.3. Let $T \in \mathcal{B}(\boldsymbol{H})$. Then T is said to be

(i) *normal* *if* $T\,T^* = T^*T$;
(ii) *selfadjoint* *if* $T = T^*$;
(iii) *positive* *if* $\langle\, T\,x\,,\,x\,\rangle \geq 0$ for all $x \in \boldsymbol{H}$
(iv) *unitary* *if* $T\,T^* = T^*T = \mathbf{1}$;
(v) *projection* *if* $T\,T^* = T^2 = T^*T$.

Characterizations of normal, selfadjoint and unitary operators are also presented in [7].

3 Inequalities with H-space operators induced by inequalities on $\mathbb{C}$ / H

3.1 Inequalities induced by operator positivity

If the adjoint operator T^* is considered, it is easy to see that *Def. 2.3* $-$ (iii) of a positive operator is equivalent to

$$\langle\, x\,,\,T^*x\,\rangle \geq 0 \;\text{ for all }\; x \in \boldsymbol{H}\,. \tag{3.1}$$

Remark 3.1. The inequality $\langle\, T\,x\,,\,x\,\rangle \geq 0$ or its equivalent of (3.1) make sense only if the respective inner products are real. This condition is often omitted, that is not explicitly stated. But part 2° of **1.3. Proposition** in [7] states that an inner product $\langle\, T\,x\,,\,x\,\rangle \in \mathbb{R}$ if and only if T is selfadjoint.

In what follows, there are implied two or more H-space operators and we are going to denote them as A, B, $C, \ldots$ instead of T . Replacing, in the definition of a positive operator, T by the difference of the operators A , B , the next definition naturally follows.

Definition 3.1. If $A, B \in \mathcal{B}(\boldsymbol{H})$, then

$$A \geq B \underset{\text{def}}{\Longleftrightarrow} (\forall\, x \in \boldsymbol{H}\,) \quad \langle (\,A - B\,)\,x\,,\,x\,\rangle \geq 0\,. \tag{3.2}$$

The positivity of a bounded operator A and the binary relation $A \geq B$ were also considered by Fuzhen Zhang in [18], with the mention that both operators should be self-adjoint. This definition implies a partial ordering over the class of bounded (selfadjoint) operators on an H-space. We state this assertion as

Proposition 3.1. *The relation defined by (3.2) is a partial ordering on* $\mathcal{B}(\boldsymbol{H})$.

Proof. (R) − Reflexivity : $A \geq A$ since

$$(\, \forall\, x \in \boldsymbol{H}\,)\ \ \langle (\, A - A\,)\, x\,,\, x\,\rangle = \langle 0\, x\,,\, x\,\rangle = \langle\, 0\,,\, x\,\rangle = 0\ . \tag{3.3}$$

The last equation in (3.3) is obvious for any inner product, and it is slightly weaker than the first axiom for any IP. See a *Note* next to (2.1). Hence $A - A = 0 \Longrightarrow A = A$, the equality case of $A \geq A$.(AS) − Antisymmetry : $A \geq B \quad \& \quad B \geq A \Longrightarrow A = B$. Indeed, according to *Def. 2.2* − *(iii)* , the two inequalities in the left side of this implication mean that

$$(\forall\, x \in \boldsymbol{H})\ \langle (A - B)x, x\rangle \geq 0\ \&\ \langle (B - A)x, x\rangle = -\langle (A - B)x, x\rangle \geq 0 \Longrightarrow$$

$$[\langle (A - B)x, x\rangle \geq 0\ \langle (A - B)x, x\rangle \leq 0] \Longrightarrow \langle (A - B)x, x\rangle = 0\ \Longrightarrow\ A = B\ .$$

(Tr) − Transitivity : $A \geq B \quad \& \quad B \geq C \quad \Longrightarrow \quad A \geq C$. In a similar way,

$$A \geq B\ \&\ B \geq C \quad \Longrightarrow \quad [\langle (A - B)x, x\rangle \geq 0\ \ \&\ \ \langle (B - C)x, x\rangle \geq 0] \quad \Longrightarrow$$

$$(\forall\, x \in H\,)\quad \langle (\, A - C\,)\, x\,,\, x\,\rangle \geq 0 \quad \Longrightarrow \quad A \geq C\,.$$

The but last inequality obviously follows by adding − side-by-side − the two inequalities inside [...]. □

Remark 3.2. In view of the earlier *Remark 3.1*, the statement of this *Proposition 3.1* would have had to be restricted to the (subspace / subalgebra of) selfadjoint operators since the order relation of (3.2) holds for this class of operators only. However, we have just noticed that, in most publications dealing with positive operators, this restriction is not explicitly stated. In fact, the operator differences that occur in (3.2) and in the subsequent equations should be selfadjoint, and not necessarily the operators A, B, C themselves. As a matter of terminology, the self-adjoint operators (or matrices) are equivalently termed *Hermitian* in Serge Lang's volume Linear Algebra (Springer-Verlag, 1987) − [13].

3.2 T. Kato's inequality involving the domains and the norms of images

Another binary relation between two operators was defined by T. Kato in an article of 1952.

Definition 3.2. ([10]) Let S , T be two linear operators. Then

$$S \ll T \;\Leftrightarrow [\mathfrak{D}_S \supseteq \mathfrak{D}_T \quad \& \quad (\forall\, x \in \mathfrak{D}_T) \;\; ||\, S\,x\,|| \;\leq ||\, T\,x\,||\,] \,. \tag{3.4}$$

Proposition 3.2. *The relation defined by (3.10) is a partial ordering on the algebra of linear operators defined on* $\boldsymbol{H}$.

Proof. The inclusion between the domains of S and T is a simple subset inclusion over the parts of $\boldsymbol{H}$ and it clearly satisfies the three properties of a partial ordering. As regards the inequality between the norms of the two images, let us recall that the norm of an element (vector) in a H-space is defined as in (2.2) : $||\, x\,|| \;=\; \sqrt{\langle\, x\,, x\,\rangle}$. Hence

$$||\,S\,x\,|| \;\;=\;\; \sqrt{\langle\, S\,x\,,\; S\,x\,\rangle} \quad \text{and} \quad ||\,T\,x\,|| \;\;=\;\; \sqrt{\langle\, T\,x\,,\; T\,x\,\rangle}. \tag{3.5}$$

It follows that $(\forall\, x \in \mathfrak{D}_T)\;\; ||\,S\,x\,|| \leq ||\,T\,x\,|| \Leftrightarrow \sqrt{\langle\, S\,x\,,\; S\,x\,\rangle} \leq \sqrt{\langle\, T\,x\,,\; T\,x\,\rangle} \Leftrightarrow$

$$\Leftrightarrow \langle\, S\,x\,,\; S\,x\,\rangle \leq \langle\, T\,x\,,\; T\,x\,\rangle \Leftrightarrow \langle\, (T - S\,)\,x\,,\; (T - S\,)\,x\,\rangle \geq 0\,. \tag{3.6}$$

The inequality in (3.10) obviously satisfies the three axioms of a partial order (R) : $||\; S\,x\;|| \;\; \leq ||\; S\,x\;||$; (AS) : $||\,S\,x\,|| \;\leq ||\,T\,x\,||$ and $||\,T\,x\,|| \;\leq ||\,S\,x\,|| \Longrightarrow$, by (3.5) , $T - S = 0 \Longrightarrow S = T$. The transitivity holds, too. □

In *Definition 3.2* and in the statement of this Proposition there are not assumed any properties of the operators thereof. However, the author asserts, for the case when S and T are selfadjoint, that $S \ll T$ is equivalent to $S^{\,2} \ll T^{\,2}$ in the sense of F. Rellich [16]. More generally, if S and T are closed and have everywhere closed domains, then

$$S \ll (\; S^*S\;)^{1/2} \ll S \quad \text{and} \quad T \ll (\; T^*T\;)^{1/2} \ll T$$

so that $S \ll T$ is equivalent to

$$(\; S^*S\;)^{1/2} \ll (\; T^*T\;)^{1/2} \;, \text{i.e. to } \; S^*S \ll T^*T\;. \tag{3.7}$$

We close these quotations from T. Kato's article [11] by a Corollary to this author's Theorem 1. Let A and B be selfadjoint operators and let $A \ll B$. If $A^{\,-1}$ exists, then $B^{\,-1}$ also exists and $B^{\,-1} \leq A^{\,-1}$.

Remark 3.3. Let us see that the operator inequalities defined in (3.2) and (3.4) do not effectively follow from some scalar inequalities, but they are defined in terms of the scalar product and norms on the images of the two operators, that hold in the H-space $\boldsymbol{H}$. The order relation between two operators $A,\ B \in \mathcal{B}(\boldsymbol{H})$ was also defined, in an equivalent way, by S.S. Dragomir as follows.

Definition 3.3. ([5]) If $A,\ B$ are selfadjoint operators on $\boldsymbol{H}$ then

$$A \le B \Leftrightarrow \langle\, A\, x\,,\, x\,\rangle \le \langle\, B\, x\,,\, x\,\rangle \quad \text{for all} \quad x \in \boldsymbol{H}\ . \tag{3.8}$$

It is obvious that the definition in (3.8) is equivalent to the one in (3.2). It simply suffices to interchange $A,\ B$ and to see that $\langle\, A\, x,\ x\,\rangle \le \langle\, B\, x,\ x\,\rangle \Leftrightarrow \langle (B{-}A)\, x\,, x\,\rangle \ge 0$. This author's *Theorem 2*, next to this definition, states that the relation $A \le B$ is reflexive, transitive and antisymmetric $-$ the properties proved in our earlier *Proposition 3.1.* The statement of S.S. Dragomir's *Theorem 2* also includes two more properties that involve a third operator C and two real scalars, as well :

4. If $A \le B$ *and* $\alpha \ge 0$ *then*

$$A + C \le B + C \quad \text{and} \quad \alpha\, A \le \alpha\, B, \qquad -\, A \ge -\, B\ . \tag{3.9}$$

5. If

$$\alpha \le \beta \quad \text{then} \quad \alpha\ A \le \beta\ A\ . \tag{3.10}$$

Under similar assumptions on one / more operators, other two theorems are stated (without proofs) by S.S. Dragomir :

Theorem 3 ([5]). *Let* A *be a positive selfadjoint operator on* H . *Then*

$$||\, A\ x\,||^2 \le ||\ A\ ||^2\ \langle\, A\, x\,,\, x\,\rangle \quad \textit{for any} \quad x \in H\ . \tag{3.11}$$

Theorem 4 ([5]). *Let* $A_n,\ B$ *with* $n \ge 1$ *be positive selfadjoint operators on* $\boldsymbol{H}$ *with the property that*

$$A_1 \le A_2 \le\ \ldots\ A_n \le\ \ldots\ B\ . \tag{3.12}$$

Then there exists a bounded selfadjoint operator A *defined on* $\boldsymbol{H}$ *such that*

$$A_n \le A \le B \quad \textit{for all} \ \ n \ge 1 \ \ \textit{and} \ \ \lim_{n \to \infty} A_n\, x = A\, x\ . \tag{3.13}$$

Remark 3.4. It is clear then the operators A_n $(\, n \ge 1)$ in the statement of the latter Theorem are the terms of an increasing sequence of operators, upper-bounded by an operator B , with respect to the order relation of (3.2) / (3.8) . The operator A of (3.13) appears to be an upper margin of $\{A_n : n \ge 1\}$. This property is similar to a feature of increasing and upper-bounded real sequences in Calculus.

3.3 Operator inequalities induced by Cauchy-Schwarz type (scalar) inequalities

One of the most important inequalities over an inner product space (which is necessarily a normed space as well) is the Cauchy – Schwarz inequality. As a detail, some authors call it the Cauchy – Schwarz – Bunjakovski inequality. Obviously, it also holds on an H-space:

$$\text{(C-S-B)}\ (\forall\, x,\, y \in \boldsymbol{H}\,)\ \ |\langle\, x,\, y\,\rangle|^2 \leq \langle\, x,\, x\,\rangle\, \langle y,\, y\rangle \Leftrightarrow |\langle\, x,\, y\,\rangle| \leq ||\, x\,||\, ||\, y\,||. \quad (3.14)$$

S.S. Dragomir presents a generalization of inequality (3.14). In each of the three inner products that occur in its first (left) version, one of its factors (which are vectors in $\boldsymbol{H}$) is replaced by its (their) image(s) through a selfadjoint and positive operator A:

$$(\forall x,\, y \in \boldsymbol{H}\,)\ |\langle\, A\, x\,,\, y\,\rangle\,|^2 \leq \langle\, A\, x\,,\, x\,\rangle\, \langle\, A\, y\,,\, y\,\rangle. \quad (3.15)$$

Remark 3.5. Let us see that the two IPs in the right side of (3.21) are positive, just because $A \geq 0$. The connection from (3.14) to (3.15) illustrates a way to obtain inequalities with operators, suggested by the title of this subsection. In fact, inequality (3.14) is a generalization of the Cauchy – Schwarz inequality since [(3.15) & $A = \mathbf{1}$] $\Longrightarrow$ (3.14). The inequality (3.15) , with T instead of A , was presented in the Introduction to F. Kittaneh's article [12] and called the *Schwarz inequality for positive operators.*

Many generalizations of certain inequalities with H-space operators were formulated and studied by taking real (and not natural or 1/2) powers of operators there involved. It comes to formulas including – for example – powers of operators of the form A^{ν}, B^{ν} or $B^{1-\nu}$ with $\nu \in \mathbb{R}_+$. Fuad Kittaneh presents, in [12], such an inequality with real powers of the absolute values of T and its adjoint T^*. In fact, F.K. mentions that it was earlier established by T. Kato in his paper [11] of 1952, and he calls it the

Mixed Schwarz inequality : If $T \in \mathcal{B}(\boldsymbol{H})$, $x\,,\, y \in \boldsymbol{H}$ and $0 \leq \alpha \leq 1$ then

$$\langle\, T\, x\,,\ y\,\rangle\,|^2 \leq \langle\, |\, T\,|^{\,2\,\alpha}\, x\,,\ x\,\rangle\, \langle\, |\, T^*\,|^{\,2(1-\alpha)}\, y\,,\ y\,\rangle\ . \quad (3.16)$$

Remark 3.6. In (3.16), $|\,T\,| = (\,T^*T\,)^{1/2}$ and $|\,T^*| = (\,T^*T\,)^{1/2}$. These two equalities, taken to (3.16) , lead to an equivalent expression of the above inequality :

$$\langle\, T\, x\,,\ y\,\rangle\,|^2 \leq \langle\, (\,T^*T\,)^{\,\alpha}\, x\,,\ x\,\rangle\, \langle\, (T^*T\,)^{\,1-\alpha}\, y\,,\ y\,\rangle\ . \quad (3.17)$$

3.4 Bohr – type inequalities with H-space operators

In certain cases, an inequality involving one or two (bounded self-adjoint or only positive) operators like T, A, B defined on a complex H-space $\boldsymbol{H}$, can be obtained from a "scalar" inequality which holds on $\mathbb{C}$ or in a Hilbert space by replacing one or two arguments like complex numbers $z, w \in \mathbb{C}$ or elements (vectors) $x, y \in \boldsymbol{H}$ by operators. Obviously, the expressions with operators thus obtained should result in H-space operators as well, based on operations with such operators like those in the algebra $\mathcal{B}(\boldsymbol{H})$. In a more general approach, new inequalities can be derived by use of certain operator functions of the form $f(A)$, with specific properties imposed to f.

For the former approach, it is claimed (in many references) that the classical Harald Bohr's inequality for scalars was presented in [2] – *Acta Mathematica* **45** (1924) – under the form :

$$|z + w|^2 \leq p\,|z|^2 + q\,|w|^2 \tag{3.18}$$

where $z, w \in \mathbb{C}$ & $p, q \in \mathbb{R}$ such that

$$p, q > 1 \text{ and } 1/p + 1/q = 1\ . \tag{3.19}$$

In several sources, the real numbers p, q are called *conjugate exponents*. However, see the next *Remark* on the original form of H. Bohr's inequality.

This expression (3.18), with conditions (3.19), of H. Bohr's inequality over $\mathbb{C}$ is also presented in the article [4] by W-S. Cheung and J. Pečarić, where it is asserted that equality occurs in (3.18) if and only if $w = (p-1)\,z$. In [4], the authors cite a generalization of the Bohr inequality (3.18), due to J. Pečarić & S.S. Dragomir [15], to the context of normed vector spaces. If $(V, ||\ \ ||)$ is such an NVS and the real p, q satisfy (3.19), then $||\,x + y\,||^2 \leq p\,||\,x\,||^2 + q\,||\,y\,||^2$ for any $x, y \in V$.

A variant of inequality (3.18) is presented, for instance, in O. Hirzallah's article [10] as well as in F. Zhang's [18] : if $a, b \in \mathbb{C}$ and $p, q \in \mathbb{R}$ also satisfy conditions (3.19), then

$$|a - b|^2 \leq p\,|a|^2 + q\,|b|^2. \tag{3.20}$$

Obviously, equivalence between (3.18) and (3.20) follows by the replacements $z \leftrightarrows a$ & $w \leftrightarrows -b$.

Remark 3.7. In many references dedicated to Bohr type inequalities, it is asserted that expression (3.18) or (3.20) of the classical H. Bohr inequality was given in the (above quoted) reference [2]. In fact, that article was dedicated to almost periodic functions, to their Fourier expansions and to Dirichlet series. We did not find such an inequality in any of the 99 pages of that study. The only similar inequality appears at page 78 of [2]:

$$|\, a+b\,|^2 \le (1+c)\,|\, a\,|^2 + \left(1+\frac{1}{c}\right)\,|\, b\,|^2 \tag{3.21}$$

where $a, b \in \mathbb{C}$ and $c > 0$. This original form of the Bohr inequality is equally presented (and exactly quoted from [2]) in the paper [14] by J. Pečarić & Th. Rassias (*J.M.A.A.*, 1993), with other symbols for the two complex numbers: $a \to z_1$ & $b \to z_2$.

The latter variant (3.20) is better suited than (3.18) for stating operator inequalities of Bohr type. In [21] , the author quotes a result of [10] which generalizes (3.20) , namely O. Hirzallah's

Theorem 1 ([10]) *If* A , $B \in \mathcal{B}(\boldsymbol{H})$, $p\,, q > 1$ *with* $1/p + 1/q = 1$ *and* $p \le q$ *then*

$$|\, A - B\,|^2 + |\,(1-p)\,A - B\,|^2 \le p\;|\,A\,|^2 + q\;|\,B\,|^2. \tag{3.22}$$

The proof of inequality (3.22) is given in [10] at page 579. O.H. presents an operator version of inequality (3.20) in the *Corollary 1* of this *Theorem 1* .

Corollary 1 ([10]) *Let* A , $B \in \mathcal{B}(\boldsymbol{H})$, $p\,,\, q > 1$ *with* $1/p + 1/q = 1$. *Then*

$$|\, A - B\,|^2 \le p\;|\,A\,|^2 + q\;|\,B\,|^2. \tag{3.23}$$

with equality if and only if $(1-p)\;A = B$.

The absolute value of an operator (which occurs in the previous two equations) has been defined in our earlier *Remark 3.6.*

We present sketches of the proofs of these two results, followed by a Proposition which brings several details. The proof of the inequality (3.22) starts by the expansions of the first term in its left side:

$$|\, A - B\,|^2 = (\,A - B\,)^*(\,A - B\,) = (\,A^* - B^*)\,(\,A - B\,) = \;\ldots$$

$$\ldots = |\,A\,|^2 - (\,A^*B + B^*A\,) + |\,B\,|^2\;. \tag{3.24}$$

A similar expansion for the second term in the left hand of (3.22) is

$$|\,(1-p)\,A - B\,|^2 = [(1-p)\,A - B\,]^*\,[(1-p)\,A - B\,] = \;\ldots$$

$$\ldots = (1-p)^2\,|\,A\,|^2 + (\,p-1)\,(A\,B^* + A^*\,B\,) + |\,B\,|^2. \tag{3.25}$$

Summing up − side by side − equations (3.24) & (3.25) , it is obtained

$$|\,A - B\,|^2 + |\,(1-p)\,A - B\,|^2 = \;\ldots$$

$$\ldots = [1+(1-p)^2]\,|\,A\,|^2 + 2\,|\,B\,|^2 - \,(2-p)\,(\,A^*B + B^*A\,) =$$

$$= (\,p^2 - 2\,p + 2)\,|\,A\,|^2 + 2\,|\,B\,|^2 + (\,p-2)\,(\,A^*B + B^*A\,)\,. \tag{3.26}$$

Next, the right side of (3.22), that is $p|\,A\,|^2{+}q|\,B\,|^2$, is subtracted from both (leftmost and rightmost) sides of (3.26), resulting in

$$(1-p)^2\,|\,A\,|^2 + |\,B\,|^2 - (\,2-p)\,(\,A^*B + B^*A\,) = \,\ldots = (\,p-2)\,(\,p-1)\,|\,A\,|^2 +$$

$$+[(\,p-2)\,/\,(\,p-1)]\;|\,B\,|^2 + (\,p-2)\,(\,A^*B + B^*A\,). \tag{3.27}$$

Obviously, $(\,p-2)$ can be taken out as a common factor from the rightmost side of (3.27), and it follows that

$$|\,A - B\,|^2 + |\,(1-p)\,A - B\,|^2 - (\,2-p)\,(\,A^*B + B^*A\,) =$$

$$= (\,p-2)\,\left[(\,p-1)\,|\,A\,|^2 + [1\,/\,(\,p-1)]\;|\,B\,|^2 + A^*B + B^*A\right]\,. \tag{3.28}$$

Other calculations lead to the conclusion that the right side of (3.28) is negative (non-positive), what implies the inequality (3.22) in the statement of this Theorem.

As regards the proof of *Corollary 1*, O. Hirzallah takes into account both possibilities for the conjugate exponents $p\,,\,q$. In fact, the author states that, if $p \le q$ then inequality (3.23) follows by *Theorem 1*. If the symmetric inequality $q \le p$ holds, then (3.28) becomes

$$|\,A - B\,|^2 + |\,(1-q)\,B - A\,|^2 \le p\;|\,A\,|^2 + q\;|\,B\,|^2. \tag{3.29}$$

In the proposition which follows, we add some details to O. Hirzallah's proofs of the previous *Theorem 1* and *Corollary 1*, a proof of (3.29), etc.

Proposition 3.3 (*Addenda to some earlier presented results and their proofs*)**.**

(i) Proof of the equality case in (3.23).

(ii) The operator version of the original H. Bohr's scalar inequality (3.21): If $a\,,b \in \mathbb{C}\,,\;\; c \in \mathbb{R}_+^*\;\;(c > 0\,)$ and $A\;,\;B \in \mathcal{B}(\boldsymbol{H})$ then

$$|\,A - B\,|^2 \le (1+c)\,|\,A\,|^2 + \left(1 + \frac{1}{c}\right)\;|\,B\,|^2. \tag{3.30}$$

(iii) Addenda to the proofs of inequalities (3.22), (3.23) and (3.29).

Proofs. (i) As we have just earlier mentioned, it is stated – in the *Corollary 1* of [10] – that the equality in (3.23) occurs if and only if $(1-p)\,A = B$.

Let us see that the implication $(3.22) \Longrightarrow (3.23)$ is immediate if $B = (1-p)\,A$ is taken inside the second term of the left side of (3.22). Next, going back to the scalar inequality (3.18), it is (more or less) easy to see that it turns into an equality if $w = (p-1)\,z$. Indeed, $|\,z + w\,|^2 \longrightarrow$

$$\longrightarrow |\,z + (p-1)\,z\,|^2 = \;\ldots\; = (\,z + (p-1)\,z)\,(\,\overline{z} + \;(p-1)\overline{z}\,) =$$

$$= |\,z\,|^2 + 2\,(p-1)\,|\,z\,|^2 + (p-1)^2\,|\,z\,|^2 = p^2\,|\,z\,|^2. \tag{3.31}$$

The right side $p\,|\,z\,|^2 + q\,|\,w\,|^2$ of (3.18) becomes
$p\;|\,z\,|^2 + q\;|(\,p-1)\,z|^2 = p\;|\,z\,|^2 + q\;\;(p-1)^2\,|\,z\,|^2 = [\,p + q\,(\,p^2 - 2\,p + 1)]\,|\,z\,|^2 =$

$$= [\;p + p\;q\;(p-2) + q]\,|\,z\,|^2 = p\;q\;(1 + p - 2)\,|\,z\,|^2 = p\;q\;(p-1)\,|\,z\,|^2. \tag{3.32}$$

The second equality on line (3.32) follows from the defining equation for the conjugate exponents p, q of (3.19) :

$$\frac{1}{p} + \frac{1}{q} = 1 \;\Longrightarrow\; \frac{p+q}{p\,q} = 1 \;\Longrightarrow\; p + q = p\,q\,. \tag{3.33}$$

Next, $(3.33) \Longrightarrow\; (p-1)\;q = p$ which implies that the leftmost side of (3.31) is just equal to the rightmost side of (3.32).

As regards the earlier presented *Theorem 1* and *Corollary 1* by O. Hirzallah [10], Eq. (3.29) would follow from (3.18) by taking operators instead of complex numbers, namely $w \to A\;\;\&\;\; z \to -\,B$. However, we are going to address this issue in part (iii) of our Proposition.

(ii) We have presented the operator version of the original Bohr inequality in (3.30. It still follows to see the connection between the positive parameter c and the conjugate exponents p, q. A simple comparison between (3.29) and (3.30) leads to

$$1 + c = p \quad \& \quad 1 + \frac{1}{c} = q\,. \tag{3.34}$$

Expressions $(3.34) \Longrightarrow p > 1,\; q > 1$ and $1\,/\,p + 1\,/\,q = 1\,/\,(1+c) + c\,/\,(c+1) = 1\,.$ Hence, the equivalence between (3.29) and (3.30) holds if we replace, in (3.21), $a \longrightarrow A$ and $b \longrightarrow -\,B$.

(iii) Some more details on the proof of O. Hirzallah's *Theorem 1* and *Corollary 1* of [10] follow. In the proof of (3.22), let us first see how the rightmost sides in (3.24) and (3.25) were obtained.

$|\,A - B\,|^2 + |\,(1-p)\,A - B\,|^2 =$ (using (3.24) & (3.25))
$= |\,A\,|^2 - (A^*B + B^*A) + |\,B\,|^2 + (1-p)^2\,|A|^2 - (1-p)\,(A^* + A^*B) + |B|^2 =$

$$= [1 + (1-p)^2]\,|\,A\,|^2 + 2\,|\,B\,|^2 - (2-\,p)\,(\,A^*B + B^*A\,) =$$

$$= (p^2 - 2\,p + 2)\,|\,A\,|^2 + 2\,|\,B\,|^2 + (\,p - 2\,)\,(\,A^*B + B^*A\,)\,. \qquad (3.35)$$

Subtracting $p\,|\,A\,|^2 + q\,|\,B\,|^2$ from both sides of (3.35) , it follows that

$$|\,A - B\,|^2 + |\,(1-p)\,A - B\,|^2 - (p\,|\,A\,|^2 + q\,|\,B\,|^2) =$$
$$= (p^2 - 2\,p + 2)\,|\,A\,|^2 + 2\,|\,B\,|^2 + (p-2)\,(\,A^*B + B^*A\,) - (p|\,A\,|^2 + q|\,B\,|^2) =$$

$$= (\,p^2 - 3\,p + 2)\,|\,A\,|^2 + (2-q)\,|\,B\,|^2 + (\,p - 2\,)\,(A^*B + B^*A\,), \qquad (3.36)$$

The first factor in the rightmost side of (3.36) can be written as $(\,p-2\,)\,(\,p-1)$ while

$$2 - q = 2 - p\,/(p-1) = (p-2)\,/\,(p-1),$$

as it follows from (3.33) $\Longrightarrow$ $(p-1)\ q = p$. Consequently, $(\,p - 2\,)$ can be taken as a common factor from (33.36), what leads to inequality (3.29). This factor is a negative real number since the assumption $p \le q$ in the statement of Theorem 1 plus (3.19) $\Longrightarrow$

$$\Longrightarrow\ 1 = \frac{1}{p} + \frac{1}{q} \le \frac{1}{p} + \frac{1}{p} = \frac{2}{p}\ \Longrightarrow\ \frac{2}{p} \ge 1\ \Longrightarrow p \le 2\,.$$

Finally, (3.28) with (3.27) would lead to the negativity of the leftmost side in (3.36) provided the sum of the three operators between [...] in the right side of (3.28) represents a positive operator. This is derived in [10] (but without a detailed proof) from

$$(\,p-1)\,|\,A\,|^2 + \left(\frac{1}{p-1}\right)|\,B\,|^2 + A^*B + B^*A = \left|\sqrt{p-1}\,A + \frac{1}{\sqrt{p-1}}\,B\right|^2. \quad (3.37)$$

The right side of (3.37) can be expanded as follows.

$$\left|\sqrt{p-1}\,A + \frac{1}{\sqrt{p-1}}\,B\right|^2 =$$
$$= \left[\sqrt{p-1}\,A^* + \frac{1}{\sqrt{p-1}}\,B^*\right]\ \left[\sqrt{p-1}\,A + \frac{1}{\sqrt{p-1}}\,B\right] =$$
$$= (\,p-1)\,|\,A\,|^2 + [1\,/\,(\,p-1\,)]\ |\,B\,|^2 + (\,A^*B + B^*A)\,.$$

The operator in the right side of (3.37) is (obviously) positive and, with its negative factor $(\,p-2)$ of (3.34), the inequality in (3.28) follows. As regards O. Hirzallah's proof for Corollary 1, in both alternatives on p, q, that is $p \leq q$ and then $q \leq p$, we do not bring any more details or comments on author's proof, but let us simply notice that (3.29) would follow from (3.28) if the operator $(1-p)^2 \,|\, A \,|^2 + |\, B \,|^2 \geq 0$. In the Remark which follows, we recall a somehow symmetric to (3.28) inequality, and we then continue with another approach to the study of Bohr-type inequalities with H-space operators, based on operator identities. □

Remark 3.8. Some additional results, other than O. Hirzallah's just presented Theorem 1 with its Corollary, were given in the article [4] by W.-S. Cheung and J. Pečarić. These authors recall Theorem 1 of [10] with its proof (earlier presented and completed), with the Remark 1 which states that their Theorem 1 – (i) is equivalent to (3.28) since $1 < p \leq 2 \Longleftrightarrow 1 < p \leq q$. Their assumptions on p, q are $1/p + 1/q = 1$ and $1 < p \leq 2$. In view of a consequence of our earlier equation (3.29), namely $q = p/(p-1) \geq p > 1$, it follows that $q > 1$ is also satisfied. But it is more significant that another inequality is stated and proved in Theorem 1 – (ii) of [4],

$$|\, A - B \,|^2 + |\, A - (1-q)\, B \,|^2 \geq p \,|\, A \,|^2 + q \,|\, B \,|^2. \tag{3.38}$$

Obviously, it is practically equivalent to inequality (3.29) of [10], but O. Hirzallah's hypothesis $q \leq p$ is no more assumed here. The proof is similar to that of (3.28). The expression of $|\, A - B \,|^2$ is just the same of (3.24). As regards the second term,

$$|\, A - (1-q)\, B \,|^2 = (1-q)\,|\, B \,|^2 + |\, A \,|^2 - (1-q)\,(A^*B + B^*A)\,. \tag{3.39}$$

Adding (again), side by side, equation (3.24) with (3.39) and subtracting $p \,|\, A|^2 + q\,|B|^2$ it is obtained

$$|\, A - B \,|^2 + |\, A - (1-q)\, B \,|^2 - p \,|\, A \,|^2 - q \,|\, B \,|^2 = \ldots$$

$$\ldots = (\, q^2 - 3\,q + 2)\,|\, B \,|^2 + (\, 2 - p\,)\,|\, A \,|^2 + (\, q - 2\,)\,(\, A^*B + B^*A\,) =$$

$$= (\, q - 2\,)\,(\, q - 1\,)\,|\, B \,|^2 + \frac{q-2}{q-1}\,|\, A \,|^2 + (\, q - 2\,)\,(\, A^*B + B^*A\,) =$$

$$= (\, q - 2\,) \left[(\, q - 1\,)\,|\, B \,|^2 + \frac{1}{q-1}\,|\, A \,|^2 + (\, A^*B + B^*A\,) \right].$$

Since $1 < p \leq 2 \implies q \geq 2$ we have

$$|\, A - B \,|^2 + |\, A - (1-q)\, B \,|^2 - p \,|\, A \,|^2 - q \,|\, B \,|^2 = (\, q - 2\,) \left| \sqrt{p-1}\, A + \frac{1}{\sqrt{p-1}}\, B \right|^2 \geq 0$$

and inequality (3.39) is thus proved. In the particular case when $p = 2 \Longrightarrow q = 2$, it follows from the previous inequalities that

$$|A-B|^2+|A+B|^2 \le 2\,|A|^2+2\,|B|^2 \le |A-B|^2+|A+B|^2 \implies$$

$$\implies |A-B|^2+|A+B|^2 = 2\,|A|^2+2\,|B|^2$$

which is the parallelogram law for H-space operators.

3.5 Bohr – type inequality via identities

In his article [18] of *J.M.A.A.* (2007), F. Zhang obtains some earlier presented Bohr type inequalities by use of operator identities and he also formulates certain extensions thereof, whose proofs involve 2-by-2 block matrices with operator entries. After quoting O. Hirzallah's Theorem 1 and Corollary 1, that is the inequalities (3.22) and (3.23), he states **Theorem 2**:

Let A, $B \in \mathcal{B}(\boldsymbol{H})$, $p, q > 1$, $1/p + 1/q = 1$. *Then*

$$|A-B|^2+\left|\sqrt{p/q}\,A+\sqrt{q/p}\,B\right|^2 = p\,|A|^2+q\,|B|^2. \tag{3.40}$$

Equivalently, for any α, $0 \le \alpha \le 1$,

$$|\alpha A+(1-\alpha)\,B|^2+\alpha\,(1-\alpha)\,|A-B|^2 = \alpha\,|A|^2+(1-\alpha)\,|B|^2. \tag{3.41}$$

The operator $|A-B|^{\,2}$ is expanded exactly like in (3.24), under the equivalent form

$$|A-B|^{\,2} = |A|^{\,2}+|B|^{\,2}-(A^*B+B^*A). \tag{3.42}$$

Similarly, the next term in the left side of (3.40) becomes

$$\left|\sqrt{p/q}\,A+\sqrt{q/p}\,B\right|^2 = (p/q)\,|A|^2+(q/p)\,|B|^2+(A^*B+B^*A). \tag{3.43}$$

Obviously, (3.43) + (3.44) $\implies$

$$|A-B|^2+\left|\sqrt{p/q}\,A+\sqrt{q/p}\,B\right|^2 = (1+p/q)\,|A|^2+(1+q/p)\,|B|^2. \tag{3.44}$$

This equation is equivalent to (3.40) since

$$1/p+1/q=1 \implies [1+p/q=p \quad \& \quad 1+q/p=q].$$

According to [21], the equivalence between (3.40) and (3.41) follows if the former inequality is divided by pq and $1/q =_{\text{not}} \alpha$. The coefficients in the right side of (3.41), after dividing it by pq, become $(p+q)/(pq)\,q = 1/q$ and (respectively) $(p+q)/(pq)\,p = 1/p = 1-1/q$. These two equalities follow from our previous equalities of (3.33). Hence, the right side of (3.41) actually becomes

$= \alpha \ |A|^2 + (1-\alpha\ )\ |B|^2$. The first term in the left side of (3.41) becomes $\alpha\ (1-\alpha\ )\ |A-B|^2$. As regards the second term in the left side of (3.41), its multiplication by $1/\ (p\,q) = (1/\sqrt{p\,q}\ )^2$ turns the coefficients inside $|\ \ldots\ |$ into

$$\sqrt{p\ /\ q}\ /\ \sqrt{p\,q} = 1\ /\ q = \alpha\ \&\ \sqrt{q\ /\ p}\ /\ \sqrt{p\,q} = 1\ /\ p = 1 - \alpha\ . \tag{3.45}$$

Hence, the identity (3.41) is equivalent to (3.40) .

Remark 3.9. The author F. Zhang asserts that the identity (3.41) gives immediately the inequality

$$|\,\alpha\ A + (1-\alpha)\ B\,|^{\,2} \le \alpha\ |\,A\,|^{\,2} + (1-\alpha)\ |\,B\,|^{\,2}, \tag{3.46}$$

that is, the *square-convexity inequality.* This inequality would follow by the positivity of $\alpha\ (1-\alpha)\ |\,A-B\,|^2$, since $|\,A-B\,|^2$ is positive and $\alpha\ (1-\alpha\ ) \in [0\ ,\ 1]$. Hence, if this term is omitted (removed) from the left side of (3.48) , the remaining term becomes less than the right side. For the equality case in (3.29) , the second term in the left side of (3.40) should vanish, that is

$$\sqrt{p\ /\ q}\ A + \sqrt{q\ /\ p}\ B\ = 0 \Longleftrightarrow (\,p\ /\ q\ )\ A + B = 0 \Longleftrightarrow B = (1-p)\ A\ ,$$

which is just the equality case in O. Hirzallah's *Corollary 1* of [10] .

As regards the proof that inequality (3.29) follows from the identity (3.40) , F. Zhang notices that it should be proved that

$$|\,(1-p)\,A - B\,|^{\,2} \le \left|\sqrt{p/q}\ A + \sqrt{q/p}\ B\ \right|^{\,2}$$

when $1 \le p \le 2$. Before continuing, let us see an equivalent form of this inequality ; it follows by the replacements $\sqrt{p/q} \to \sqrt{p-1}$ and $\sqrt{q\ /\,p} \to \sqrt{q-1}$:

$$|\,(1-p)\,A - B\,|^{\ 2} \le \left|\sqrt{p-1}\ A + \sqrt{q-1}\ B\ \right|^{2}. \tag{3.47}$$

But the author states and proves two Lemmas that lead to a more general inequality :

Lemma 1 ([18]) *Let* $A,\ B \in \mathcal{B}(\boldsymbol{H}\ )$. *If* $a,\ b > 0\ ,\ \ c \in \mathbb{R}$ *and* $a\ b \ge c^2$, *then*

$$a|\,A\,|^{\,2} + b|\,B\,|^{\,2} + c\,(\,A^*B + B^*A\,) \ge 0\ . \tag{3.48}$$

The proof of (3.48) makes use of 2-by-2 matrices with both scalar and operator entries. The latter operator matrices were considered in P. Halmos's monograph

[6] as well in other sources. A 2-by-2 operator matrix acts on the direct sum of a Hilbert space by itself,

$$\boldsymbol{H} \oplus \boldsymbol{H} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix} : x \, , \; y \in \boldsymbol{H} \right\} \tag{3.49}$$

Such a direct sum is a Hilbert space as well. An operator acting on it, involving four H-space operators $A\,,\,B\,,\,C\,,\,D$ looks like and it is defined by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A\,x + B\,y \\ C\,x + Dy \end{bmatrix}.$$

The 2-by-2 matrix with real entries in the statement of this Lemma is positive, that is

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} \geq 0 \quad \text{since} \quad a\,b \geq c^2. \tag{3.50}$$

Another positive matrix (with operator entries) is

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} |\,A\,|^2 & A^*B \\ B^*A & |\,B\,|^2 \end{bmatrix} \geq 0 \Longrightarrow \begin{bmatrix} a\,|\,A\,|^2 & c\,A^*B \\ c\,B^*A & b\,|\,B\,|^2 \end{bmatrix} \geq 0\,. \tag{3.51}$$

Thus,

$$\begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} a\,|\,A\,|^2 & c\,A^*B \\ c\,B^*A & b\,|\,B\,|^2 \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} = a\,|\,A\,|^2 + b\,|\,B\,|^2 + c\,(\,A^*B + B^*A\,) \geq 0\,. \tag{3.52}$$

Before recalling another lemma of [21], let us notice that the positivity of the 2-by-2 matrix (over $\mathbb{R}$) means that it is positive definite. The notations in the subsequent equations slightly differ from those of [21] . For instance, F. Zhang's notation $(\,I\,,\,I\,)$ looks like the one specific to an ordered pair, but it denotes (in fact) the row vector $[\,I \quad I\,]$, with $I =$ the identity operator (also denoted as **1**). As regards the first operation which occurs in (3.52), it is not an actual (matrix) product but a conventional way to obtain a 2-by-2 matrix from the "product" of a 2-by-1 matrix (column vector) with a 1-by-2 matrix (row vector). We used a similar way for defining the matrix of a BLF $-$ a bilinear form f in a basis $A = [\,a_1 \;\; \ldots \;\; a_i \;\; \ldots \;\; a_j \;\; \ldots \;\; a_n\,]$ spanning an $n-$ dimensional vector space V (in our textbook [3]): in such a basis, written as a row of vectors, the matrix is defined as $F_A = f\,(A^{\mathrm{T}},\,A) = [\,f\,(\,a_i\,,\,a_j\,)]_{n\,\times\,n}$. The next **Lemma 2** of [18] states that :

If $A,\,B \in \mathcal{B}(\boldsymbol{H})$ *and* $x,\,y,\,s,\,t \in \mathbb{R}$ *such that* $|\,x\,| \;\leq |\,s\,|, |\,y\,| \;\leq |\,t\,|$ & $x\,t = s\,y$ *then*

$$|\,x\,A + y\,B\,|^2 \leq |\,s\,A + t\,B\,|^2. \tag{3.53}$$

This inequality involving two operators and four real parameters follows by expanding the two sides of (3.53), moving the terms from the left to the right side and then applying *Lemma 1* with

$$a = s^2 - x^2, \quad b = t^2 - y^2 \quad \text{and} \quad c = s\,t - x\,y\,. \tag{3.54}$$

We have earlier seen that $1 \le p \le q \implies 1 \le p \le 2$. Taking $x = p-1$, $y = 1$, $s = \sqrt{p\,/\,q}$, $t = \sqrt{q\,/\,p}$ in (3.53), the implication $|\,(1-p)\,A - B\,|^2 \le \left|\sqrt{p\,/\,q}\;A + \sqrt{q\,/\,p}\;B\,\right|^2 \implies$ (3.22) immediately follows.

The author also remarks that his *Theorem 2*, that is the identity (3.40), can generate a whole variety of inequalities similar to O. Hirzallah's *Theorem 1*, that is (3.22). Indeed, putting $s = \sqrt{p/q}$, $t = \sqrt{q/p}$, for any real numbers x & y satisfying $x\,q = y\,p$ and $x^2 \le p\,/\,q = p-1$ in (3.53), it follows that

$$|\,A - B\,|^{\,2} + |\;x\;A + y\;B\;|^{\,2} \le p\;|\,A\,|^{\,2} + q\;|\,B\,|^{\,2}. \tag{3.55}$$

Remark 3.10. (i) The proof of Lemma 2, that is of inequality (3.53), follows from the expansion of the two sides in (3.53) and from

$$|\,x\,| \;\le |\,s\,|\,, \quad |\,y\,| \;\le |\,t\,| \quad \& \quad x\,t = s\;y \implies x^2 \le s^2\,, \quad y^2 \le t^2 \implies$$

$$\implies \begin{cases} a\;b = \;\;\ldots\;\; = s^2\,t^2 + x^2\,t^2 - (\,s^2\,y^2 + t^2\,x^2), \\ \qquad\quad c^2 = s^2\,t^2 + x^2\,t^2 - 2\,s\,t\,x\,y \end{cases} \implies \ldots \implies a\;b \ge c^2. \tag{3.56}$$

It would follow that inequality (3.53) holds. However, the implications in the above formulas (3.56) are not quite immediate. We are going to give more details in our Proposition which follows.

(ii) Another extension of the Bohr inequality, which is implied by (3.55), holds for $1 \le p \le q$, $x = (\,p-1\,)^k$ and $y = (\,p-1\,)^{k-1}$, where k is any positive integer :

$$|\,A - B\,|^2 + |\;(\,p-1)^k\;A + (\,p-1)^{k-1}\;B\;|^2 \le p\;|\,A\,|^2 + q\;|\,B\,|^2. \tag{3.57}$$

The implication from (3.55) to (3.57) is true since

$$x = (\,p-1)^k = y\;(\,p-1) = (\,p\,/\,q\,)\,y \implies x\,q = y\,p\;, \tag{3.58}$$

and, with the notations in *Lemma 2* and *Theorem 2* of [18], also taking into account that $1 \le p \le q \implies 1 \le p \le 2 \implies p-1 \le 1$,

$$\begin{cases} \quad s = \sqrt{p\,/\,q} \implies s^2 = p\,/\,q\;, \\ x = (\,p-1)^k \implies x^2 = (\,p-1)^{2\,k} \end{cases} \implies$$

$$\Longrightarrow\ x^2 = (p-1)^{2k-1}(p-1) \le p-1 = p\,/\,q = s^2.$$

Similarly, $y^2 = (p-1)^{2k-2}$ and $t^2 = q\,/\,p \Longrightarrow$

$$\Longrightarrow y^2 = (p-1)^{2k}\,/\,(p-1)^2 = (p-1)^{2\,k}\,(q\,/p) \le t^2.$$

Therefore, the hypotheses in *Lemma 2* are satisfied, inequalities (3.55) and (3.56) hold, and they imply (3.57) . It is obvious that inequality (3.57) reduces to (3.28) when $k = 1$.

Proposition 3.4. *(Explicit proof of inequality* $a\,b \ge c^2$ *in* (3.56)*).*

Proof. It follows from the statement of F. Zhang's *Lemma 2*, namely from

$$|\,x\,| \le |\,s\,|\,,\quad |\,y\,| \le |\,t\,| \quad \& \quad x\,t = s\,y\,, \tag{3.59}$$

that $|\,x\ t\,| = |\,s\ y|\ \Longrightarrow\ |\,x\,|\,|\,t\,| = |\,s\,|\,|y|$. But the two inequalities of (3.59) imply

$$|\,x\,|\,|\,y\,| \le |\,x\,|\,|\,t\,| \le |\,s\,|\,|\,t\,| \Longrightarrow |\,x\,|^2\,|\,y\,|^2 \le |\,s\,|^2\,|\,t\,|^2 \Longrightarrow x^2\,y^2 \le s^2\,t^2. \tag{3.60}$$

Next, the expressions of a & b in (3.54) lead to

$$a\ b = (s^2 - x^2)\ (t^2 - y^2) = s^2\,t^2 + x^2\ y^2 - (x^2\,t^2 + s^2\ y^2). \tag{3.61}$$

Expression (3.61) plus the equation in (3.59) lead to

$$a\ b = s^2\,t^2 + x^2\ y^2 - 2\ t^2\,x^2 = s^2\,t^2 + x^2\ y^2 - 2\ s^2\,y^2. \tag{3.62}$$

On another hand, $c = s\,t - x\ y \Longrightarrow c^2 = (s\,t - x\ y\,)^2 = s^2t^2 + x^2\ y^2 - 2\ s\,t\,x\ y =$

$$= s^2\,t^2 + x^2\,y^2 - 2\ (x\,t)(s\ y) = s^2\,t^2 + x^2\,y^2 - 2\ s^2t^2. \tag{3.63}$$

Now, if the equation (3.63) is side-by-side subtracted from (3.61), that is leftmost / rightmost sides, it follows that

$$a\ b - c^2 = \ \ldots\ = -\,2\ s^2\,y^2 + 2\ s^2\,t^2 = 2\ s^2(t^2 - y^2) \underset{(3.59)}{\ge} 0$$

and this proves that inequality $a\ b \ge c^2$ holds. □

4 Concluding remarks

This paper offers a survey of some inequalities defined on Hilbert spaces. The literature dedicated to this area of research is quite rich. We found more than 40 papers and monographs approaching this subject, but it was not possible to make reference to more than 21 of them. Several classical inequalities like those due to Harald Bohr (of 1924), F. Rellich (1950),T. Kato (1952) attracted the interest of many authors who established equivalent expressions and (especially) extensions/generalizations of such inequalities. Some authors deserve mention in this respect: F. Hansen, F. Kittaneh, O. Hirzallah, J. Pečarić, F. Zhang are only a few of them. In **Section 2** we recalled a series of basic definitions and notations on Hilbert space operators, classification of some H-space operators, etc. The subsequent **Section 3** with its subsections present some ways to obtain operator inequalities by specific methods/ways. We approached several inequalities obtained from inequalities on the algebra of operators and on the field of (complex) scalars $\mathbb{C}$. In subsection **3.1** we dealt with operator inequalities induced by operator positivity and we checked that $A \geq B$ is an actual partial order relation. We showed, in subsection **3.2**, that T.Kato's relation $S \ll T$ is a partial order as well. Inequalities of Cauchy-Schwarz type are presented in subsection **3.3**. The next two subsections **3.4**, **3.5** were dedicated to Harald Bohr's inequality and its many equivalent expressions and generalizations. In **3.5** we recalled F. Zhang's method to obtain Bohr-type inequalities from certain identities. In this *Section 3* we gave some completions to a series of proofs, we tried to find connections between certain inequalities, in ten Remarks and four Propositions. It may be considered, as our *main results*, the details added to O. Hirzallah's proofs of his Theorem 1 and Corollary 1 of [10], mainly in the three parts of our Proposition 3.3. Other completions were given in *Remark 3.10*, on F. Zhang's formula (3.53) and its connections with earlier considered Bohr-type inequalities, we also proved the inequality $ab > c^2$ in the last Proposition 3.4.

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A description of collineations-groups of an affine plane

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Abstract: Based on the following very interesting work in the past [2], [3], [4], [9], [12], this article becomes a description of collineations in the affine plane [10]. We are focusing at the description of translations and dilatations, and we make a detailed description of them. We describe the translation group and dilatation group in affine plane [11]. A detailed description we have given also for traces of a dilatation. We have proved that translation group is a normal subgroup of the group of dilatations, wherein the translation group is a commutative group and the dilatation group is just a group. We think that in this article have brings about an innovation in the treatment of detailed algebraic structures in affine plane.

Keywords: Affine plane, collineations, translation, dilatation, trace of points, dilatation group.

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1 The collineation group

Definition 1.1. Let $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be an affine plane and $\mathcal{S}=\{\psi : \mathcal{P} \to \mathcal{P}|$ where $\psi-$is bijection$\}$ set of bijections to set points $\mathcal{P}$ on yourself. Collineation of affine plane A called a bijection $\psi \in \mathcal{S}$, such that

$$\forall \ell \in \mathcal{L}, \psi(\ell) \in \mathcal{L}, \tag{1.1}$$

Otherwise, a collineation of the affine plane $\mathcal{A}$ is a bijection of set $\mathcal{P}$ on yourself [14], that preserves lines. It is known that the set of bijections to a set over itself is a group on associated with the binary action " $\circ$ " of composition in it, which is known as total group or symmetric groups [1], [6], [7], [8], [13].

In a collineation ψ of an affine plans, image $\psi(P)$ to a point P to plans often mark briefly P'.

Proposition 1.2. *Every bijections of set to points $\mathcal{P}$ on yourself to affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a his collineation.*

Proof. Let it be a bijection $\psi : \mathcal{P} \to \mathcal{P}$ and an line $\ell \in \mathcal{L}$. From [11],[12] , the line ℓ is incident with two different points $P, Q \in \mathcal{P}$, which the bijection ψ leads in two different points $P' = \psi(P)$ and $Q' = \psi(Q)$. From axiom A1 of the affine plane [10], [11] these points define a single line $P'Q' \in \mathcal{L}$, namely, $P'Q' = \psi(PQ) = \psi(\ell) \in \mathcal{L}$. □

Corollary 1.3. *Collineation set* $\mathbf{Col}_{\mathcal{A}}$ *of the affine plane* $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ *forms symmetrical groups about the composition* "$\circ$", *namely* $(\mathbf{Col}_{\mathcal{A}}, \circ)$ *is symmetric group.*

It is clear that identical bijections $id_{\mathcal{P}} : \mathcal{P} \to \mathcal{P}$ is a collineation of the affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, we call identical collineations of $\mathcal{A}$. In this collineation, every point of $\mathcal{P}$ passes itself, as well as every line $\mathcal{L}$ passes itself.

Definition 1.4. An point P of the affine plan $\mathcal{A}$ called fixed point his associated with a collineation δ, if coincides with the image itself $\delta(P)$, briefly when

$$P = \delta(P).$$

According to this definition, we have this

Proposition 1.5. *Every point of the affine plane, is a fixed point related to his identical collineation.*

2 The dilatation group

Definition 2.1. [3], [4] Dilatation of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ called a its collineation δ such that

$$\forall P \neq Q \in \mathcal{P}, \delta(PQ) \| PQ \tag{2.1}$$

According to axiom A1 of the affine plane definitions [10], [11] , [12]. , the line that passes through two different points P, Q we have written PQ. From the fact that dilatations δ is bijection, worth implication

$$P \neq Q \Leftrightarrow \delta(P) \neq \delta(Q) \tag{2.2}$$

therefore line $\delta(PQ)$ also written $\delta(P)\delta(Q)$ and definition 2.1, in these circumstances, takes the view

$$\forall P \neq Q \in \mathcal{P}, \delta(P)\delta(Q) \| PQ \tag{2.3}$$

It is clear that identical collineations $id_{\mathcal{P}}$ of an affine plan $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, is an dilatation of his, who called his identical dilation.

However, by being bijection an dilatation δ of an affine plane, the his inverse δ^{-1} is also bijection. It's a dilatation his affine plan the bijections δ^{-1}? By (2.3), easily shown that is true this

Proposition 2.2. *Inverse bijections δ^{-1} of a dilatations δ of an affine plan is also an dilatation of that plan.*

Proposition 2.3. *Composition of two dilatations of a affine plan is agin an dilatations of his.*

Proof. Let's be δ_1 and δ_2 two dilatations of the affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$. For any two points of the plan, according to (2.2) have:

$$P \neq Q \Longleftrightarrow \delta_1(P) \neq \delta_1(Q)$$

and

$$\delta_1(P) \neq \delta_1(Q) \Longleftrightarrow \delta_2(\delta_1(P)) \neq \delta_2(\delta_1(Q))$$

From the first equivalence according to (2.3), we have

$$\delta_1(P)\,\delta_1(Q) \,\|PQ$$

and from the second we have

$$\delta_2(\delta_1(P))\,\delta_2(\delta_1(Q)) \,\|\delta_1(P)\,\delta_1(Q)$$

the parallelism relation in $\mathcal{A}$ is a equivalence relation [4], [10], [11] therefore by its transition properties take:

$$\delta_2(\delta_1(P))\,\delta_2(\delta_1(Q)) \,\|PQ$$

otherwise,

$$(\delta_2 \circ \delta_1)(P)\,(\delta_2 \circ \delta_1)(Q) \,\|PQ$$

In conclusion,

$$\forall P \neq Q \in \mathcal{P}, (\delta_2 \circ \delta_1)(P)\,(\delta_2 \circ \delta_1)(Q) \,\|PQ$$

$\forall P \neq Q \in \mathcal{P}, (\delta_2 \circ \delta_1)(P)\,(\delta_2 \circ \delta_1)(Q) \,\|PQ$

which according to (2.3), indicates that the composition $\delta_2 \circ \delta_1$ of dilatations δ_2 and δ_1 of affine plane $\mathcal{A}$ is also its dilation. □

Let it be $\mathbf{Dil}_{\mathcal{A}} = \{\delta \in \mathbf{Col}_{\mathcal{A}} | \delta -$ is a dilatation of $\mathcal{A}\}$ the dilatation set of affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$. As such it is the subset of collineations $\mathbf{Col}_{\mathcal{A}}$. Propositions 2.3, indicates that $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ is a sub-structure of the symmetric group $(\mathbf{Col}_{\mathcal{A}}, \circ)$ of collineations of the affine plane $\mathcal{A}$. Propositions 2.2, indicates that this sub-structure is a sub-group of the group $(\mathbf{Col}_{\mathcal{A}}, \circ)$, [1], [6], [7], [8], [13]. Is obtained in that way this

Theorem 2.4. *The dilatation set* $\mathbf{Dil}_{\mathcal{A}}$ *of affine plane* $\mathcal{A}$ *forms a group with respect to composition* $\circ$.

Definition 2.5. Let it be δ an dilatation of affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, P his one point. Lines that passes by P and $\delta(P)$, called trace of points P regarding dilatations δ.

If $P \neq \delta(P)$, then according to axiom A1, trace $P\delta(P)$ is the only. And when $P = \delta(P)$, trace the point P regarding δ will mark $P\delta(P)$. In this case the trace $P\delta(P)$ is not only after, according to a proposition in the affine plane [5], [10], each point is incident with at least three lines, which have from this, the point P has at least three traces associated with dilatation δ.

Theorem 2.6. *For an point* P *to an affine plane* $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, *not fixed related to an his dilatation* δ , *is true propositions*

$$\forall Q \in \mathcal{P} - \{P\}, Q \in P\delta(P) \Longrightarrow \delta(Q) \in P\delta(P).$$

Otherwise every point of a traces of a not-fixed point, to an affine plane associated with its dilation has its own image associated with that dilation in the same traces.

Proof. Given that $Q \neq P$ and δ is a dilatation of plane $\mathcal{A}$, then according to (2.2), $PQ \parallel \delta(P)\delta(Q)$. From condition, $Q \in P\delta(P)$, that implies $Q, P, \delta(P) \in PQ$. By parallelism of lines PQ, $\delta(P)\delta(Q)$ and by the fact that points $\delta(P)$ is the common point of their, results:

$$PQ = \delta(P)\delta(Q)$$

that implies

$$\delta(Q) \in PQ = \delta(P)\delta(Q)$$

□

Now formulate a constructive character theorem of an dilatations for affine plane. Mark with ℓ_m^P the line who fulfills the conditions $P \in \ell_m^P$ and $\ell_m^P \parallel m$.

Theorem 2.7. *[3] If two different assigned points P, Q of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, are defined their image $P' = \delta(P)$ and $Q' = \delta(Q)$ by an his dilatations $\delta \neq id_{\mathcal{A}}$, then image $R' = \delta(R)$ of an other points $R \in \mathcal{P} - \{P, Q\}$ determined as follows:*

$$R \notin PQ \Longrightarrow \delta(R) = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'}$$
$$R \in PQ \Longrightarrow \exists S \in \mathcal{P}, S \notin PQ, \delta(R) = \ell_{RP}^{P'} \cap \ell_{RS}^{S'}$$

Proof. **I)** Examine the first case when the point R is not line incidents PQ: $(R \notin PQ)$. Distinguish three sub-cases.

Case 1: $R \notin PP'$ and $R \notin QQ'$. In this case we have $P \neq P'$ and $Q \neq Q'$, after that, if the accept for example $Q = Q'$, then $RQ = RQ'$ and by (2.2) we have $R'Q' || RQ$. Hence:

$$R'Q' || RQ' \overset{A.1}{\Longrightarrow} R'Q' = RQ' = RQ \Longrightarrow R \in R'Q'.$$

contrary to condition.

Constructing in P' the line $\ell_{RP}^{P'}$, that is parallel to RP. From (2.2), $R'P' || RP$. Hence, the axiom A2 of affine plane we have that $\ell_{RP}^{P'} = R'P'$. Consequently $R' \in \ell_{RP}^{P'}$. Constructing now Q' the line $\ell_{RQ}^{Q'}$, that is parallel with RQ.

Same as above comes out that $R' \in \ell_{RQ}^{Q'}$. Well,

$$R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'}$$

Results in this manner the true proposition

$$\forall R \in \mathcal{P} - \{P, Q\}, R \notin PQ \Longrightarrow \delta(R) = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'} \tag{2.4}$$

indicating that $R' = \delta(R)$, defined as cutting points of the lines:

$$\ell_{RP}^{P'} = \delta(RP) \quad and \; \ell_{RQ}^{Q'} = \delta(RQ).$$

Case 2: $R \in PP'$ or $R \in QQ'$, concrete terms with the first. From (2.2), $R'P' || RP$, implicates that $R' \in RP = \ell_{RP}^{P'}$. As in the case 1, shows that $R' \in \ell_{RQ}^{Q'}$. Consequently we have $R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'}$.

Case 3: R is incident with both tracks PP', QQ'. Since $R \notin PQ$, then PP', QQ' are different, therefore, according to A1, they meet at point $R : R = PP' \cap QQ'$. Just like the case 2 proved that

$$R \in PP' \Longrightarrow R' \in RP = \ell_{RP}^{P'}$$

and

$$R \in QQ' \Longrightarrow R' \in RQ = \ell_{RQ}^{Q'}$$

hence

$$R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'} = RP \cap RQ = R. \tag{2.5}$$

II) Now examine another case, when the point R is incident with the line PQ: $R \in PQ$. According to axiom A3, in affine plane has a point S, non-incident with PQ. Is clear that $R \notin PS$ (otherwise PS and PQ they will be coincide according to Axiom A1, because they will pass by two different points P and R, which would implicate the wrong conclusion that $S \in PQ$).

Since $S \notin PQ$, according to (2.4), constructed $S' = \ell_{SP}^{P'} \cap \ell_{SQ}^{Q'}$, $\delta(S) = S'$. But also $R \notin PS$, hence according to (2.4) next constructed the $R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'}$. In summary the construction of point $R' = \delta(R)$ in these conditions is presented in the form

$$\forall R \in \mathcal{P} - \{P, Q\}, R \in PQ \Longrightarrow \exists S \notin PQ, S' = \ell_{SP}^{P'} \cap \ell_{SQ}^{Q'} \text{ and } R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'} \tag{2.6}$$

□

Is clear that when the tracks $P\delta(P), Q\delta(Q)$ the two points P, Q the affine plane are cutting, then definitely $P \neq Q$. Therefore, from (2.5) and this proves true

Corollary 2.8. *If the tracks $P\delta(P), Q\delta(Q)$ the two points P, Q of an affine plane are expected, then their cutting points $P\delta(P) \cap Q\delta(Q)$ is a fixed point related to his dilatations δ.*

Corollary 2.9. *If an point Q of affine plans is to trace $P\delta(P)$ to an his point P, then the its image $\delta(Q)$ locates at the a trace.*

Otherwise, in an affine plan, line, which is an a trace of his points by an dilatation, is a trace for every other points of it

Proof. We distinguish two cases.

Case 1: The point Q is fixed-point in connection with dilatation δ. In this case we have to:

$$\delta(Q) = Q$$

Since from corollary condition, we have to point $Q \in P\delta(P)$, we have to

$$\delta(Q) \in P\delta(P)$$

Case 2: The point Q is not fixed-point in connection with dilatation δ. The corollary of proof we have from Theorem 2.6.

□

Corollary 2.10. *Two dilatations $\delta_1 \neq id_{\mathcal{P}}$ and $\delta_2 \neq id_{\mathcal{P}}$ of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, are equal if and only if, when two points $P \neq Q \in \mathcal{P}$ are simultaneously true equations*

$$\delta_1(P) = \delta_2(P) \text{ and } \delta_1(Q) = \delta_2(Q) \tag{2.7}$$

Otherwise, an dilatation $\delta \neq id_{\mathcal{P}}$ of an affine plane is completely determined by giving his image according to two different points of the plans.

Proof. For two dilatations δ_1 and δ_2, valid the implication

$$\delta_1 = \delta_2 \Longleftrightarrow \forall R \in \mathcal{P}, \delta_1(R) = \delta_2(R) \tag{2.8}$$

after being reflections, for two reflections $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$, valid the implication [8]

$$f = g \Leftrightarrow \forall x \in X, f(x) = g(x). \tag{2.9}$$

From here, when dilations $\delta_1 \neq id_{\mathcal{P}}$ and $\delta_2 \neq id_{\mathcal{P}}$ are equally, particularly for points $P \neq Q \in \mathcal{P}$, we have

$$\delta_1(P) = \delta_2(P) \textit{ and } \delta_1(Q) = \delta_2(Q).$$

Conversely, let's have $\delta_1(P) = \delta_2(P)$ *and* $\delta_1(Q) = \delta_2(Q)$, and prove to $\delta_1 = \delta_2$. From the condition, equation is tru for $R = P$ and for $R = Q$. On a different point, ie a point $R \in \mathcal{P} - \{P, Q\}$, in case when $R \notin PQ$, according to (2.4) and (2.6), we have

$$\delta_1(R) = \ell_{RP}^{\delta_1(P)} \cap \ell_{RQ}^{\delta_1(Q)} = \ell_{RP}^{\delta_2(P)} \cap \ell_{RQ}^{\delta_2(Q)} = \delta_2(R)$$

but also in case when $R \in PQ$, according to (2.5), (2.6) and to above, we have

$$\exists S \notin PQ, \delta_1(S) = \ell_{SP}^{\delta_1(P)} \cap \ell_{SQ}^{\delta_1(Q)} = \ell_{SP}^{\delta_2(P)} \cap \ell_{SQ}^{\delta_2(Q)} = \delta_2(S)$$

Well, $\forall R \in P, \delta_1(R) = \delta_2(R)$, that according to (2.7), shows that $\delta_1 = \delta_2$. □

Theorem 2.11. *For every dilatation $\delta \neq id_{\mathcal{P}}$ of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, exists in the plane least two not fixed points about what dilatation.*

Proof. The fact that dilatation $\delta \neq id_{\mathcal{P}}$ imply the existence of at least one point P in plane $\mathcal{A}$ that is not the fixed-point connected to δ, namely $\delta(P) \neq P$. Of course exist also another point $Q \in \mathcal{P}$, such that $\delta(Q) \neq Q$.

On the contrary, if $\forall Q \in \mathcal{P} - \{P\}$ we would have $\delta(Q) = Q$, then

$$\delta(PQ) \parallel PQ \Longrightarrow \delta(P) = P,$$

in contradiction the fact that the point P is not the fixed-point. □

Theorem 2.12. *If an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, has two fixed points about an dilatation then he dilatation is identical dilatation $id_{\mathcal{P}}$ of his.*

Proof. Let's be P, Q two fixed points, in relation to an dilatation δ and R another point of affine plane $\mathcal{A}$. If $R \notin PQ$, according to (2.4), the image of her R' is

$$R' = \ell_{RP}^{P'} \cap \ell_{RQ}^{Q'} = \ell_{RP}^{P} \cap \ell_{RQ}^{Q},$$

because P, Q are the fixed points, in relation to an dilatation δ. But $\ell_{RP}^{P} \cap \ell_{RQ}^{Q} = R$, that imply $R' = R$. If the point $R \in PQ$, from axioms A2, exist a point S such that $S \notin PQ$. According to (2.4), image of her S' is

$$S' = \ell_{SP}^{P'} \cap \ell_{SQ}^{Q'} = \ell_{SP}^{P} \cap \ell_{SQ}^{Q} = S,$$

that imply $S' = S$. Then, according to (2.5),

$$R' = \ell_{RP}^{P} \cap \ell_{RS}^{S} = R.$$

□

According to this theorem, if related to an dilatation $\delta \neq id_{\mathcal{P}}$ of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, plan has a fixed point, he can not have any other fixed point, because otherwise, it would be identical dilatation $id_{\mathcal{P}}$. So we have this

Corollary 2.13. *For every dilatation $\delta \neq id_{\mathcal{P}}$ to an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ if in the plan has an fixed point with respect to that dilatation, then it is only.*

Theorem 2.14. *For every dilatation $\delta \neq id_{\mathcal{P}}$ to an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, which has a fixed point V associated with it dilatation, is true propositions*

$$\forall P \in \mathcal{P}, V \in P\delta(P),$$

otherwise, all the tracks regarding with dilatation δ crossed in the point V.

Proof. The proposition (1.5) is evident for $P = V$. For $P \neq V$ have $P \neq \delta(P)$, because, according to corollary, the fixed-point V associated with that dilatation δ is the only in the plane $\mathcal{A}$. Whereas, the fact that δ is a dilatation, imply $VP \parallel V\delta(P)$, because $V = \delta(V)$. The lines VP and $V\delta(P)$ have in common point V, therefore $VP = V\delta(P)$, that imply $V \in P\delta(P)$. □

Theorem 2.15. *An affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, has not fix point related to an dilatation $\delta \neq id_{\mathcal{P}}$ then and only then, when all the tracks $P\delta(P)$ for all $P \in \mathcal{P}$ are parallel between themselves.*

Proof. If the tracks by dilation δ in plane $\mathcal{A}$, are parallel between them, then he plane has not fixed point by dilatation δ, when on the contrary, by corollary of Theorem 2.12, those would not be parallel.

Conversely, we accept that the plan $\mathcal{A}$ has not fixed point by dilatation $\delta \neq id_{\mathcal{P}}$ and prove that all the tracks $P\delta(P)$ for all $P \in \mathcal{P}$ are parallel between them. Let's be P, Q two random points of the affine plane $\mathcal{A}$. The case where $P = Q$ is evident. And when $P \neq Q$, again their tracks are parallel, because if we accept that $P\delta(P) \nparallel Q\delta(Q)$, then their cross cutting, according to corollary 2.8, of the Theorem 2.7, will be the fixed point of $\mathcal{A}$, associated with dilations δ, that is in contradiction with the condition. Further, the parallelism between all traces derived from the fact that the parallel lines of a affine plane, is equivalence relation. □

This Theorem we can give also this wording:

For every dilatation $\delta \neq id_{\mathcal{P}}$ of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, worth propositions

$$\text{"}\mathcal{A} \text{ has not fixed point by } \delta\text{"} \Longleftrightarrow \forall P, Q \in \mathcal{P}, P\delta(P) \parallel Q\delta(Q) \tag{2.10}$$

The last two theorems summarized in this

Proposition 2.16. *In an affine plane related to dilatation $\delta \neq id_{\mathcal{P}}$ all traces $P\delta(P)$ for all $P \in \mathcal{P}$, or cross the by a single point, or are parallel between themselves.*

3 The traslations groups

By Propositions 2.16, in an affine plane all traces related an dilatation of his or cross the by a single point, or are parallel between themselves. This fact leads us to this

Definition 3.1. [3],[4],[9],[12], Translation of an affine plans $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, called identical dilatation $id_{\mathcal{P}}$ his and every other of its dilatation, about which heaffine plane has not fixed points.

If σ is an translation different from identical translation $id_{\mathcal{P}}$, then, by Theorems 2.15, all traces related to σ form the a set of parallel lines. According to a proposition in the affine plane [4], [10], which have from this, at every point $P \in \mathcal{P}$ pass at least three lines out $\mathcal{L}$, among which only one is its a trace of translations σ. Because $\parallel$ parallelism relation on $\mathcal{L}$, is an equivalence relation, see [10], then $\pi = \mathcal{L}/\parallel$ is an a cleavage of $\mathcal{L}$ in the equivalence classes by parallelism, see [10], [11], [12]. Each class has representative an line that passes from of random point P.

Definition 3.2. For one translation $\sigma \neq id_{\mathcal{P}}$, equivalence classes of the cleavage $\pi = \mathcal{L}/\|$, which contained tracks by σ of points of the plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ called the direction of his translations σ and marked π_{σ}.

So, for $\sigma \neq id_{\mathcal{P}}$, the direction π_{σ} represented by single the trace by σ every point $P \in \mathcal{P}$, for translation $id_{\mathcal{P}}$. We say that there are undefined direction. Otherwise we say that has the same the direction with every other translations σ of the plane $\mathcal{A}$, namely true accept the propositions:

$$\text{For every translation } \sigma \text{ of the plane } \mathcal{A}, \pi_{id_{\mathcal{P}}} = \pi_{\sigma} \tag{3.1}$$

Subject of review at this point would be the set of translations of the affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$:

$$\mathbf{Tr}_{\mathcal{A}} = \{\sigma \in \mathbf{Dil}_{\mathcal{A}} \mid \sigma \textit{ is translation of } \mathcal{A}\}.$$

Let it be $\alpha : \mathbf{Tr}_{\mathcal{A}} \longrightarrow \mathbf{Tr}_{\mathcal{A}}$, an whatever application of $\mathbf{Tr}_{\mathcal{A}}$, on yourself. For every translation σ, its image $\alpha(\sigma)$ is again an translation, that can be $\alpha(\sigma) = id_{\mathcal{P}}$ or $\alpha(\sigma) \neq id_{\mathcal{P}}$. So there is a certain direction or indefinitely. The first equation, in the case where $\sigma = id_{\mathcal{P}}$, takes the view $\alpha(id_{\mathcal{P}}) = id_{\mathcal{P}}$, and the second $\alpha(\sigma) \neq id_{\mathcal{P}}$, that it is not possible to α is application. To avoid this, yet accept that for every application $\alpha : \mathbf{Tr}_{\mathcal{A}} \longrightarrow \mathbf{Tr}_{\mathcal{A}}$, is true equalization:

$$\alpha(id_{\mathcal{P}}) = id_{\mathcal{P}}. \tag{3.2}$$

Theorem 3.3. *If a certain point P of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$. Its image is determined $P' = \sigma(P)$ according to an his translations $\sigma \neq id_{\mathcal{P}}$, then image $Q' = \sigma(Q)$ a other point $Q \in \mathcal{P} - \{P\}$ determined as follows:*

$$Q \notin PP' \Longrightarrow \sigma(Q) = \ell^{Q}_{PP'} \cap \ell^{P'}_{PQ} \tag{3.3}$$

$$Q \in PP' \Longrightarrow \exists S \notin PP', S' = \ell^{S}_{PP'} \cap \ell^{P'}_{PS} \textit{ and } \sigma(Q) = PP' \cap \ell^{S'}_{SQ} \tag{3.4}$$

Proof. First consider the case when the point Q is not the point of the traces PP' of points P according to σ. Then, by Theorem 2.15, the QQ' its trace is parallel to PP', therefore $QQ' = \ell^{Q}_{PP'}$, indicates that $Q' \in \ell^{Q}_{PP'}$. But by being translations, σ is a dilatation therefore PQ is parallel to $P'Q' = \ell^{P'}_{PQ}$, also indicates that $Q' \in \ell^{P'}_{PQ}$. Results so that the image Q' is a cross cutting of the lines $\ell^{Q}_{PP'}$ and $\ell^{P'}_{PQ}$ so $Q' = \ell^{Q}_{PP'} \cap \ell^{P'}_{PQ}$.

Consider now another case, when the point Q is a point of trace PP' of points P according to σ. According to axiom A3, in affine plane, see [4], [5], [10], [11],[12] exists a point S no incidents with trace PP'. According to (3.3), constructed its image $S' = \ell^S_{PP'} \cap \ell^{P'}_{PS}$. But also Q not is a incidents with trace SS', therefore according to (3.3), further constructed also $Q' = \ell^Q_{SS'} \cap \ell^{S'}_{SQ}$. In summary the construction of image $Q' = \sigma(Q)$, in these conditions is presented in the form:
$Q \in PP' \Longrightarrow \exists S \notin PP', S' = \ell^S_{PP'} \cap \ell^{P'}_{PS}$ and $\sigma(Q) = PP' \cap \ell^{S'}_{SQ}$. □

Corollary 3.4. *Two translations $\sigma_1 \neq id_{\mathcal{P}}$, $\sigma_2 \neq id_{\mathcal{P}}$, of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, are equal only when for a point $P \in \mathcal{P}$, is true equalization*

$$\sigma_1(P) = \sigma_2(P) \tag{3.5}$$

Proof. From the definition of translations that have: since the translations σ_1 and σ_2 are different from the identical translation, then these translations not have the fixed points. If have which $\sigma_1 = \sigma_2$ it is obvious which $\forall P \in \mathcal{P}, \sigma_1(P) = \sigma_2(P)$.
Conversely: Let's have a point $P \in \mathcal{P}$, to which is true equalization $\sigma_1(P) = \sigma_2(P)$.
Now take another whatever point $Q \in \mathcal{P}$, by Theorem 3.3, we have which $\sigma_1(Q)$ and $\sigma_2(Q)$ defined by the equations (3.3) and (3.4).

In the case where $Q \notin P\sigma_1(P) = P\sigma_2(P)$, by the equations (3.4) we have:

$$\sigma_1(Q) = \ell^Q_{P\sigma_1(P)} \cap \ell^{\sigma_1(P)}_{PQ} = \ell^Q_{P\sigma_2(P)} \cap \ell^{\sigma_2(P)}_{PQ} = \sigma_2(Q) \Longrightarrow \sigma_1(Q) = \sigma_2(Q)$$

In the case where $Q \in P\sigma_1(P) = P\sigma_2(P)$, by the equations (3.4) we have:

$$Q \in P\sigma_1(P) = P\sigma_2(P) \Longrightarrow \exists S \notin P\sigma_1(P) = P\sigma_2(P),$$

$$S' = \ell^S_{P\sigma_1(P)} \cap \ell^{\sigma_1(P)}_{PS} = \ell^S_{P\sigma_2(P)} \cap \ell^{\sigma_2(P)}_{PS},$$

and

$$\sigma_1(Q) = P\sigma_1(P) \cap \ell^{S'}_{SQ} = P\sigma_2(P) \cap \ell^{S'}_{SQ} = \sigma_2(Q).$$

So we have which:

$$\forall Q \in \mathcal{P}, \sigma_1(Q) = \sigma_2(Q) \Longrightarrow \sigma_1 = \sigma_2.$$

□

Otherwise, *one translations $\sigma \neq id_{\mathcal{P}}$ of an affine plane $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, is completely determined by giving her the likeness to an point according to the plane.*

Proposition 3.5. *The inverse translation σ^{-1} of an translation σ to an affine plane $\mathcal{A}$=($\mathcal{P}$,$\mathcal{L}$,$\mathcal{I}$), is also an translation in the affine plane.*

Proof. If $\sigma = id_{\mathcal{P}}$, then, by $id_{\mathcal{P}}^{-1} = id_{\mathcal{P}}$ as bijection, it turns out that σ is translation. If $\sigma \neq id_{\mathcal{P}}$, then according to her in the plan has not fixed point. Suppose that σ^{-1} is not translation. As an dilatation, from the Proposition 2.16, in the plane has a fixed point P according to σ^{-1}, for which we have $\sigma^{-1}(P) = P$. But

$$\sigma\left(\sigma^{-1}(P)\right) = \sigma(P) \Longleftrightarrow P = \sigma(P),$$

proving that the P is fixed-point also for σ, in contradiction with condition. □

Inasmuch as $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = id_{\mathcal{P}}, \sigma^{-1}$ is the inverse translation of the translation σ.

Corollary 3.6. *For every translation σ to an affine plane $\mathcal{A}$=($\mathcal{P}$,$\mathcal{L}$,$\mathcal{I}$), σ and σ^{-1} have the same direction.*

Proof. It is evident that $\sigma \neq id_{\mathcal{P}} \Longleftrightarrow \sigma^{-1} \neq id_{\mathcal{P}}$. Well, the translations σ *and* σ^{-1} they have determined directions when $\sigma \neq id_{\mathcal{P}}$. The trace of a point P according to σ is the line $P\sigma(P)$, which represents the direction π_σ, while the trace of point $\sigma(P) = P'$ according to σ^{-1} is the line $P'\sigma^{-1}(P')$, which represents the direction $\pi_{\sigma^{-1}}$. But $\sigma(P) = P' \Longleftrightarrow P = \sigma^{-1}(P')$, hence $P'\sigma^{-1}(P') = P\sigma(P)$, which represents the direction $\pi_\sigma = \pi_{\sigma^{-1}}$. According to (3.2), this equation is true even when $\sigma = id_{\mathcal{P}}$. □

Proposition 3.7. *The composition of the two translations in affine plane is agin a translation of his.*

Proof. Let's be σ_1, σ_2 two translations of a affine plane. Having been bijections, if one from the translations σ_1, σ_2 is equal to $id_{\mathcal{P}}$, then $\sigma_2 \circ \sigma_1 = id_{\mathcal{P}}$. In this case, from the Definition 3.1, that production is a translation. Even if $\sigma_1 \neq id_{\mathcal{P}}$ and $\sigma_2 \neq id_{\mathcal{P}}$, again $\sigma_2 \circ \sigma_1$ is translation. Suppose the *contrary*, that $\sigma_2 \circ \sigma_1$ is not translation. Then, as dilatation, from the proposition 2.16, in the plan has a fixed point P according to $\sigma_2 \circ \sigma_1$, for which we have $(\sigma_2 \circ \sigma_1)(P) = P \Longleftrightarrow \sigma_2(\sigma_1(P)) = P \Longleftrightarrow \sigma_1(P) = \sigma_2^{-1}(P)$. According to corollary of the Theorem 3.3, results that $\sigma_1 = \sigma_2^{-1}$, that implicates $\sigma_2 \circ \sigma_1 = id_{\mathcal{P}}$, in contradiction with supposition. □

Proposition 3.8. *If translations σ_1 and σ_2 have the same direction with translation σ to a affine plane $\mathcal{A}$=($\mathcal{P}$,$\mathcal{L}$,$\mathcal{I}$), then and composition $\sigma_2 \circ \sigma_1$ has the same the direction, otherwise*

$$\forall \sigma_1, \sigma_2, \sigma \in \mathbf{Tr}_{\mathcal{A}}, \pi_{\sigma_1} = \pi_{\sigma_2} = \pi_\sigma \Longrightarrow \pi_{\sigma_2 \circ \sigma_1} = \pi_\sigma. \tag{3.6}$$

Proof. According to (3.2) and the corollary of Proposition 3.5, easily indicated veracity of propositions in cases where at least one from the translations $\sigma_1, \sigma_2, \sigma$ is $id_{\mathcal{P}}$ or when $\sigma_1 = \sigma_2^{-1}$. In the case where $\sigma_2 \circ \sigma_1 \neq id_{\mathcal{P}}$, $\sigma \neq id_{\mathcal{P}}$, from the condition, for a point $P \in \mathcal{P}$, the traces $P\sigma(P), P\sigma_1(P)$ and $P\sigma_2(P)$ are parallel and have a common point P, hence they coincide. This means that in the trace $P\sigma(P)$, are also images of the point P, according σ_1 and σ_2. We mark $\sigma_1(P) = P_1$, then, on the track $P\sigma(P)$, is also the image

$$\sigma_2(P_1) = \sigma_2(\sigma_1(P)) = (\sigma_2 \circ \sigma_1)(P).$$

This proves that

$$P\sigma(P) = (\sigma_2 \circ \sigma_1)(P),$$

namely

$$\pi_{\sigma_2 \circ \sigma_1} = \pi_\sigma.$$

□

Theorem 3.9. *Set* $\mathbf{Tr}_{\mathcal{A}}$ *of translations to an affine plane* $\mathcal{A}$ *form a group about the composition* $\circ$, *which is a sub-group of the group* $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ *to dilatations of affine plane* $\mathcal{A}$.

Proof. According to Definition 3.1, the translation set $\mathbf{Tr}_{\mathcal{A}}$ of the affine plane $\mathcal{A}$ is sub-set, of $\mathbf{Dil}_{\mathcal{A}}$. The Proposition 3.7, proves that $(\mathbf{Tr}_{\mathcal{A}}, \circ)$ is the sub-structure of the Dilatation group $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ of the affine plane $\mathcal{A}$. Proposition 3.5, proves that this sub-structure is a sub-group of the sub-group $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ of the group $(\mathbf{Col}_{\mathcal{A}}, \circ)$, see [7] and [8]. □

Theorem 3.10. *Group* $(\mathbf{Tr}_{\mathcal{A}}, \circ)$ *of translations to the affine plane* $\mathcal{A}$ *is normal sub- group of the group of dilatations* $(\mathbf{Dil}_{\mathcal{A}}, \circ)$ *of him plane.*

Proof. For this, according to a theorem [7] it suffices to prove that:

$$\forall \delta \in \mathbf{Dil}_{\mathcal{A}}, \forall \sigma \in \mathbf{Tr}_{\mathcal{A}} \text{ have } \delta^{-1} \circ \sigma \circ \delta \in \mathbf{Tr}_{\mathcal{A}}.$$

If $\sigma = id_{\mathcal{P}}$, then
$\delta^{-1} \circ \sigma \circ \delta = \delta^{-1} \circ (id_{\mathcal{P}} \circ \delta) = \delta^{-1} \circ \delta = id_{\mathcal{P}} \in \mathbf{Tr}_{\mathcal{A}}$.

If $\sigma \neq id_{\mathcal{P}}$, we mark $\sigma_1 = \delta^{-1} \circ \sigma \circ \delta$, which is a dilatation, while $\delta^{-1}, \sigma, \delta$, are dilatations. If $\sigma_1 = id_{\mathcal{P}}$, then again $\delta^{-1} \circ \sigma \circ \delta \in \mathbf{Tr}_{\mathcal{A}}$. But even if the $\sigma_1 \neq id_{\mathcal{P}}$, $\delta^{-1} \circ \sigma \circ \delta \in \mathbf{Tr}_{\mathcal{A}}$. Suppose the contrary, that in this case $\delta^{-1} \circ \sigma \circ \delta \notin \mathbf{Tr}_{\mathcal{A}}$. Then, as dilatation from Proposition 2.16, in the plan has a fixed point P according to σ_1, for which we have

$\sigma_1(P) = P \Longleftrightarrow \left(\delta^{-1} \circ \sigma \circ \delta\right)(P) = P \Longleftrightarrow \delta^{-1}(\sigma(\delta(P))) = P \Longleftrightarrow \sigma(\delta(P)) = \delta(P)$.

The last equation proves that the point $\delta(P)$ is the fixed point of the plane relating to σ, in the contradiction with the fact that $\sigma \neq id_{\mathcal{P}}$. □

Corollary 3.11. *For every dilatations $\delta \in \mathbf{Dil}_{\mathcal{A}}$ and for every translations $\sigma \in \mathbf{Tr}_{\mathcal{A}}$ of affine plane $\mathcal{A}$=($\mathcal{P}$,$\mathcal{L}$,$\mathcal{I}$), translations σ and $\delta^{-1} \circ \sigma \circ \delta$ of his have the same direction.*

Proof. From the above results for us $\delta \neq id_{\mathcal{P}}, \sigma \neq id_{\mathcal{P}} \Longleftrightarrow \delta^{-1} \circ \sigma \circ \delta \neq id_{\mathcal{P}}$. Well translations σ and $\delta^{-1} \circ \sigma \circ \delta$ have determined directions, when $\sigma \neq id_{\mathcal{P}}$ and $\delta \neq id_{\mathcal{P}}$. For judging for its directions π_σ and $\pi_{\delta^{-1} \circ \sigma \circ \delta}$, examine the tracks according σ for points of plane $\delta(P) = P'$ and according $\sigma_1 = \delta^{-1} \circ \sigma \circ \delta$ for its points P. If these traces are two parallel lines of the plan, then these belong to the same equivalence class, namely $\pi_\sigma = \pi_{\delta^{-1} \circ \sigma \circ \delta}$. Trace of point P' according to σ is the line $P'\sigma(P')$, while the trace of point P according to σ_1 is $P\sigma_1(P)$.

For dilatation δ have

$$P = \left(\delta^{-1} \circ \delta\right)(P) = \delta^{-1}(\delta(P)) = \delta^{-1}\left(P'\right),$$

whereas

$$\sigma_1(P) = \left(\delta^{-1} \circ \sigma \circ \delta\right)(P) = \delta^{-1}(\sigma(\delta(P))) = \delta^{-1}\left(\sigma\left(P'\right)\right).$$

But δ^{-1} is dilatation hence the different plane points $P', \sigma(P')$ with image

$$\delta^{-1}\left(P'\right) = P, \delta^{-1}\left(\sigma\left(P'\right)\right) = \sigma_1(P),$$

We have

$$P'\sigma\left(P'\right) \parallel P\sigma_1(P).$$

Easily proved, according (3.2), that this equation is true even when at least one of dilatations, δ, σ is equal to $id_{\mathcal{P}}$. □

According to the understanding of the normal sub-group, see [7], [8], [13], from this Theorem also it shows that there is true the implication

$$\forall(\delta, \sigma) \in \mathbf{Dil}_{\mathcal{A}} \times \mathbf{Tr}_{\mathcal{A}}, \delta \circ \sigma = \sigma \circ \delta.$$

Because $\delta \in \mathbf{Tr}_{\mathcal{A}} \Longrightarrow \delta \in \mathbf{Dil}_{\mathcal{A}}$, from that comes true the implication

$$\forall(\delta, \sigma) \in \mathbf{Tr}_{\mathcal{A}} \times \mathbf{Tr}_{\mathcal{A}}, \delta \circ \sigma = \sigma \circ \delta. \tag{3.7}$$

This indicates that is true this

Corollary 3.12. *The translations group* $(\mathbf{Tr}_{\mathcal{A}}, \circ)$ *of an affine plane* $\mathcal{A}$ *is (abelian) commutative.*

By definition of an Abelian Groups, see [7], [8], this means that besides (3.7), are true even these propositions:

$$\forall \sigma_1, \sigma_2, \sigma_3 \in \mathbf{Tr}_{\mathcal{A}}, (\sigma_1 \circ \sigma_2) \circ \sigma_3 = \sigma_1 \circ (\sigma_2 \circ \sigma_3) \tag{3.8}$$

$$\forall \sigma \in \mathbf{Tr}_{\mathcal{A}}, \sigma \circ id_{\mathcal{P}} = id_{\mathcal{P}} \circ \sigma = \sigma \tag{3.9}$$

$$\forall \sigma \in \mathbf{Tr}_{\mathcal{A}}, \exists \sigma^{-1} \in \mathbf{Tr}_{\mathcal{A}}, \sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = id_{\mathcal{P}} \tag{3.10}$$

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Academician Professor RADU MIRON at 90th Birthday: a Life for Mathematics

Mihai Anastasiei

Academician Professor Radu Miron has reached the venerable age of ninety years, sixty-five of which he dedicated to Mathematics education and research, representing the Romanian mathematics school as a Professor at Alexandru Ioan Cuza University in Iaşi, as well as a member of the Romanian Academy. His vast and compact scholarly work has brought him international recognition and has established him as a leader of the Romanian school of geometry.

As a former student, including the doctoral level and then as a collaborator of Acad. Radu Miron in writing papers and books, I had the privilege of witnessing his outstanding career. I dedicate the following lines, with admiration and love to my Professor, Radu Miron.

As a fresh student I noticed in the faculty hallways the presence of a professor who impressed by his confidence and authority. His name was Radu Miron, a young Associate Professor at the Department of Geometry and researcher at the Iaşi Branch of the Romanian Academy. He had just received the prestigious award "Gheorghe Ţiţeica" of the Romanian Academy for the monograph "The Geometry of Myller Configurations" (in Romanian).

Three months later his lectures on Analytical Geometry strongly impressed all my colleagues. In each lesson, he briefly recalled the issues of the previous ones.

Then a new problem followed and its solution was obtained in a cascade of definitions, lemmas, theorems, corollaries and applications. In the end everything became clear and easy. All students learned with pleasure and gained great marks.

I have taken three more geometry courses of his and I have always enjoyed his noble academic attitude, the clarity and the scientific accuracy of his lessons.

Professor Radu Miron and his book "The Geometry of Myller Configurations" have contributed crucially to my decision to study Geometry more thoroughly and to do a doctorate in this discipline under his supervision. This happened between 1972-1977. In the meantime he became Full Professor (1969) and was elected as the Dean of the Faculty of Mathematics.

Despite some new teaching and administrative duties, full of energy, Professor Radu Miron continued to research interesting and difficult topics. Thus he extended the notion of Myller configuration to spaces with affine connection and developed a theory of distributions in such spaces.

It has been known since a long time that for the study of the Finsler spaces of dimensions 2 and 3 some special ortonormal frames are used. These have been introduced by Berwald and Moor, respectively.

At a Geometry Conference in Debrecen, Professor Radu Miron wondered if such a frame can be built for any dimension. The problem appeared as interesting to Professor Mokoto Matsumoto, the leader of Japan's Finsler School of Geometry, a prominent participant in that Conference.

The issue has been extensively discussed during the Conference, but no solution has been found. Returning to the country, Professor Radu Miron finds a solution he sends to Matsumoto. There is an exchange of letters in which the solution was refined and published in a joint paper in Periodica Mathematica Hungarica 8(1977).

This paper was taken over entirely by Matsumoto in his monograph dedicated to Finsler spaces by speaking of "the Miron frames". This was the first important contribution of Professor Radu Miron to the Finsler geometry.

Such fortunate happenings have appeared several times in the research work of Professor Radu Miron so he is right when he says: *I congratulate myself on the courage to get into new problems that many could not solve. There are some famous problems that have remained unresolved for hundreds of years.*

That first meeting with Matsumoto was for Professor Radu Miron the first step towards studying Finsler Geometry. Soon he will include the theory developed by the Japanese school in a larger one and provide it with modern and efficient methods.

The second step was a talk *Finsler Geometry.Romanian Mathematician's Contributions* he gave in 1977 at the National Conference on Geometry and Topology in Timişoara (Romania). His talent of revealing the beauty of a subject matter stimulated the interest in the study of this Geometry and many young participants at this Conference decided to approach topics suggested by Professor Miron' lecture, taking the advantage of his advice and help.

The third step was his decision to present the monograph of M. Matsumoto, *Foundation of Finsler Geometry and Special Finsler Spaces*, still unpublished, in the Geometry Department. At the same time he asked me to study and to present in the same Department the point of view of French mathematicians in approaching Finsler geometry as it appeared in papers by A. Lichnerowicz, J. Klein, P. Dazord and J. Grifone's PhD thesis.

So he managed to accumulate in a short time a large amount of knowledge that he integrated providing a synthesis and an overview of the development stage of Finsler geometry. These were the basis for the generalizations that he later proposed.

In the winter of 1980, the first National Seminar on Finsler Geometry took place at the University of Braşov according to a proposal of Prof. Radu Miron. In a four-hour lecture, a text published later on 53 pages, Professor Radu Miron proposed an original viewpoint on the geometry of Finsler spaces, fusing together the influences of the earlier Romanian, Japanese and French researchers. In this lecture, the role of the nonlinear connections is fully clarified as well as that of geometric object of Finsler type. A novelty with great impact later on was the introduction of spaces with metric Finsler structures, called also generalized Finsler spaces.

The introduction and the study of metric Finsler structures is Professor Miron's second major original contribution to the theory of Finsler spaces.

This contribution actually modified the framework of the Finsler geometry and led to new generalization and new points of view.

An interesting one belongs to Professor Radu Miron. He noticed that the techniques from the geometry of generalized Finsler spaces can be also used in the study of the geometry of the total space of any vector bundle. He developed, by using such techniques, an elegant theory of geometric structures and of connections compatible with these, on the total space of a vector bundle, geometrically, with easy-to-follow calculus. His co-workers adapted and applied his idea to various frameworks.

In the early 1980s Prof. Matsumoto and his collaborators have become more and more interested in the new ideas promoted by Professor Miron. Some of them (M. Hashiguchi, Y. Ichijio) repeatedly visited the Faculty of Mathematics in Iaşi. During these visits it was organized the Romanian - Japanese Colloquium on Finsler Geometry which was held at the universities from Iaşi and Braşov in August 1984.

This Colloquium was an important step in expanding the collaboration of Romanian geometers with Japanese geometers, as embodied in numerous joint papers. For instance, Prof. Radu Miron has written joint papers with M. Matsumoto, M. Hashiguchi, Y. Ichijio, H. Izumi, S. Kikuchi, S. Watanabe, S. Ikeda. There was no a Japanese-Romanian colloquium. It would not have been possible before 1989, and after December 1989, Romanian geometers entered the global choir of geometers and no unilateral meeting was justified. They met Japanese colleagues at international conferences organized in different countries, including Japan.

At the fourth National Seminar on Finsler Geometry, that was held also at the University of Braşov (Romania) in January 1986, Prof. Radu Miron proposes the study of spaces endowed with Lagrangians that are no longer homogeneous in the directional variable as is the case with Finsler spaces. He calls them Lagrange spaces and presents the basics of their geometry. The study of Lagrange spaces was enthusiastically continued by many participants at that Seminar.

After 1970, there has appeared in the Theoretical Physics the interest in developing a Finslerian Theory of Relativity that should offer the possibility of describing anisotropy properties of space. As to this matter Professor Radu Miron has a simple idea, as all great ideas: to consider the Einstein equations in Lagrange spaces as the Einstein equations associated to the canonical metrical connection from the almost Hermitian model. By decomposing the Einstein equations from the model in the adapted frames to nonlinear connection, he obtains two sets of Einstein equations.

Prof.Dr.S.Ikeda from the University of Sciences in Tokyo explained the physical foundations of the entire theory in a work published as the last chapter of the monograph *Vector bundles. Lagrange spaces. Applications to Relativity*, published in Romanian by the Romanian Academy in 1987.

The introduction of the notion of Lagrange space and the establishment of the basic properties of these spaces is the third major contribution of Professor Radu Miron in the Finsler type geometries. The fourth significant contribution is to introduce and establish the main properties of the Hamilton spaces. The idea is coming from Mechanics. To any Lagrangian corresponds, by the Legendre duality, a Hamiltonian. Professor Radu Miron has studied the Hamiltonian spaces as dual to the Lagrange spaces. A difficult step was to determine a nonlinear connection which depends on the Hamiltonian only.

Two years later, in the same National Seminar on Finsler Geometry, Prof. Radu Miron defines and sketches the geometry of two new spaces: higher order Lagrange spaces and higher-order Hamilton spaces, thought to be dual.

The duality of these spaces will be clarified later in a one-night discussion with a famous Japanese engineer, K. Kondo. In the same framework he solves the famous problem of the prolongation of order $k > 1$ of the Riemannian space, brings a solid contribution to the foundation of the Mechanics of the Lagrangians which depends on the higher order accelerations and creates some new geometrical models for the theory of physical fields.

These two notions complete a Finsler-centered painting not only with more and more general notions but also with new techniques useful in the particular case of Finsler spaces as well. By 1988 Prof. Radu Miron left to others the detailed study of these spaces and focused on the possible applications of their geometries.

Of course, developing applications he has also had to solve new geometrical problems. This has clearly happened when he developed a theory of electromagnetism and studied the geometrical optics based on a generalization of a metric due to J.L.Synge, by applying the geometry of generalized Lagrange spaces.

The research of Prof. Radu Miron, his innovative ideas, the efforts to organize a Finsler geometry school in our country, the scientific collaborations with other countries, especially with Japan, were remarked by the members of the Mathematical Section from the Romanian Academy. Consequently, he was proposed and elected correspondent (1991) and then full member (1993) of the Romanian Academy.

This high appreciation of the Romanian Academy has been an additional reason for Academician Radu Miron to continue.

The change of political regime in Romania opened up new perspectives and possibilities for Romanian mathematicians. Not only did he use these possibilities, but also mobilized many Romanian mathematicians to promote the mathematical ideas and techniques developed in Romania.

The '90s were for Academician Radu Miron years with many achievements. I mention those which, in my opinion, are most relevant.

• He has published on his own or in collaboration five scientific monographs in English, over one thousand pages, at international publishing houses.

• In the decade 1990-2000 he published about one-third of all scientific papers, around five per year.

• He presented scientific papers at 2-3 scientific conferences each year. For this purpose he has repeatedly traveled to Japan, Canada, USA, Egypt, India and almost all European countries. I have been accompanying him for some of these journeys for scientific purposes.

He is an ideal traveling companion. He is always optimistic, he is not afraid of possible difficulties, he is organized and with care for his traveling companions. He also likes to travel for tourism purposes. He visited China by train passing through the former Soviet Union and Mongolia.

• He has completed several research visits to universities in Canada, Japan, Greece, Hungary. Remarkable is his collaboration with Professor Peter Louis Antonelli of Alberta University in Edmonton, Canada. In a congratulatory message, Antonelli wrote:

I first met Prof. Miron in 1991 in Debrecen, Hungary, at a conference on Finsler and Lagrange Geometry. At that meeting, Miron and I set the stage for our 5-year official cooperative research plan, between "Alexandru Ioan Cuza" University of Iaşi in Romania and University of Alberta, in Canada. This involved writing books and papers, organizing conferences and training post- doctoral fellows of which, I. Bucataru, D. Hrimiuc, S. Rutz, T. Zastawniak and B. Lackey, are outstanding examples. There were also extended visits from Profs. Anastasiei and Miron and numerous other scholars over the 5 years.

In the same message, appreciating the value of the books published by Acad. Radu Miron, he wrote:

It is my personal opinion that Prof. Miron's books satisfy a need of those wishing to get to the heart of geometry with clarity, elegance and brevity of exposition... a mighty achievement.

• He taught the students at the Faculty of Mathematics until the age of seventy when he retired. He continued to teach math lessons at the Economics School of a private university until the age of eighty.

• He was involved in the center-right politics locally in the memory of his father who had been liberal between the two world wars. He was several times a municipal or county councilor.

Academician Radu Miron is an open person who easily establishes relationships with interlocutors and offers his friendship through a handshake. He likes to invite friends to lunches or dinners in his family's apartment. I was invited together with my wife many times. At the conferences organized in Iaşi he also used to invite guests from abroad.

He has two daughters. One is an oculist and the other is a math teacher. They are married and each one has two children. Some of them already have children. So he has a pretty big family. I met many of its members. His wife was a great lady (she died a few days before her husband was ninety years old), very nice to the guests, appreciating the work of her husband very much. She accompanied him to some scientific conferences.

Academician Radu Miron loves good meals and quality wines. Sometimes, for a guest from abroad, he prepares himself a special kind of Romanian food that he is very proud of. It was customary for autumn to prepare very tasty sausages with sheep meat and many spices. He prepares also various pickles, especially watermelons.

If the last decade of the 20th century was a decade of intense work, exciting travels and friendships, the first two decades of the 21st century were for Academician Radu Miron decades of rewards. Here are a few.

He was elected as a honorary member of the Academy of Sciences of the Republic of Moldova, he was awarded the title of "doctor honoris causa" at several Romanian universities, he received "Omnia Opera" award for the whole scientific activity and several diplomas of excellence as well.

He received *V.Pogor* award from City Hall of Iaşi and was declared an honorary citizen of the city of Iaşi. Also, Academician Radu Miron was celebrated on his 70th, 75th, 80th,85th anniversaries by Symposiums and international Conferences organized in his honor.

The Conferences were devoted to the main field that he is interested in: Finsler, Lagrange and Hamilton geometries and their applications to Mechanics and Physics. Romanian journals dedicated special volumes to him.

The academic community from Iaşi, together with the Romanian mathematicians, as well as many geometers around the world enjoyed to celebrate the 90th anniversary of Academician Radu Miron by a Symposium on his life and work, organized by the Iaşi branch of the Romanian Academy.

The conference ICAPM 2017 was also dedicated to him. Many geometers from Romania and abroad sent beautiful congratulatory messages.

Dear Academician Professor Radu Miron, Happy Birthday to You!

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