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Bounds for global solutions of nonlinear heat equations with nonlinear boundary conditions

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Abstract: In this paper, we consider the initial-boundary value problem for nonlinear heat equations with nonlinear boundary conditions of radiation type. The local well-posedness of this problem is shown by applying an abstract theory for the evolution equation governed by subdifferential operators. Moreover results on the uniform boundedness for time global solutions are obtained.

Keywords: Nonlinear boundary conditions, Bounds for global solutions.

MSC2010: Primary 35K20; Secondary 35K59, 35B40

1 Introduction

In this paper, we are concerned with the asymptotic behavior of global solutions of the following initial boundary value problem for the nonlinear heat equation:

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-2}u & t > 0,\ x \in \Omega, \\ \partial_\nu u + |u|^{q-2}u = 0 & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega. \end{cases} \tag{P}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$; $T > 0$ is a given constant; $p \in (2, \infty)$, $q \in (1, p)$ are given numbers; $u : [0, T] \times \Omega \to \mathbb{R}$ is a real-valued unknown function. This problem (P) is a prototype of nonlinear heat equations with nonlinear boundary conditions of radiation type.

When one tries to set up mathematical models for describing actual nonlinear phenomena, it is crucial to determine right ruling nonlinear structures in domains where the phenomena occur, but it is also very important to pay careful attention to the choice of the boundary conditions. In other words, when we are concerned with the diffusion equations, it should be noted that the standard boundary condition such as Dirichlet or Neumann boundary condition is realistic only for the case where there is some artificial control of the flux on the boundary. For a large scale system, however, it is impossible to give such a control on the boundary. If there is no control of heat flux on the boundary, there is a prototype model in physics well known as Stefan-Boltzmann's law, which says that the heat energy radiation from the surface of the body is proportional to the fourth power of the difference of temperatures between the inside and outside of the body. In this sense, from a physical point of view, it could be more natural to consider nonlinear boundary conditions rather than the linear boundary conditions such as the homogeneous Dirichlet or Neumann boundary condition.

There are large amounts of works concerning the asymptotic behavior of solutions of the following nonlinear heat equation with the homogeneous Dirichlet boundary condition:

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-2}u & t > 0,\ x \in \Omega, \\ u = 0 & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega. \end{cases} \tag{1.1}$$

Uniform bounds of global solutions of (1.1) was first studied by [14] as an abstract equation of the form $u_t + \partial\psi^1(u) - \partial\psi^2(u) = 0$ in $L^2(\Omega)$. Here $\partial\psi^i$ are subdifferentials of lower semi-continuous convex and homogeneous functionals ψ^i $(i = 1, 2)$ on $L^2(\Omega)$, where it is shown that every global solution of (1.1) is uniformly bounded in $H^1_0(\Omega)$ with respect to time for $p \in (2, p_S)$. Here p_S is the Sobolev critical exponent defined by $p_S = \infty$ for $N = 1, 2$; $p_S = \frac{2N}{N-2}$ for $N \geq 3$. Cazenave-Lions [5] showed that every global solution (allowing sing-changing) solution is bounded in $L^\infty(\Omega)$ uniformly in time provided that $p \in (2, p_{CL})$, where $p_{CL} = \infty$ when $N = 1$; $p_{CL} = 2 + \frac{12}{3N-4}$ when $N \geq 2$. (Note that $p_{CL} \leq p_S$ for any $N \in \mathbb{N}$). Giga [6] removed this restriction on p for positive global solutions. Namely the uniform boundedness of every positive global solution of (1.1) in $L^\infty(\Omega)$ was shown for any $p \in (2, p_S)$. Quittner [16] extended this result for sign-changing solutions. The main tool in [6] is the rescaling argument and [16] relies on the bootstrap argument based on the interpolation and the maximal regularity theory. However it seems to be difficult to apply these devices for our problem (P) because of the presence of the nonlinear boundary condition.

The main purpose of this paper is to derive the uniform boundedness in $H^1(\Omega)$ and $L^\infty(\Omega)$ for every global solution of (P) by following the same strategy as that in [14]. However, we can not directly apply arguments in [14], since the functional associated with the Laplacian with nonlinear boundary conditions is not homogeneous, which is one of basic tools used in [14]. Nevertheless by introducing a new substitutive argument to avoid the use of the homogeneity of functionals, we are able to derive uniform bounds for global solutions in $H^1(\Omega)$. Moreover with the aid of Moser's iteration scheme, the uniform bound in $L^\infty(\Omega)$ is also obtained.

This paper is composed as follows. In the next section, we deal with the local well-posedness of (P) in $H^1(\Omega)$ and $L^\infty(\Omega)$. In order to work in $H^1(\Omega)$, we reduce

(P) to abstract evolution equations in a real Hilbert space governed by subdifferential operators and apply a nonmonotone perturbation theory developed in [15]. For the analysis in $L^\infty(\Omega)$, we rely on the L^∞-energy method given in [14].

In §3, our main theorem is stated and its proof is given by following the strategy in [14] and by relying on Moser's iteration scheme, whose main tool is proved in Appendix.

2 Preliminaries and Local Well-posedness

In this section, we are going to show the local well-posedness for (P) by applying the theory of the evolution equation governed by subdifferential operators. Throughout this paper, H designates a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $\Phi(H)$ be the set of all convex and lower-semicontinuous functionals $\phi : H \to (-\infty, +\infty]$ such that its effective domain $D(\phi) := \{u \in H; \phi(u) < \infty\}$ is nonempty. For each $u \in D(\phi)$, we call the set

$$\partial\phi(u) = \{f \in H; \phi(v) - \phi(u) \geq (f, v-u) \quad \forall v \in D(\phi)\}$$

subdifferential of ϕ at u. Then $\partial\phi : H \to 2^H$ becomes a (possibly multivalued) maximal monotone operator with domain $D(\partial\phi) := \{u \in D(\phi); \partial\phi(u) \neq \emptyset\}$, which is called by subdifferential operator. We remark that $D(\partial\phi) \subset D(\phi) \subset \overline{D(\phi)} = \overline{D(\partial\phi)}$ holds (for the proofs see [2, 3]).

Define the functional φ on $L^2(\Omega)$ by

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 dx + \dfrac{1}{q}\displaystyle\int_{\partial\Omega} |u|^q d\sigma & u \in D(\varphi) := \{u \in H^1(\Omega); u \in L^q(\partial\Omega)\}, \\ +\infty & u \in L^2(\Omega) \setminus D(\varphi), \end{cases}$$

Then we can see that $\varphi \in \Phi(L^2(\Omega))$ and the subdifferential operator associated with φ is given as follows (see [4]):

$$\begin{cases} D(\partial\varphi) = \{u \in H^2(\Omega)\,;\, \partial_\nu u + |u|^{q-2}u = 0 \quad \text{a.e. on } \partial\Omega\}, \\ \partial\varphi(u) = -\Delta u. \end{cases} \tag{2.1}$$

Furthermore the following elliptic estimate for $\partial\varphi$ holds, i.e., there exist some constants c_1, $c_2 > 0$ such that

$$\|u\|_{H^2(\Omega)} \leq c_1\| - \Delta u + u\|_{L^2(\Omega)} + c_2 \quad \forall u \in D(\partial\varphi) \tag{2.2}$$

and $\overline{D(\partial\varphi)} = L^2(\Omega)$ (see [4, 2]).

Hereafter we denote the L^p norm by $\|\cdot\|_p$ $(1 \leq p \leq \infty)$, a general constant by $C > 0$ which may vary from place to place and Sobolev critical exponent by 2^*, i.e., $2^* = \frac{2N}{N-2}$ for $N \geq 3$, $2^* = \infty$ for $N = 1, 2$.

2.1 Local well-posedness in $D(\varphi)$

We first show the existence of time local solutions of (P) for the initial values which belong to the effective domain $D(\varphi)$ of φ (note that $D(\varphi) \subset H^1(\Omega)$). We here emphasize that even though $\partial\varphi(u) = -\Delta u$ looks like a linear operator, this is not the case since $D(\partial\varphi)$ does not have the linear structure. Therefore we can not rely on Duhamel's principle. Instead we here rely on the following abstract theory of nonlinear evolution equations associated with subdifferential operator.

Proposition 2.1. ([15]) *Let $\phi \in \Phi(H)$ and the following assumptions* (A1) - (A3) *be satisfied.*

(A1) For any $L > 0$, the set $\{u \in H\,;\, \phi(u) + \|u\|^2 \leq L\}$ is compact in H.

(A2) $B : H \to H$ satisfies the following ϕ-demiclosedness condition:
If $u_n \to u$ strongly in $C([0,T];H)$, $\partial\phi(u_n) \rightharpoonup \partial\phi(u)$ weakly in $L^2(0,T;H)$ and $B(u_n) \rightharpoonup b$ weakly in $L^2(0,T;H)$, then $b = B(u)$ holds a.e. in $t \in [0,T]$.

(A3) There exist a monotone increasing function $\ell(\cdot) : [0,\infty) \to [0,\infty)$ and $k \in [0,1)$ such that

$$\|B(u)\|^2 \leq k\|\partial\phi(u)\|^2 + \ell(\phi(u) + \|u\|) \qquad \forall u \in D(\partial\phi).$$

Then for any $u_0 \in D(\phi)$, there exists a positive number $T_0 = T_0(\|u_0\|, \phi(u_0)) \in [0,T]$ such that the following abstract Cauchy problem in H;

$$\frac{d}{dt}u(t) + \partial\phi(u(t)) + B(u(t)) = 0, \quad t > 0, \qquad u(0) = u_0,$$

possesses a strong solution $u \in C([0,T_0];H)$ such that

$$\frac{d}{dt}u,\ \partial\phi(u),\ B(u) \in L^2(0,T_0;H). \tag{2.3}$$

Then we can apply Proposition 2.1 for the existence part of the following result.

Proposition 2.2. *Let $p \in (2, 2^*)$ and $u_0 \in D(\varphi)$. Then there exists $T_0 = T_0(\varphi(u_0)) > 0$ such that* (P) *possesses a unique solution u satisfying the following regularity*

$$u \in C([0,T_0];L^2(\Omega)); \quad \partial_t u,\ \Delta u,\ |u|^{p-2}u \in L^2(0,T_0;L^2(\Omega)). \tag{2.4}$$

Proof. (Existence) Put $B(u) = -|u|^{p-2}u$, then (P) is reduced to the following abstract evolution equation in $H := L^2(\Omega)$:

$$\frac{d}{dt}u(t) + \partial\varphi(u(t)) + B(u(t)) = 0, \qquad u(0) = u_0. \tag{2.5}$$

In order to show the existence of a solution of (2.5), we are going to apply Proposition 2.1. To do this, we have only to check three assumptions (A.1), (A.2) and (A.3). It is clear that (A1) follows from the boundedness of the domain Ω and the Rellich-Kondrachov compactness theorem. Since $-B(u)$ is maximal monotone and the maximal monotone operator satisfies the demiclosedness property (in the standard sense), assumption (A2) is also satisfied. To verify (A3), we note that there exists $\lambda = \lambda(p, N) \in (0, 2]$ such that

$$\|u\|_{2(p-1)}^{2(p-1)} \leq C\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \qquad \forall u \in H^2(\Omega), \tag{2.6}$$

which will be proved in the next section (see Lemma 3.6).

Then by virtue of (2.6), the elliptic estimate (2.2) and Young's inequality, we obtain

$$\begin{aligned}
\|B(u)\|_2^2 &= \|u\|_{2(p-1)}^{2(p-1)} \\
&\leq C\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \\
&\leq C\left(\|-\Delta u + u\|_2^{2-\lambda} + 1\right)\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \\
&\leq C\left(\|-\Delta u\|_2^{2-\lambda} + \|u\|_2^{2-\lambda} + 1\right)\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \\
&\leq \varepsilon\|-\Delta u\|_2^2 + \frac{\lambda}{2}\left(\frac{2-\lambda}{2\varepsilon}\right)^{\frac{2-\lambda}{\lambda}} C^{\frac{2}{\lambda}}\|u\|_{H^1(\Omega)}^{\frac{2(2p-4+\lambda)}{\lambda}} + C\left(\|u\|_2^{2-\lambda} + 1\right)\|u\|_{H^1(\Omega)}^{2p-4+\lambda},
\end{aligned}$$

for every $\varepsilon > 0$. Hence since $\|u\|_{H^1(\Omega)}^2 \leq 2\varphi(u) + \|u\|_2^2$, in view of (2.1), we can assure (A3). Thus by Proposition 2.1 we observe that (P) admits a local solution $u \in C([0, T_0]; L^2(\Omega))$ satisfying (2.3).

(Uniqueness) Let u and v be two solutions of (P) on $[0, T_0]$ with the initial values $u_0 \in D(\varphi)$ and $v_0 \in D(\varphi)$, respectively. Then $w := u - v$ satisfies

$$\begin{cases} \partial_t w - \Delta w = |u|^{p-2}u - |v|^{p-2}v & t > 0,\ x \in \Omega, \\ \partial_\nu w + |u|^{q-2}u - |v|^{q-2}v = 0 & t > 0,\ x \in \partial\Omega, \\ w(0, x) = u_0(x) - v_0(x) & x \in \Omega. \end{cases} \tag{P$_w$}$$

Multiplying (P$_w$) by w and using integration by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \|\nabla w(t)\|_2^2 + \int_{\partial\Omega}\left(|u|^{q-2}u - |v|^{q-2}v\right) w\, d\sigma \\
&\qquad = \int_{\Omega}\left(|u|^{p-2}u - |v|^{p-2}v\right) w\, dx.
\end{aligned}$$

Since $u \mapsto |u|^{q-2}u$ is monotone increasing, $\int_{\partial\Omega} \left(|u|^{q-2}u - |v|^{q-2}v\right) w \, d\sigma \geq 0$.

Moreover we note

$$\left||x|^{p-2}x - |y|^{p-2}y\right| = \left|\int_x^y (p-1)|s|^{p-2}ds\right| \leq (p-1)\left(|x|^{p-2} + |y|^{p-2}\right)|x-y|$$

for all x, $y \in \mathbb{R}^1$. Hence by Hölder's inequality, we obtain

$$\begin{aligned}\int_\Omega \left(|u|^{p-2}u - |v|^{p-2}v\right) w \, dx &\leq (p-1)\int_\Omega \left(|u|^{p-2} + |v|^{p-2}\right) w^2 dx \\ &\leq (p-1)\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)\|w(t)\|_p^2.\end{aligned}$$

We here recall the following Gagliardo-Nirenberg interpolation inequality (see [11])

$$\|u\|_p \leq C\left(\|\nabla u\|_2^\eta \|u\|_2^{1-\eta} + \|u\|_2\right) \qquad \forall u \in H^1(\Omega),$$

where $\eta \in (0,1)$ is determined by $\frac{1}{p} = \eta\left(\frac{1}{2} - \frac{1}{N}\right) + (1-\eta)\frac{1}{2}$. By this inequality and Young's inequality, we obtain

$$\begin{aligned}&\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)\|w(t)\|_p^2 \\ &\leq C\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)\left(\|\nabla w(t)\|_2^{2\eta}\|w(t)\|_2^{2(1-\eta)} + \|w(t)\|_2^2\right) \\ &\leq \frac{1}{2(p-1)}\|\nabla w(t)\|_2^2 + C\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)^{\frac{1}{1-\eta}}\|w(t)\|_2^2 \\ &\quad + C\left(\|u(t)\|_p^{p-2} + \|v(t)\|_p^{p-2}\right)\|w(t)\|_2^2.\end{aligned}$$

Since u and v satisfy the regularity (2.3) of Proposition 2.1, $\varphi(u)$ and $\varphi(v)$ are absolute continuous on $[0,T_0]$ (see [3]). Noting that $p \in (2,2^*)$ implies $\|u\|_p \leq C(\varphi(u) + \|u\|_2^2)^{1/2}$, we deduce that $\|u\|_p$ and $\|v\|_p$ are bounded above by some constant $M > 0$ uniformly on $[0,T_0]$. Thus we get

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \frac{1}{2}\|\nabla w(t)\|_2^2 \leq C\left((2M^{p-2})^{\frac{1}{1-\eta}} + 2M^{p-2}\right)\|w(t)\|_2^2.$$

Then by Gronwall's inequality, we obtain

$$\|u(t) - v(t)\|_2^2 \leq \|u_0 - v_0\|_2^2 \, e^{2C\left((2M^{p-2})^{\frac{1}{1-\eta}} + 2M^{p-2}\right)t} \qquad \forall t \in [0,T_0],$$

whence follows the uniqueness. □

2.2 Local well-posedness in $L^\infty(\Omega)$

In this subsection, we are going to show the local well-posedness in $L^\infty(\Omega)$ without any restriction on the growth order p. The main tool here is "L^∞-energy method" developed in [13], for which we need prepare the following lemma (see Lemma 2.2 of [13]).

Lemma 2.3. *Let $y(t)$ be a bounded measurable non-negative function on $[0,T]$ and suppose that there exists $y_0 \geq 0$ and a monotone non-decreasing function $\ell(\cdot) : (0,+\infty) \to (0,+\infty)$ such that*

$$y(t) \leq y_0 + \int_0^t \ell(y(s))ds \qquad a.e.\ t \in (0,T).$$

Then for any $y_1 > y_0$, there exists a number $T_0 = T_0(y_0, y_1, \ell(\cdot)) \in (0,T]$ such that

$$y(t) \leq y_1 \qquad a.e.\ t \in [0,T_0]. \tag{2.7}$$

Proof. Put $z(t) = y_0 + \displaystyle\int_0^t \ell(y(s))ds$, then $z(t) \in C([0,T];\mathbb{R}^1)$ and $y(t) \leq z(t)$. So $z(t)$ satisfies

$$z(t) \leq y_0 + \int_0^t \ell(z(s))ds \qquad \text{for all } t \in [0,T]. \tag{2.8}$$

We here claim that

$$z(t) \leq y_1 \qquad \text{for all } t \in [0,T_0], \quad T_0 = \min\Big(\frac{y_1 - y_0}{2\,\ell(y_1)}, T\Big). \tag{2.9}$$

In fact, suppose that (2.9) does not hold, i.e., there exists $t_0 \in (0,T_0]$ such that $z(t_0) > y_1$, then since $z(t)$ is continuous on $[0,T]$ and $z(0) = y_0 < y_1$, there exists $t_1 \in (0,t_0)$ such that $z(t_1) = y_1$ and $z(t) < y_1 \quad \forall t \in [0,t_1)$. Then, by (2.8), we get

$$\begin{aligned} y_1 = z(t_1) &\leq y_0 + \int_0^{t_1} \ell(z(s))ds \\ &\leq y_0 + \ell(y_1)\,T_0 \leq y_0 + \frac{y_1 - y_0}{2} < y_1, \end{aligned}$$

which leads to a contradiction. Thus (2.9) is verified and hence (2.7) is derived from the fact that $y(t) \leq z(t)$ for all $t \in [0,T]$. □

Now the local well-posedness of (P) in $L^\infty(\Omega)$ can be stated as follows.

Proposition 2.4. *Let* $u_0 \in L^\infty(\Omega)$*, then there exists* $T_0 = T_0(\|u_0\|_\infty) > 0$ *such that* (P) *possesses a unique solution* u *satisfying the following regularity*

$$\begin{gathered} u \in C([0,T_0]; L^2(\Omega)) \cap L^\infty(0,T_0; L^\infty(\Omega)), \\ \sqrt{t}\,\partial_t u,\ \sqrt{t}\,\Delta u,\ \sqrt{t}\,|u|^{p-2}u \in L^2(0,T_0; L^2(\Omega)). \end{gathered} \tag{2.10}$$

Proof. (Uniqueness) Let u and v be two solutions of (P) with the same initial data $u_0 \in L^\infty(\Omega)$ satisfying the regularity (2.10). Then $w := u - v$ satisfies (P$_w$) with $w(0) = 0$. Multiplying (P$_w$) by w, we now get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 &\le \int_\Omega \left(|u|^{p-2}u - |v|^{p-2}v\right) w\,dx \\ &\le (p-1)\int_\Omega \left(|u|^{p-2} + |v|^{p-2}\right) w^2 dx \\ &\le (p-1)\left(\|u\|^{p-2}_{L^\infty(0,T;L^\infty(\Omega))} + \|v\|^{p-2}_{L^\infty(0,T;L^\infty(\Omega))}\right)\|w(t)\|_2^2 \\ &\le C\,\|w(t)\|_2^2, \end{aligned}$$

whence follows from Gronwall's inequality

$$\|w(t)\|_2^2 \le \|w(0)\|_2^2 e^{2CT} = 0 \qquad \forall t \in (0,T).$$

(Existence) We consider the following auxiliary problem:

$$\begin{cases} \partial_t u - \Delta u = |[u]_M|^{p-2}\,u & t > 0,\ x \in \Omega, \\ \partial_\nu u + |u|^{q-2}u = 0 & t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases} \tag{2.11}$$

Here

$$M = \|u_0\|_\infty + 2 \tag{2.12}$$

and $[u]_M$ is a cut-off function of u defined by

$$[u]_M = \begin{cases} M & u \ge M, \\ u & |u| \le M, \\ -M & u \le -M. \end{cases}$$

Set $B_M(u) = -|[u]_M|^{p-2}u$, then the auxiliary problem (2.11) can be reduced to the following evolution equation in $L^2(\Omega)$:

$$\frac{d}{dt}u(t) + \partial\varphi(u(t)) + B_M(u(t)) = 0, \qquad u(0) = u_0. \tag{2.13}$$

Since $B_M : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous, we know that (2.13) has a unique global solution $u \in C([0,T]; L^2(\Omega))$ for $u_0 \in L^2(\Omega)$ satisfying the same regularity (except L^∞-estimate) of Proposition 2.4 with T_0 replaced by T by applying the abstract theory developed by H. Brézis (see Proposition 3.12 in [3]).

Furthermore we can show that $u_0 \in L^\infty(\Omega)$ assures $u(t) \in L^\infty(\Omega)$ for all $t \geq 0$. To see this, put $v(t) := e^{-M^{p-2}t}u(t)$, then $v(t)$ satisfies

$$\partial_t v(t) - \Delta v(t) = \Big(|[u]_M|^{p-2} - M^{p-2}\Big) v(t), \quad v(0) = u_0. \tag{2.14}$$

Multiplying (2.14) by $[v(t) - M]^+ := \max(v(t) - M, 0)$ and noting that $|[u]_M|^{p-2} - M^{p-2} \leq 0$, we get

$$\frac{1}{2}\frac{d}{dt}\left\|[v(t) - M]^+\right\|_2^2 + \int_\Omega |\nabla [v(t) - M]^+|^2\, dx \leq 0. \tag{2.15}$$

Here we used the fact that

$$\begin{aligned} -\int_\Omega \Delta v [v - M]^+ dx &= \int_\Omega |\nabla [v - M]^+|^2 dx - \int_{\partial\Omega} \partial_\nu v [v - M]^+ d\sigma \\ &= \int_\Omega |\nabla [v - M]^+|^2 dx + \int_{\partial\Omega} |u|^{q-2} v [v - M]^+ d\sigma \\ &\geq \int_\Omega |\nabla [v - M]^+|^2 dx + \int_{\partial\Omega} |u|^{q-2} M [v - M]^+ d\sigma \\ &\geq \int_\Omega |\nabla [v - M]^+|^2 dx. \end{aligned}$$

Hence $\|[v(t) - M]^+\|_2 \leq \|[u_0 - M]^+\|_2 = 0$ (which is assured by (2.12)), i.e., $v(t) \leq M$ so we get $u(t) \leq M e^{M^{p-2}t}$ for a.e. $t \in [0, \infty)$.

Multiply again (2.14) by $[v(t) + M]^- = \max\,(-v(t) - M, 0)$. Then in parallel with (2.15), we get

$$\frac{1}{2}\frac{d}{dt}\left\|[v(t) + M]^-\right\|_2^2 + \int_\Omega |\nabla [v(t) + M]^-|^2\, dx \leq 0, \tag{2.16}$$

whence follows $u(t) \geq -M e^{M^{p-2}t}$. Thus we get $|u(t)|_{L^\infty} \leq M e^{M^{p-2}t}$. In particular, we observe that $u(t) \in L^\infty$ for a.e. $t \in [0, \infty)$. Hence noticing that $|u|^{r-2}u \in L^2(\Omega)$ and $|\,|[u]_M|^{p-2}\,| \leq |u|^{p-2}$, we multiply (2.11) by $|u|^{r-2}u$ to obtain

$$\begin{aligned} &\frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + (r-1)\int_\Omega |\nabla u|^2 |u|^{r-2} dx + \int_{\partial\Omega} |u|^{q+r-2} d\sigma \\ &= \int_\Omega |[u]_M|^{p-2} |u|^r dx \\ &\leq \int_\Omega |u|^{p+r-2} dx \leq \|u(t)\|_\infty^{p-2} \|u(t)\|_r^r. \end{aligned}$$

Since the second term and third term of left hand side are nonnegative,

$$\|u(t)\|_r^{r-1}\frac{d}{dt}\|u(t)\|_r \leq \|u(t)\|_\infty^{p-2}\|u(t)\|_r^r.$$

Divide both sides by $\|u(t)\|_r^{r-1}$ and integrate with respect to t on $[0,t]$, then we get

$$\|u(t)\|_r \leq \|u_0\|_r + \int_0^t \|u(\tau)\|_\infty^{p-2}\|u(\tau)\|_r d\tau$$

Note that even though $\|u(t)\|_r^{r-1}$ attains zero, we can justify this argument by Proposition 1 in [9]. Letting r tend to ∞, we derive

$$\|u(t)\|_\infty \leq \|u_0\|_\infty + \int_0^t \|u(\tau)\|_\infty^{p-1} d\tau.$$

Hence applying Lemma 2.3 with $y_0 = \|u_0\|_\infty,\ y_1 = y_0 + 1$ and $\ell(y) = y^{p-1}$, we see that there exists $T_0 > 0$ such that

$$\|u(t)\|_\infty \leq \|u_0\|_\infty + 1 \qquad \text{a.e. } t \in [0, T_0].$$

Since $M > \|u_0\|_\infty + 1$ by (2.12), we can see that u gives a solution for (P) on $[0, T_0]$ by the definition of cut-off function $[u]_M$. □

Remark 2.5. If $\|u_0\|_\infty > 0$, then applying Lemma 2.3 with $y_0 = \|u_0\|_\infty,\ y_1 = 2\,y_0$ and $\ell(y) = y^{p-1}$ and choosing $T_0 = \frac{1}{2^p\|u_0\|_\infty^{p-2}}$, we can show

$$\|u(t)\|_\infty \leq 2\,\|u_0\|_\infty \qquad \text{a.e. } t \in [0, T_0].$$

From this observation we can deduce that the maximal existence time $T_m(u)$ of u is larger than $\frac{1}{2^p\|u_0\|_\infty^{p-2}}$, which can be sufficiently large for sufficiently small $\|u_0\|_\infty > 0$.

3 Uniform Bounds for Global Solutions

In this section, we discuss the existence of uniform bounds for global solutions of (P). In order to investigate this, we make most use of a variational structure of our problem, which can be characterized by the following functionals. Set

$$\psi(u) = \frac{1}{p}\|u\|_p^p, \tag{3.1}$$

$$J(u) = \varphi(u) - \psi(u) \tag{3.2}$$

and

$$j(u) = -\|\nabla u\|_2^2 - \int_{\partial\Omega} |u|^q d\sigma + \|u\|_p^p. \tag{3.3}$$

Let u be a global solution of (2.5) satisfying (2.4). Then multiplying (2.5) by u and $du(t)/dt$, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 = j(u(t)) \qquad \forall t \in [0, \infty) \tag{3.4}$$

and

$$\frac{d}{dt}J(u(t)) + \left\|\frac{du}{dt}(t)\right\|_2^2 = 0 \qquad a.e.\ t \in (0, \infty). \tag{3.5}$$

Hence, in particular, it is obvious that J is monotone non-increasing in $(0, \infty)$ and

$$J(u(t)) \le J_0 := J(u_0) \qquad \text{for all } \ t \ge 0. \tag{3.6}$$

Now our main theorems can be stated as follows.

Theorem 3.1. *Assume that* $q \in (1, p)$, $p \in (2, 2^*)$ *and* $u_0 \in D(\varphi)$. *Let* u *be a global strong solution of* (P) *satisfying* (2.4). *Then we have*

$$\|u(t)\|_2 \le \left[\frac{q_2\, p\, J_0\, |\Omega|^{\frac{p-2}{2}}}{p - q_2}\right]^{1/p} \qquad \forall t \ge 0, \tag{3.7}$$

$$\sup_{t\ge 0} \varphi(u(t)) < \infty, \tag{3.8}$$

where $q_2 := \max\,(2, q)$.

Theorem 3.2. *Assume that* $q \in (1, p)$, $p \in (2, 2^*)$ *and* $u_0 \in L^\infty(\Omega)$. *Let* u *be a global strong solution of* (P) *satisfying* (2.10). *Then there exists* $C_\infty = C_\infty(p, q, |\Omega|)$ *such that*

$$\|u(t)\|_2 \le C_\infty\, \|u_0\|_\infty \qquad \forall t \ge 0, \tag{3.9}$$

$$\sup_{t\ge 0} \|u(t)\|_\infty < \infty. \tag{3.10}$$

To prove these theorems, we prepare some lemmas.

Lemma 3.3. *Let* $q_2 = \max\,(2, q) < p$ *and let* u *be a global solution of* (2.5) *satisfying* (2.4). *Then we have*

$$0 \le J(u(t)) \le J_0 \qquad \forall t \ge 0, \tag{3.11}$$

$$\|u(t)\|_2 \le B_{L^2} := \left[\frac{q_2\, p\, J_0\, |\Omega|^{\frac{p-2}{2}}}{p - q_2}\right]^{1/p} \qquad \forall t \ge 0. \tag{3.12}$$

Furthermore there exists a constant C_0 depending only on p, q, J_0 and $|\Omega|$ such that

$$\sup_{t\geq 0}\int_t^{t+1}\left(\psi(u(s))^2+\varphi(u(s))^2\right)\,ds \leq C_0. \tag{3.13}$$

Proof. By (3.4), (3.3)and (3.6), we get

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_2^2 &= -2\left(\|\nabla u(t)\|_2^2+\int_{\partial\Omega}|u(t)|^q d\sigma-\|u(t)\|_p^p\right)\\
&\geq -2\left(\frac{q_2}{2}\|\nabla u(t)\|_2^2+\frac{q_2}{q}\int_{\partial\Omega}|u(t)|^q d\sigma-\frac{q_2}{p}\|u(t)\|_p^p\right)+\frac{2(p-q_2)}{p}\|u(t)\|_p^p\\
&\geq -2\,q_2\,J(u(t))+\frac{2(p-q_2)}{p}\|u(t)\|_p^p \qquad (3.14)\\
&\geq -2\,q_2\,J(u(t))+\frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \qquad (3.15)\\
&\geq -2\,q_2\,J_0+\frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \quad \forall t\in[0,\infty). \qquad (3.16)
\end{aligned}$$

Suppose that $J(u(t_1)) < 0$ for some $t_1 \in [0,\infty)$, then by (3.5) we get $J(u(t)) < 0$ for all $t \in [t_1,\infty)$, which together with (3.15) yields

$$\frac{d}{dt}\|u(t)\|_2^2 \geq \frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \qquad \forall t\in[t_1,\infty). \tag{3.17}$$

Since $p > q_2 \geq 2$ and $J(u(t_1)) < 0$ implies $\|u(t_1)\|_2 > 0$, it follows from (3.17) that $\|u(t)\|_2$ blows up in finite time, which leads to a contradiction. Thus (3.11) is derived.

Suppose now that $\|u(t_2)\|_2 > B_{L^2}$ for some $t_2 \in [0,\infty)$, i.e., $\frac{d}{dt}\|u(t_2)\|_2^2 > 0$, then $\|u(t)\|_2$ is monotone increasing in the neighborhood of $t = t_2$. Hence, by (3.16), we can easily see that

$$\frac{d}{dt}\|u(t)\|_2^2 \geq \delta := -2J_0+\frac{2(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t_2)\|_2^p > 0 \quad \forall t\in[t_2,\infty),$$

which implies that $\|u(t)\|_2$ is strictly monotone increasing and tends to ∞ as $t \to \infty$. Hence there exists $t_3 > t_2$ such that

$$\frac{d}{dt}\|u(t)\|_2^2 \geq \frac{(p-q_2)}{p}\,|\Omega|^{\frac{2-p}{2}}\,\|u(t)\|_2^p \qquad \forall t\in[t_3,\infty).$$

This leads to a contradiction as before. Thus (3.12) is verified.

Furthermore, since $d\|u(t)\|_2^2/dt = 2\,(u(t), du(t)/dt)_{L^2} \le 2\,\|u(t)\|_2\,\|du(t)/dt\|_2$, (3.5), (3.6) and (3.12) assure that $\int_t^{t+1} |d\|u(s)\|_2^2/ds|^2 ds$ is uniformly bounded. Hence, in view of (3.2) and (3.6), we can derive (3.13) from (3.14). □

As a consequence of lemma 3.3 and monotonicity of $J(u(t))$, we can conclude that

$$\lim_{t\to\infty} J(u(t)) =: J_\infty \ge 0. \tag{3.18}$$

Remark 3.4. Estimate (3.12) implies that if $J_0 = 0$, then there is no global solution of (2.5) except the trivial solution $u(t) \equiv 0$.

Lemma 3.5. *Let* $q_2 := \max\ (2, q) < p$ *and let* u *be a global solution of* (2.5) *satisfying* (2.4). *Then we have*

$$\liminf_{t\to\infty} \varphi(u(t)) \le \frac{pJ_0 + 1}{p - q_2}. \tag{3.19}$$

Proof. Suppose that

$$\liminf_{t\to\infty} \varphi(u(t)) > \frac{pJ_0 + 1}{p - q_2}.$$

Then we can see that there exists $t_0 > 0$ such that

$$\varphi(u(t)) \ge \frac{pJ_0 + 1}{p - q_2} \qquad \forall t \ge t_0. \tag{3.20}$$

By (3.4) and (3.20), it holds that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 &= j(u(t))\\ &= -\|\nabla u(t)\|_2^2 - \int_{\partial\Omega} |u(t)|^q d\sigma + \|u(t)\|_p^p\\ &\ge -\frac{q_2}{2}\|\nabla u(t)\|_2^2 - \frac{q_2}{q}\int_{\partial\Omega} |u(t)|^q d\sigma + \|u(t)\|_p^p\\ &= -q_2\varphi(u(t)) + p\psi(u(t))\\ &= -pJ(u(t)) + (p - q_2)\varphi(u(t))\\ &\ge -pJ_0 + (p - q_2)\varphi(u(t)) \ge 1 \qquad \forall t \ge t_0.\end{aligned} \tag{3.21}$$

Hence we get

$$\|u(t)\|_2^2 \ge \|u(t_0)\|_2^2 + 2(t - t_0) \qquad \forall t \ge t_0,$$

whence it follows that $\|u(t)\|_2 \to \infty$ as $t \to \infty$, which contradicts (3.12). □

Lemma 3.6. *Let $p \in (2, 2^*)$, then there exists a constant $\lambda = \lambda(N,p) \in (0,2]$ such that*

$$\|u\|_{2(p-1)}^{2(p-1)} \le C\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} \qquad \forall u \in H^2(\Omega) \tag{3.22}$$

for some $C > 0$.

Proof. First of all, if $N = 1, 2$; or $N \ge 3$ and $p \le \frac{2(N-1)}{N-2}$, then we can take $\lambda = 2$ by Sobolev's embedding $H^1(\Omega) \subset L^{2(p-1)}(\Omega)$. For the case of $N \ge 3$ and $p > \frac{2(N-1)}{N-2}$, we note that the following Gagliardo-Nirenberg inequality holds:

$$\|v\|_{2(p-1)} \le C\|v\|_{H^2(\Omega)}^{\theta}\|v\|_{\frac{2N}{N-2}}^{1-\theta} \qquad \forall v \in H^2(\Omega), \tag{3.23}$$

where $\theta \in (0,1)$ satisfies

$$\tfrac{1}{2(p-1)} = \theta\left(\tfrac{1}{2} - \tfrac{2}{N}\right) + (1-\theta)\tfrac{N-2}{2N}.$$

Then we see that $\frac{2(N-1)}{N-2} < p < \frac{2N}{N-2}$ implies $0 < \theta = \frac{(N-2)p-2N+2}{2(p-1)} < 1$ and $0 < 2(p-1)\theta = (N-2)p - 2N + 2 < 2$. Since $H^1(\Omega)$ is continuously embedded in $L^{\frac{2N}{N-2}}(\Omega)$, it follows from (3.23) that (3.22) holds with $\lambda = 2N-(N-2)p \in (0,2)$. □

Lemma 3.7. *Let $p \in (2, 2^*)$ and u be a global solution of* (P)*. Then there exists a monotone decreasing function $T_0(\cdot) : [0,\infty) \to (0,\infty)$ such that for every $t_0 > 0$*

$$\varphi(u(t)) \le \varphi(u(t_0)) + 1 \qquad \forall t \in [t_0, t_0 + T_0(\varphi(u(t_0)))].$$

Proof. Multiplying (P) by $-\Delta u = \partial\varphi(u(t))$, we get by (3.22),

$$\begin{aligned}\frac{d}{dt}\varphi(u(t)) + \|\Delta u(t)\|_2^2 &\le \int_\Omega |\Delta u||u|^{p-1}dx \\ &\le \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{2}\|u(t)\|_{2(p-1)}^{2(p-1)} \\ &\le \frac{1}{2}\|\Delta u(t)\|_2^2 + C\|u(t)\|_{H^2(\Omega)}^{2-\lambda}\|u(t)\|_{H^1(\Omega)}^{2p-4+\lambda}.\end{aligned}$$

By (2.2) and Young's inequality, for any $\eta > 0$, there exists C_η such that

$$\begin{aligned}\|u\|_{H^2(\Omega)}^{2-\lambda}\|u\|_{H^1(\Omega)}^{2p-4+\lambda} &\le \eta\,\|u\|_{H^2(\Omega)}^2 + C_\eta\|u\|_{H^1(\Omega)}^{\frac{2(2p-4+\lambda)}{\lambda}} \\ &\le \eta\,C\left(\|\Delta u\|_2^2 + \|u\|_2^2 + 1\right) + C_\eta\|u\|_{H^1(\Omega)}^{\frac{2(2p-4+\lambda)}{\lambda}} \\ &\le \eta\,C\|\Delta u\|_2^2 + M_\eta(\varphi(u)),\end{aligned}$$

where $M_\eta(\cdot)$ is a monotone increasing function on $\mathbb{R}^+$ of the form $M(s) = C_\eta(s+1)^{\frac{2p-4+\lambda}{\lambda}} + C\,\eta(s+1)$ and we used the fact that $\|u\|^2_{H^1(\Omega)} \le C(\varphi(u)+1)$, which is verified by the Poincaré-Friedrichs inequality, that is, $\|u\|_2^2 \le C(\|\nabla u\|_2^2 + \int_{\partial\Omega}|u|^q d\sigma + 1)$ holds for any $q \in (1,\infty)$. Thus, taking $\eta > 0$ sufficiently small, we obtain

$$\frac{d}{dt}\varphi(u(t)) \le M_\eta(\varphi(u(t))).$$

Hence by applying Lemma 2.3, we can conclude the claim of this lemma (cf. [13]). □

Lemma 3.8. *Let* $q_2 := \max\ (2,q) < p$ *and let* u *be a global solution of* (2.5) *satisfying* (2.4). *Then we have*

$$\limsup_{t\to\infty}\varphi(u(t)) \le \frac{pJ_0+1}{p-q_2} + 3. \tag{3.24}$$

Proof. Suppose that

$$\limsup_{t\to\infty}\varphi(u(t)) > \frac{pJ_0+1}{p-q_2} + 3.$$

Then, by (3.19) of Lemma 3.5, there exist a couple of sequences $\{t_n^i\}_{n=1}^\infty$, $\{t_n^s\}_{n=1}^\infty$ such that

$$t_n^i < t_n^s < t_{n+1}^i, \qquad t_n^i \to \infty \quad \text{as } n\to\infty, \tag{3.25}$$

$$\varphi(u(t_n^i)) = \frac{pJ_0+1}{p-q_2} + 1, \qquad \varphi(u(t_n^s)) = \frac{pJ_0+1}{p-q_2} + 3, \tag{3.26}$$

$$\varphi(u(t)) \ge \frac{pJ_0+1}{p-q_2} + 1 \qquad \forall t \in [t_n^i, t_n^s]. \tag{3.27}$$

Integrating (3.5) over $[0,t]$, we obtain

$$\int_0^t \left\|\frac{du}{d\tau}(\tau)\right\|_2^2 = J_0 - J(u(t)) \le J_0 - J_\infty.$$

Therefore $\frac{du}{dt} \in L^2(0,\infty;L^2(\Omega))$. Hence

$$\varepsilon(t) := \left\|\frac{du}{d\tau}\right\|_{L^2(t,\infty;L^2(\Omega))} \to 0 \qquad \text{as } t\to\infty. \tag{3.28}$$

In view of (3.20) and (3.27), by the same argument as for (3.21), we have

$$1 < \frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 \le \|u(t)\|_2 \left\|\frac{du}{dt}(t)\right\|_2 \qquad \forall t \in [t_n^i, t_n^s]. \tag{3.29}$$

Hence $\|u(t)\|_2^2$ is monotone increasing in $t \in [t_n^i, t_n^s]$, so we get

$$\|u(t)\|_2^2 \leq \|u(t_n^s)\|_2^2 \leq C\,(\varphi(u(t_n^s)) + 1) \qquad \forall t \in [t_n^i, t_n^s]. \tag{3.30}$$

Integrating (3.29) over $[t_n^i, t_n^s]$ and making use of (3.30), we get

$$\begin{aligned}
t_n^s - t_n^i &< \int_{t_n^i}^{t_n^s} \|u(\tau)\|_2 \Big\| \frac{du}{d\tau}(\tau) \Big\|_2 d\tau \\
&\leq C\,(\varphi(u(t_n^s)) + 1) \int_{t_n^i}^{t_n^s} \Big\| \frac{du}{d\tau}(\tau) \Big\|_2 d\tau \\
&\leq C\,(\varphi(u(t_n^s)) + 1) \left(\int_{t_n^i}^{t_n^s} \Big\| \frac{du}{d\tau}(\tau) \Big\|_2^2 d\tau \right)^{\frac{1}{2}} \left(\int_{t_n^i}^{t_n^s} d\tau \right)^{\frac{1}{2}} \\
&\leq C\Big(\frac{pJ_0 + 1}{p - q_2} + 4 \Big) \sqrt{t_n^s - t_n^i}\; \varepsilon(t_n^i).
\end{aligned}$$

Therefore from (3.28), we can derive that $t_n^s - t_n^i \to 0$ as $n \to \infty$, which contradicts Lemma 3.7 and (3.26) with a sufficiently large n. □

Now we are ready to give a proof of Theorem 3.1.

Proof of Theorem 3.1. The assertion (3.7) is nothing but (3.12) given in Lemma 3.3. By (3.24) of Lemma 3.8, there exists $T_1 > 0$ such that $\sup_{t \geq T_1} \ \varphi(u(t)) \leq \frac{p\, J_0 + 1}{p - q_2} + 4$. Since $\varphi(u(t))$ is continuous on $[0, \infty)$, we have $\sup_{0 \leq t \leq T_1} \ \varphi(u(t)) < \infty$. Hence (3.8) is verified. □

In order to discuss the uniform bounds of solutions in $L^\infty(\Omega)$, we prepare the following device, which is a variant of results by Alikakos [1] and Nakao [10]. Its proof is given in Appendix and can be done along essentially the same lines in the proof of Lemma 3.1 in [10].

Lemma 3.9. *Let $w \in W^{1,2}_{loc}([0, \infty); L^2(\Omega)) \cap L^\infty_{loc}([0, \infty); L^\infty(\Omega) \cap H^1(\Omega))$ and assume that w satisfies*

$$\frac{d}{dt}\|w(t)\|_r^r + c_0 r^{-\theta_0} \| |w(t)|^{\frac{r}{2}} \|_{H^1(\Omega)}^2 \leq c_1 r^{\theta_1} \|w(t)\|_r^r \qquad a.e.\ t \in (0, \infty) \tag{3.31}$$

for all $r \in [2, \infty)$, where $c_0 > 0$ and c_1, θ_0, $\theta_1 \geq 0$. Then there exist some positive constants a, b, c such that

$$\sup_{t \geq 0} \|w(t)\|_\infty \leq a^{\frac{1}{2}} 2^{\theta_1 + (\theta_0 + \theta_1) b} M_0,$$

where $M_0 = \max\ (1,\ c\,\|w(0)\|_\infty,\ \sup_{t \geq 0} \|w(t)\|_2)$.

Proof of Theorem 3.2. If $\|u_0\|_\infty = 0$, then the unique solution of (P) is the trivial solution $u(t) \equiv 0$, so (3.10) is obvious. Let $\|u_0\|_\infty > 0$, then as is stated in Remark 2.5, we have

$$\|u(t)\|_\infty \le 2\,\|u_0\|_\infty \qquad a.e.\ t \in [0, T_0] \quad \text{with } T_0 = \frac{1}{2^p \|u_0\|_\infty^{p-2}}. \tag{3.32}$$

In order to apply results prepared for the proof of Theorem 3.1, we are going to derive a priori bounds for $\varphi(u(t))$. Multiplying (2.5) by u, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_2^2 + \varphi(u(t)) \le \|u(t)\|_p^p \le \|u(t)\|_\infty^p\,|\Omega|,$$

where we used the fact that $\varphi(0) = 0$ and the definition of subdifferential yield $\varphi(u) \le (\partial\varphi(u), u)_{L^2}$. Integrating this over $(0, T_0)$ and using (3.32), we obtain

$$\int_0^{T_0} \varphi(u(t))\,dt \le 2^p\,\|u_0\|_\infty^p |\Omega|\,\frac{1}{2^p\|u_0\|_\infty^{p-2}} + \frac{1}{2}\|u_0\|_\infty^2 = (|\Omega| + \frac{1}{2})\,\|u_0\|_\infty^2. \tag{3.33}$$

We now multiply (2.5) by $t\,du(t)/dt$ to get

$$t\,\|\frac{du}{dt}(t)\|_2^2 + t\,\frac{d}{dt}\varphi(u(t)) \le \frac{t}{2}\,\|\frac{du}{dt}(t)\|_2^2 + \frac{t}{2}\,\|u(t)\|_{2(p-1)}^{2(p-1)}.$$

Integrating this over $(0, T_0)$, we get

$$T_0\,\varphi(u(T_0)) \le \int_0^{T_0} \varphi(u(t))\,dt + \frac{T_0^2}{4} \sup_{0\le t\le T_0} \|u(t)\|_\infty^{2(p-1)}\,|\Omega|.$$

Hence in view of (3.32) and (3.33), we obtain

$$\begin{aligned}\varphi(u(T_0)) &\le 2^p\,\|u_0\|_\infty^{p-2}\,(|\Omega| + \frac{1}{2})\|u_0\|_\infty^2 + 2^{p-4}\,\|u_0\|_\infty^p\,|\Omega| \\ &\le 2^{p+1}\,(|\Omega| + \frac{1}{2})\,\|u_0\|_\infty^p.\end{aligned} \tag{3.34}$$

Consequently, from (3.34) and (3.12) of Lemma 3.3, we can derive

$$\sup_{T_0\le t<\infty} \|u(t)\|_2 \le \Big[\frac{q_2\,p\,|\Omega|^{\frac{p-2}{2}}\,2^{p+1}\,(|\Omega| + \frac{1}{2})}{p - q_2}\Big]^{1/p}\,\|u_0\|_\infty. \tag{3.35}$$

Hence since $\|u(t)\|_2 \le \|u(t)\|_\infty\,|\Omega|^{1/2} \le 2\,\|u_0\|_\infty\,|\Omega|^{1/2}$ for all $t \in [0, T_0]$, (3.9) is derived. In order to derive the uniform bound of solutions in $L^\infty(\Omega)$ on $[T_0, \infty)$, we rely on Lemma 3.9.

To do this, we rewrite (P) in the following way:

$$\partial_t u - \Delta u + u = |u|^{p-2}u + u. \tag{3.36}$$

Multiplying (3.36) by $|u|^{r-2}u$ ($r \geq 2$), we obtain

$$\frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r - \int_\Omega |u|^{r-2}u\Delta u dx + \|u(t)\|_r^r = \int_\Omega |u|^{p+r-2}dx + \|u(t)\|_r^r. \tag{3.37}$$

We first note that the left-hand side of (3.37), denoted by (LHS), can be estimated from below as follows:

$$\begin{aligned}
\text{(LHS)} &= \frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + (r-1)\int_\Omega |\nabla u|^2 |u|^{r-2}dx + \int_{\partial\Omega} |u|^{q+r-2}d\sigma + \|u(t)\|_r^r \\
&\geq \frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + \frac{4(r-1)}{r^2}\int_\Omega \left|\nabla |u|^{\frac{r}{2}}\right|^2 dx + \||u(t)|^{\frac{r}{2}}\|_2^2 \\
&\geq \frac{1}{r}\frac{d}{dt}\|u(t)\|_r^r + \frac{4(r-1)}{r^2}\||u(t)|^{\frac{r}{2}}\|_{H^1(\Omega)}^2,
\end{aligned}$$

Here in order to give an estimate for the right-hand side of (3.37), denoted by (RHS), we use Hölder's inequality of the following form:

$$\int_\Omega |u|^{p+r-2}dx \leq \|u\|_r^{r(1-\alpha)}\,\|u\|_p^{p-2}\,\|u\|_{\frac{sr}{2}}^{\alpha r} \quad \text{with} \quad \alpha = \tfrac{(p-2)s}{p(s-2)}. \tag{3.38}$$

This is valid for all $\alpha \in (0,1)$, which holds if and only if $p < s$. So we take $s = 2^*$ for $N = 3$ and $s = 2p$ for $N = 2$ to get

$$\|u\|_{\frac{sr}{2}}^{\alpha r} = \|\,|u|^{\frac{r}{2}}\|_s^{2\alpha} \leq C\|\,|u|^{\frac{r}{2}}\|_{H^1(\Omega)}^{2\alpha}. \tag{3.39}$$

Then, recalling that $\|u\|_p \leq C(\varphi(u)+1)^{1/2}$ which is uniformly bounded by (3.8), we obtain by (3.38) and (3.39)

$$\begin{aligned}
\text{(RHS)} &\leq \|u(t)\|_r^{r(1-\alpha)}\|u(t)\|_p^{p-2}\|u(t)\|_{\frac{sr}{2}}^{\alpha r} + \|u(t)\|_r^r \\
&\leq C\|u(t)\|_r^{r(1-\alpha)}\Big(\sup_{t\geq T_0}\varphi(u(t))+1\Big)^{\frac{p-2}{2}}\|\,|u(t)|^{\frac{r}{2}}\|_s^{2\alpha} + \|u(t)\|_r^r \\
&\leq \tfrac{2(r-1)}{r^2}\|\,|u(t)|^{\frac{r}{2}}\|_{H^1(\Omega)}^2 + C\left(\tfrac{2(r-1)}{r^2}\right)^{-\frac{\alpha}{1-\alpha}}\|u(t)\|_r^r + \|u(t)\|_r^r,
\end{aligned}$$

Thus since $\frac{r^2}{2(r-1)} \leq r$ and $\frac{2(r-1)}{r} \geq 1$ for all $r \geq 2$, from (3.37) we deduce

$$\frac{d}{dt}\|u(t)\|_r^r + \left\|\,|u(t)|^{\frac{r}{2}}\right\|_{H^1(\Omega)}^2 \leq Cr^{\frac{1}{1-\alpha}}\|u(t)\|_r^r \qquad \forall t \in [T_0,\infty). \tag{3.40}$$

Then (3.40) implies that u satisfies (3.31) with $c_0 = 1$, $c_1 = C$, $\theta_0 = 0$ and $\theta_1 = \frac{1}{1-\alpha}$. Thus the desired bound of u in $L^\infty([T_0,\infty); L^\infty(\Omega))$ is derived from Lemma 3.9 and (3.9). □

Remark 3.10. It is possible to show that the global bounds of $\varphi(u(t))$ and $\|u(t)\|_\infty$ depend only on initial data $\varphi(u_0)$ and $\|u_0\|_\infty$ (as well as on $p, q, |\Omega|$) respectively, if p satisfies the following more restrictive condition: $2 < p < 2_*$, where $2_* = \infty$ for $N = 1$ and $2_* = 2 + \frac{12}{3N-4}$ for $N \geq 2$ ($2_* < 2^*$ for $N \geq 2$, see [5] and [12]).

Appendix

We here give a proof of Lemma 3.9.

Proof of Lemma 3.9. For each $k \in \mathbb{N}$, setting

$$r_k = 2^{k+1}, \quad \alpha_k = c_1 r_k^{\theta_1}, \quad \nu_k = c_0 r_k^{-\theta_0}, \quad v = w^{2^k},$$

by (3.31), we get the following inequality

$$\frac{d}{dt}\|v(t)\|_2^2 \leq -\nu_k \|v(t)\|_{H^1(\Omega)}^2 + \alpha_k \|v(t)\|_2^2. \tag{A.1}$$

We here note that the following Gagliardo-Nirenberg interpolation inequality

$$\|v\|_2^2 \leq C\|v\|_{H^1(\Omega)}^{2\theta} \|v\|_1^{2(1-\theta)} \leq \epsilon_k \|v\|_{H^1(\Omega)}^2 + C_{\epsilon_k}\|v\|_1^2$$

holds with $\theta = \frac{N}{N+2}$. Here $C_{\epsilon_k} = C^{\frac{1}{1-\theta}} \epsilon_k^{-\frac{\theta}{1-\theta}}$ and we take $\epsilon_k > 0$ sufficiently small so that $\epsilon_k \alpha_k + \epsilon_k^2 \leq \nu_k$ and $C_{\epsilon_k} \geq 1$. Then we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |w|^{r_k} dx &\leq -\epsilon_k^2 \|v(t)\|_{H^1(\Omega)}^2 + \alpha_k C_{\epsilon_k} \|v(t)\|_1^2 \\
&\leq -\epsilon_k \|v(t)\|_2^2 + (\epsilon_k + \alpha_k)\, C_{\epsilon_k} \|v(t)\|_1^2 \\
&\leq -\epsilon_k \int_\Omega |w|^{r_k} dx + (\epsilon_k + \alpha_k)\, C_{\epsilon_k} \Big(\int_\Omega |w|^{r_{k-1}} dx\Big)^2 \\
&\leq -\epsilon_k \int_\Omega |w|^{r_k} dx + (\epsilon_k + \alpha_k)\, C_{\epsilon_k} \Big(\sup_{t\geq 0}\int_\Omega |w|^{r_{k-1}} dx\Big)^2,
\end{aligned}$$

whence follows

$$\sup_{t\geq 0}\int_\Omega |w(t)|^{r_k} dx \leq \max\Big\{\delta_k \Big(\sup_{t\geq 0}\int_\Omega |w(t)|^{r_{k-1}} dx\Big)^2, \int_\Omega |w(0)|^{r_k} dx\Big\}, \tag{A.2}$$

where $\delta_k = \frac{(\epsilon_k + \alpha_k) C_{\epsilon_k}}{\epsilon_k} \geq 1$. Indeed, it is easy to see that $y'(t) \leq -\epsilon y(t) + C$ implies $\sup_{t\geq 0} y(t) \leq \max\{\frac{C}{\epsilon}, y(0)\}$.

Then the iterative use of (A.2) gives

$$\begin{aligned}&\int_\Omega |w|^{r_k} dx \leq \delta_k\, \delta_{k-1}^2 \cdots \delta_1^{2(k-1)}\, M_0^{r_k},\\ &M_0 := \max\,(1, c\,\|w(0)\|_\infty, \sup_{t\geq 0} \|w(t)\|_2) \quad \text{with} \quad c = \max\,(1, |\Omega|).\end{aligned} \tag{A.3}$$

Set $\epsilon_k = \eta\, 2^{-(\theta_0+\theta_1)k}$ and choose $\eta > 0$ sufficiently small so that $\epsilon_k \alpha_k + \epsilon_k^2 \leq \nu_k$ and $C_{\epsilon_k} \geq 1$ are satisfied, then rewriting $C_{\epsilon_k} = C\epsilon_k^{-\gamma}$ with $\gamma = \frac{\theta}{1-\theta} > 0$, we have

$$\begin{aligned}\delta_k = \frac{(\epsilon_k + \alpha_k) C_{\epsilon_k}}{\epsilon_k} &= C(\epsilon_k + \alpha_k)\epsilon_k^{-\gamma-1}\\ &\leq C\,\nu_k\,\epsilon_k^{-\gamma-2}\\ &\leq Cc_0\,2^{-\theta_0(k+1)}\eta^{-(\gamma+2)}\,2^{(\theta_0+\theta_1)(\gamma+2)k}\\ &= C\,2^{-\theta_0}c_0\,\eta^{-(\gamma+2)}\,2^{\{\theta_1+(\theta_0+\theta_1)(\gamma+1)\}k}\\ &=: a\,2^{\{\theta_1+(\theta_0+\theta_1)b\}k},\end{aligned}$$

where we put $a = C\,2^{-\theta_0}c_0\,\eta^{-(\gamma+2)}$ and $b = \gamma + 1$. Then by virtue of (A.3) with inductive reasoning, we easily obtain

$$\|w(t)\|_{r_k} \leq a^{p_k} 2^{q_k} M_0, \tag{A.4}$$

where

$$p_k = \frac{2^k - 1}{2^{k+1}}, \qquad q_k = \frac{(2^{k+1} - k - 2)\{\theta_1 + (\theta_0 + \theta_1)b\}}{2^{k+1}}.$$

Since

$$p_k \uparrow \frac{1}{2}, \qquad q_k \uparrow \theta_1 + (\theta_0 + \theta_1)b \qquad \text{as} \quad k \uparrow \infty,$$

from (A.4) we can derive (see [13])

$$\|w(t)\|_\infty \leq a^{\frac{1}{2}} 2^{\{\theta_1+(\theta_0+\theta_1)b\}} M_0 \qquad a.e.\ t \in [0, \infty).$$

□

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Inexact infinite products of nonexpansive mappings with nonsummable errors

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Abstract: Given a sequence of nonexpansive mappings which map a closed subset of a complete metric space into the space, we study the convergence of its inexact infinite products to its common fixed point set in the case where the errors are nonsummable. Previous results in this direction concerned nonexpansive self-mappings of a complete metric space and inexact iterates with summable errors.

Keywords: Complete metric space, fixed point, inexact infinite product, nonexpansive mapping.

MSC2010: 47H09, 47H10, 54E50

1 Introduction and Preliminaries

For almost six decades now, there has been a lot of research activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [2, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 21, 22] and the references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers, in particular, the convergence of (inexact) orbits of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this area including, for example, studies of feasibility and common fixed point problems, which find important and diverse applications in the physical, medical and engineering sciences [4, 6, 17, 19, 20, 21, 22].

For instance, in [3] it was shown that if every exact orbit of a nonexpansive mapping converges to one of its fixed points, then this convergence property also holds for all its inexact orbits with summable errors. This result was established for a nonexpansive self-mapping of a complete metric space.

In the present paper we are concerned with a sequence of nonexpansive mappings which map a closed subset of a complete metric space into the space. We study the convergence of its inexact infinite products to its common fixed point set in the case where the errors are *nonsummable.* Our paper contains four results. Prototypes of the first two of them can be found in [18], where we were concerned with inexact orbits of nonexpansive mappings.

2 First result

Let (X, ρ) be a complete metric space. For each $x \in X$ and each $r > 0$, set

$$B(x, r) := \{y \in X : \ \rho(x, y) \le r\}.$$

For each $x \in X$ and each nonempty set $A \subset X$, put

$$\rho(x, A) := \inf\{\rho(x, y) : \ y \in A\}.$$

Let $K \subset X$ be a nonempty closed set and let the mappings $T_i : K \to X$, $i = 1, 2, \dots,$ satisfy

$$\rho(T_i(x), T_i(y)) \le \rho(x, y) \text{ for all } x, y \in K \text{ and all integers } i \ge 1. \tag{2.1}$$

Fix a point $\theta \in K$ and suppose that $F \subset K$ is a nonempty closed set such that

$$T_i(x) = x \text{ for all } x \in F \text{ and all integers } i \ge 1 \tag{2.2}$$

and that the following property holds:

(P1) for every positive number ϵ and every positive number M, there exists an integer $n(M, \epsilon) \ge 1$ such that if $x \in B(\theta, M)$, $p \ge 1$ is an integer and $\prod_{i=1}^{n(M,\epsilon)} T_{p+i}(x)$ exists, then

$$\rho(\prod_{i=1}^{n(M,\epsilon)} T_{p+i}(x), F) \le \epsilon.$$

Note that property (P1) indeed holds for a sequence of strict contractions and for many infinite products of nonexpansive mappings of contractive type [16].

The following theorem is our first main result.

Theorem 2.1. *Assume that a sequence* $\{x_i\}_{i=0}^{\infty} \subset K$ *is bounded,*

$$\lim_{i \to \infty} \rho(x_{i+1}, T_{i+1}(x_i)) = 0 \tag{2.3}$$

and that there exists a positive number r *such that*

$$B(x_i, r) \subset K$$

for all sufficiently large natural numbers i. *Then*

$$\lim_{i \to \infty} \rho(x_i, F) = 0.$$

Proof. By assumption, there exists a number $M > 0$ for which

$$x_i \in B(\theta, M) \text{ for every integer } i \geq 0. \tag{2.4}$$

There also exists a natural number p_0 such that

$$B(x_i, r) \subset K \text{ for all integers } i \geq p_0. \tag{2.5}$$

Let ϵ be a positive number. By property (P1), there exists an integer $n_0 \geq 1$ such that the following property holds:

(P2) for every point $x \in B(\theta, M+4)$ and every integer $p \geq 1$ such that $\prod_{i=1}^{n_0} T_{p+i}(x)$ exists, we have

$$\rho(\prod_{i=1}^{n_0} T_{p+i}(x), F) \leq \epsilon/4.$$

Choose a positive number

$$r_0 \leq n_0^{-1} \min\{2^{-1}r, 4^{-1}\epsilon\}. \tag{2.6}$$

Equation (2.3) implies that there exists an integer $n_1 \geq 1$ such that

$$\rho(x_{i+1}, T_{i+1}(x_i)) \leq r_0 \text{ for every integer } i \geq n_1. \tag{2.7}$$

Assume that

$$n \geq n_0 + n_1 + p_0 \tag{2.8}$$

is an integer. We claim that

$$\rho(x_n, F) \leq \epsilon.$$

To see this, consider first the point x_{n-n_0}. Equations (2.5) and (2.8) imply the inclusion

$$B(x_{n-n_0}, r) \subset K. \tag{2.9}$$

It follows from (2.7) and (2.8) that

$$n - n_0 \geq n_1 + p_0 \tag{2.10}$$

and that

$$\rho(x_{n-n_0+1}, T_{n-n_0+1}(x_{n-n_0})) \leq r_0. \tag{2.11}$$

In view of (2.5), (2.6) and (2.11), we have

$$B(T_{n-n_0+1}(x_{n-n_0}), r - r_0) \subset B(x_{n-n_0+1}, r) \subset K. \tag{2.12}$$

Assume that

$$k \in \{1,\dots,n_0\} \setminus \{n_0\}$$

and that for every integer $i \in \{1,\dots,k\}$,

$$\prod_{j=1}^{i} T_{j+n-n_0}(x_{n-n_0}) \in K \text{ is defined,}$$

$$\rho(x_{i+n-n_0}, \prod_{j=1}^{i} T_{j+n-n_0}(x_{n-n_0})) \le ir_0 \le 2^{-1}r \tag{2.13}$$

and

$$B(\prod_{j=1}^{i} T_{j+n-n_0}(x_{n-n_0}), r - ir_0) \subset B(x_{n-n_0+i}, r) \subset K. \tag{2.14}$$

(Note that by (2.11)–(2.13), our assumption does hold for $k = 1$.) Equations (2.1), (2.5)–(2.8) and (2.13) imply that

$$\rho(x_{k+n-n_0+1}, \prod_{j=1}^{k+1} T_{j+n-n_0}(x_{n-n_0}))$$

$$\le \rho(x_{k+n-n_0+1}, T_{k+n-n_0+1}(x_{k+n-n_0}))$$

$$+\rho(T_{k+n-n_0+1}(x_{k+n-n_0}), \prod_{j=1}^{k+1} T_{j+n-n_0}(x_{n-n_0}))$$

$$\le r_0 + \rho(x_{k+n-n_0}, \prod_{j=1}^{k} T_{j+n-n_0}(x_{n-n_0}))$$

$$\le r_0 + kr_0 \le n_0 r_0 \le 2^{-1}r$$

and

$$B(\prod_{j=1}^{k+1} T_{j+n-n_0}(x_{n-n_0}), r - (k+1)r_0) \subset B(x_{k+1+n-n_0}, r) \subset K.$$

This means that the assumption made for k also holds for $k+1$. Therefore, by using induction, we have shown that our assumption holds for $k = n_0$,

$$\prod_{j=1}^{n_0} T_{j+n-n_0}(x_{n-n_0}) \in K \text{ is defined,}$$

$$\rho(x_n, \prod_{j=1}^{n_0} T_{j+n-n_0}(x_{n-n_0})) \leq n_0 r_0 \leq 2^{-1} r \tag{2.15}$$

and

$$B(\prod_{j=1}^{n_0} T_{j+n-n_0}(x_{n-n_0}), 2^{-1} r) \subset K. \tag{2.16}$$

By property (P2), (2.4) and (2.16), we have

$$\rho(\prod_{j=1}^{n_0} T_{j+n-n_0}(x_{n-n_0}), F) \leq \epsilon/4.$$

When combined with (2.6) and (2.15), this inequality implies that

$$\rho(x_n, F) \leq n_0 r_0 + \epsilon/4 \leq \epsilon.$$

Thus for all integers $n \geq n_0 + n_1 + p_0$, we have

$$\rho(x_n, F) \leq \epsilon,$$

as claimed. Since ϵ is an arbitrary positive number, this completes the proof of Theorem 2.1. □

3 Second result

In this section we continue to use the assumptions and notation introduced in Section 2.

Theorem 3.1. *Assume that a sequence* $\{x_i\}_{i=0}^{\infty} \subset K$ *satisfies*

$$\lim_{i\to\infty} \rho(x_{i+1}, T_{i+1}(x_i)) = 0, \tag{3.1}$$

$$\liminf_{i\to\infty} \rho(x_i, X \setminus K) > 0 \tag{3.2}$$

and that it has a bounded subsequence $\{x_{i_p}\}_{p=1}^{\infty}$. *Then*

$$\liminf_{i\to\infty} \rho(x_i, F) = 0.$$

Proof. There exists a number $M > 0$ for which

$$x_{i_p} \in B(\theta, M) \text{ for every integer } p \geq 1. \tag{3.3}$$

In view of (3.2), there exist $r > 0$ and a natural number p_0 such that

$$B(x_i, r) \subset K \text{ for every integer } i \geq p_0 \tag{3.4}$$

Let ϵ be a positive number. It follows from property (P1) that there exists a natural number n_0 such that the following property holds:

(P3) for every point $x \in B(\theta, M + 4)$ and every integer $p \geq 1$ such that $\prod_{i=1}^{n_0} T_{p+i}(x)$ exists, we have

$$\rho(\prod_{i=1}^{n_0} T_{p+i}(x), F) \leq \epsilon/4.$$

Choose a positive number

$$r_0 \leq n_0^{-1} \min\{2^{-1}r, 4^{-1}\epsilon\}. \tag{3.5}$$

Equation (3.1) implies that there exists an integer $n_1 \geq 1$ for which

$$\rho(x_{i+1}, T_{i+1}(x_i)) \leq r_0 \text{ for every integer } i \geq n_1. \tag{3.6}$$

Assume now that an integer $p \geq 1$ satisfies

$$i_p \geq n_1 + p_0. \tag{3.7}$$

We claim that

$$\rho(x_{i_p+n_0}, F) \leq \epsilon.$$

Indeed, in view of (3.4) and (3.7),

$$B(x_{i_p}, r) \subset K. \tag{3.8}$$

By (3.6) and (3.7), we have

$$\rho(x_{i_p+1}, T_{i_p+1}(x_{i_p})) \leq r_0 \tag{3.9}$$

and

$$B(T_{i_p+1}(x_{i_p}), r - r_0) \subset B(x_{i_p+1}, r) \subset K. \tag{3.10}$$

Assume that

$$k \in \{1, \dots, n_0\} \setminus \{n_0\}$$

and that for each $j \in \{1, \dots, k\}$,

$$\prod_{i=1}^{j} T_{i+i_p}(x_{i_p}) \in K \text{ is defined,}$$

$$\rho(x_{i_p+j}, \prod_{i=1}^{j} T_{i+i_p}(x_{i_p})) \le jr_0 \le 2^{-1}r \tag{3.11}$$

and

$$B(\prod_{i=1}^{j} T_{i+i_p}(x_{i_p}), r - jr_0) \subset B(x_{i_p+j}, r) \subset K. \tag{3.12}$$

(Note that in view of (3.9) and (3.10) our assumption does hold for $k = 1$.) It follows from (2.1), (3.5)–(3.7) and (3.11) that

$$\rho(x_{i_p+k+1}, \prod_{i=1}^{k+1} T_{i_p+i}(x_{i_p}))$$

$$\le \rho(x_{i_p+k+1}, T_{i_p+k+1}(x_{i_p+k})) + \rho(T_{i_p+k+1}(x_{i_p+k}), \prod_{i=1}^{k+1} T_{i_p+i}(x_{i_p}))$$

$$\le r_0 + kr_0 = (k+1)r_0$$

and

$$B(\prod_{i=1}^{k+1} T_{i_p+i}(x_{i_p}), r - (k+1)r_0) \subset B(x_{i_p+k+1}, r) \subset K.$$

In other words, the assumption made for k also holds for $k+1$. Therefore, by using induction, we have shown that our assumption holds for $k = n_0$,

$$\prod_{i=1}^{n_0} T_{i_p+i}(x_{i_p}) \in K \text{ is defined,}$$

$$\rho(x_{i_p+n_0}, \prod_{i=1}^{n_0} T_{i_p+i}(x_{i_p})) \le n_0 r_0 \tag{3.13}$$

and

$$B(\prod_{i=1}^{n_0} T_{i_p+i}(x_{i_p}), 2^{-1}r) \subset K. \tag{3.14}$$

It follows from property (P3), (3.3) and (3.14) that

$$\rho(\prod_{i=1}^{n_0} T_{i_p+i}(x_{i_p}), F) \le \epsilon/4.$$

When combined with equations (3.5) and (3.13), the above inequality implies that

$$\rho(x_{i_p+n_0}, F) \le n_0 r_0 + \epsilon/4 \le \epsilon,$$

as claimed. Since the above inequality holds for all integers $p \geq 1$ such that $i_p \geq n_1 + p_0$, we conclude that

$$\liminf_{i \to \infty} \rho(x_i, F) \leq \epsilon.$$

Since ϵ is an arbitrary positive number, this completes the proof of Theorem 3.1. □

4 Third result

Theorem 4.1. *Assume that a sequence* $\{x_i\}_{i=0}^{\infty} \subset K$ *satisfies*

$$\lim_{i \to \infty} \rho(x_{i+1}, T_{i+1}(x_i)) = 0, \tag{4.1}$$

$r > 0$,

$$B(x_i, r) \subset K \tag{3.2}$$

for all sufficiently large natural numbers i *and that there exists a bounded subsequence* $\{x_{i_p}\}_{p=1}^{\infty}$ *such that*

$$\sup\{i_{p+1} - i_p : \ p = 1, 2, \dots\} < \infty \tag{4.3}$$

Then

$$\lim_{i \to \infty} \rho(x_i, F) = 0.$$

Proof. By Theorem 2.1, it suffices to show that the sequence $\{x_i\}_{i=0}^{\infty}$ is bounded.

To see this, we first note that, by assumption, there exists a number $M > 0$ such that

$$x_{i_p} \in B(\theta, M), \ p = 1, 2, \dots. \tag{4.4}$$

In view of (4.1), there exists $\Delta > 0$ such that

$$\rho(x_{i+1}, T_{i+1}(x_i)) \leq \Delta. \tag{4.5}$$

Equation (4.3) implies that there exists a natural number q such that

$$i_{p+1} < i_p + q, \ p = 1, 2 \dots. \tag{4.6}$$

Fix

$$\theta_0 \in F. \tag{4.7}$$

Let $j \geq i_1$ be an integer. By (4.6), there exists a natural number p such that

$$i_p \leq j < i_{p+1} < i_p + q. \tag{4.8}$$

Equation (4.4) implies that

$$\rho(\theta_0, x_{i_p}) \le \rho(\theta_0, \theta) + \rho(\theta, x_{i_p}) \le \rho(\theta_0, \theta) + M. \tag{4.9}$$

It follows from (2.1), (2.2), (4.5) and (4.7) that for each $i \in \{i_p, \dots, i_{p+1} - 1\}$, we have

$$\begin{aligned} \rho(\theta_0, x_{i+1}) &\le \rho(\theta_0, T_{i+1}(x_i)) + \rho(T_{i+1}(x_i), x_{i+1}) \\ &\le \rho(\theta_0, x_i) + \Delta. \end{aligned}$$

When combined with (4.8) and (4.9), this implies that

$$\rho(\theta_0, x_j) \le q\Delta + \rho(\theta_0, x_{i_p}) \le q\Delta + M + \rho(\theta_0, \theta)$$

and

$$\rho(\theta_0, x_j) \le q\Delta + M + \rho(\theta_0, \theta)$$

for all integers $j \ge i_1$. Therefore the sequence $\{x_i\}_{i=0}^{\infty}$ is indeed bounded. This completes the proof of Theorem 4.1. □

5 Fourth result

Theorem 5.1. *Assume that the set F is bounded, a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ satisfies*

$$\lim_{i\to\infty} \rho(x_{i+1}, T_{i+1}(x_i)) = 0, \tag{5.1}$$

$r > 0$,

$$B(x_i, r) \subset K \tag{5.2}$$

for all sufficiently large natural numbers i and that there exists a bounded subsequence $\{x_{i_p}\}_{p=1}^{\infty}$. Then

$$\lim_{i\to\infty} \rho(x_i, F) = 0.$$

Proof. By assumption, there exists a number $M > 0$ such that

$$F \subset B(\theta, M) \tag{5.3}$$

and

$$x_{i_p} \in B(\theta, M),\ p = 1, 2, \dots. \tag{5.4}$$

In view of (5.2), there exists a natural number q_0 such that

$$B(x_i, r) \subset K \text{ for every integer } i \ge q_0. \tag{5.5}$$

It follows from property (P1) that there also exists a natural number n_0 such that the following property holds:

(P4) for every point $x \in B(\theta, 2M+4)$ and every integer $p \geq 1$ such that $\prod_{i=1}^{n_0} T_{p+i}(x)$ exists, we have

$$\rho(\prod_{i=1}^{n_0} T_{p+i}(x), F) \leq 1.$$

Choose a positive number

$$r_0 \leq n_0^{-1} \min\{2^{-1}r, 2^{-1}\}. \tag{5.6}$$

Equation (5.1) implies that there exists an integer $n_1 \geq 1$ for which

$$\rho(x_{i+1}, T_{i+1}(x_i)) \leq r_0 \text{ for every integer } i \geq n_1. \tag{5.7}$$

Next, assume that an integer m satisfies

$$m \geq n_1 + q_0, \tag{5.8}$$

and

$$\rho(x_m, \theta) \leq 2M + 4. \tag{5.9}$$

In view of (5.5) and (5.8),

$$B(x_m, r) \subset K.$$

By (5.7) and (5.8), we have

$$\rho(x_{m+1}, T_{m+1}(x_m)) \leq r_0 \tag{5.10}$$

and

$$B(T_{m+1}(x_m), r - r_0) \subset B(x_{m+1}, r) \subset K. \tag{5.11}$$

Assume that

$$k \in \{1, \dots, n_0\} \setminus \{n_0\}$$

and that for each $j \in \{1, \dots, k\}$,

$$\prod_{i=1}^{j} T_{i+m}(x_m) \in K \text{ is defined,}$$

$$\rho(x_{m+j}, \prod_{i=1}^{j} T_{m+i}(x_m)) \leq j r_0 \leq 2^{-1} r \tag{5.12}$$

and

$$B(\prod_{i=1}^{j} T_{i+m}(x_m), r - jr_0) \subset B(x_{m+j}, r) \subset K. \tag{5.13}$$

(Note that in view of (5.10) and (5.11) our assumption does hold for $k = 1$.) It follows from (2.1), (5.7), (5.8) and (5.12) that

$$\rho(x_{m+k+1}, \prod_{i=1}^{k+1} T_{m+i}(x_m))$$

$$\leq \rho(x_{m+k+1}, T_{m+k+1}(x_{m+k})) + \rho(T_{m+k+1}(x_{m+k}), \prod_{i=1}^{k+1} T_{m+i}(x_m))$$

$$\leq r_0 + kr_0 = (k+1)r_0$$

and

$$B(\prod_{i=1}^{k+1} T_{m+i}(x_m), r - (k+1)r_0) \subset B(x_{m+k+1}, r) \subset K.$$

In other words, the assumption made for k also holds for $k+1$. Therefore, by using induction, we have shown that our assumption holds for $k = n_0$,

$$\prod_{i=1}^{n_0} T_{m+i}(x_m) \in K \text{ is defined,}$$

$$\rho(x_{m+n_0}, \prod_{i=1}^{n_0} T_{m+i}(x_m)) \leq n_0 r_0 \tag{5.14}$$

and

$$B(\prod_{i=1}^{n_0} T_{m+i}(x_m), 2^{-1}r) \subset K. \tag{5.15}$$

It now follows from property (P4), (5.9) and (5.15) that

$$\rho(\prod_{i=1}^{n_0} T_{m+i}(x_m), F) \leq 1. \tag{5.16}$$

By (5.3), (5.6), (5.14) and (5.16), we have

$$\rho(x_{m+n_0}, F) \leq \rho(x_{m+n_0}, \prod_{i=1}^{n_0} T_{m+i}(x_m)) + \rho(\prod_{i=1}^{n_0} T_{m+i}(x_m), F)$$

$$\leq n_0 r_0 + 1 \leq 2.$$

It follows from the inequality above and (5.3) that

$$\rho(x_{m+n_0}, \theta) \leq \rho(x_{m+n_0}, F) + \sup\{\rho(z, \theta) : \ z \in F\} \leq 2 + M.$$

Thus we have shown that the following property holds:

(a) if an integer $m \geq n_1 + q_0$ and $\rho(x_m, \theta) \leq 2M + 4$, then

$$\rho(x_{m+n_0}, \theta) \leq M + 2.$$

There exists an integer $p_0 \geq 1$ such that

$$i_{p_0} \geq n_1 + q_0.$$

By this inequality, (5.4) and property (a),

$$\rho(x_{i_{p_0}+kn_0}, \theta) \leq M + 2$$

for all integers $k \geq 1$. Now the assertion of the theorem follows from Theorem 4.1. □

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On t-Balancers, t-Balancing Numbers and Lucas t-Balancing Numbers

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Abstract: In this work, we determined the general terms of t-balancers, t-balancing numbers and Lucas t-balancing numbers in terms of balancing and Lucas-balancing numbers by solving the Pell equation $2x^2-y^2 = 2t^2+4t+1$ for some integer $t \geq 1$.

Keywords: balancing numbers, t-balancing numbers, Pell equation.
MSC2010: 11B37, 11B39, 11D09, 11D79.

1 Introduction

A positive integer n is called a balancing number ([2]) if the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r) \tag{1.1}$$

holds for some positive integer r which is called balancer corresponding to n. If n is a balancing number with balancer r, then from (1.1)

$$n^2=\frac{(n+r)(n+r+1)}{2} \quad \text{and} \quad r=\frac{-2n-1+\sqrt{8n^2+1}}{2}. \tag{1.2}$$

From (1.2), they noted that n is a balancing number if and only if n^2 is a triangular number and $8n^2+1$ is a perfect square. Though the definition of balancing numbers suggests that no balancing number should be less than 2. But from (1.2), Behera and Panda noted that $8(0)^2+1=1$ and $8(1)^2+1=3^2$ are perfect squares. So they accepted 0 and 1 to be balancing numbers. Let B_n denote the n^{th} balancing number. Then $B_0=0, B_1=1$, $B_2=6$ and $B_{n+1}=6B_n-B_{n-1}$ for $n \geq 2$.

Later Panda and Ray ([12]) defined that a positive integer n is called a cobalancing number if the Diophantine equation

$$1+2+\cdots+n=(n+1)+(n+2)+\cdots+(n+r) \tag{1.3}$$

holds for some positive integer r which is called cobalancer corresponding to n. If n is a cobalancing number with cobalancer r, then from (1.3)

$$n(n+1)=\frac{(n+r)(n+r+1)}{2} \quad \text{and} \quad r=\frac{-2n-1+\sqrt{8n^2+8n+1}}{2}. \tag{1.4}$$

From (1.4), they noted that n is a cobalancing number if and only if $n(n+1)$ is a triangular number and $8n^2+8n+1$ is a perfect square. Since $8(0)^2+8(0)+1=1$ is a perfect square, they accepted 0 to be a cobalancing number just like Behera and Panda accepted 0 and 1 to be balancing numbers. Let b_n denote the n^{th} cobalancing number. Then $b_0=b_1=0, b_2=2$ and $b_{n+1}=6b_n-b_{n-1}+2$ for $n\geq 2$.

It is clear from (1.1) and (1.3) that every balancing number is a cobalancer and every cobalancing number is a balancer, that is, $B_n=r_{n+1}$ and $R_n=b_n$ for $n\geq 1$, where R_n is the n^{th} the balancer and r_n is the n^{th} cobalancer. Since $R_n=b_n$, we get from (1.1) that

$$b_n=\frac{-2B_n-1+\sqrt{8B_n^2+1}}{2} \quad \text{and} \quad B_n=\frac{2b_n+1+\sqrt{8b_n^2+8b_n+1}}{2}. \tag{1.5}$$

Thus from (1.5), B_n is a balancing number if and only if $8B_n^2+1$ is a perfect square and b_n is a cobalancing number if and only if $8b_n^2+8b_n+1$ is a perfect square. Thus

$$C_n=\sqrt{8B_n^2+1} \quad \text{and} \quad c_n=\sqrt{8b_n^2+8b_n+1} \tag{1.6}$$

are integers which are called the n^{th} Lucas-balancing number and n^{th} Lucas-cobalancing number, respectively (Note that $C_0=c_0=1$).

Let $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$ be the roots of the characteristic equation for Pell numbers which are the numbers defined by $P_0=0, P_1=1$ and $P_n=2P_{n-1}+P_{n-2}$ for $n\geq 2$. Ray ([17]) derived some nice results on balancing numbers and Pell numbers his Phd thesis. Since x is a balancing number if and only if $8x^2+1$ is a perfect square, he set $8x^2+1=y^2$ for some integer $y\geq 1$. Then he get

$$y^2-8x^2=1 \tag{1.7}$$

which is a Pell equation ([1, 3, 9]). The fundamental solution of (1.7) is $(y_1,x_1)=(3,1)$. So $y_n+x_n\sqrt{8}=(3+\sqrt{8})^n$ for $n\geq 1$ and similarly $y_n-x_n\sqrt{8}=(3-\sqrt{8})^n$. Let $\gamma=3+\sqrt{8}$ and $\delta=3-\sqrt{8}$. Then he get $x_n=\frac{\gamma^n-\delta^n}{\gamma-\delta}$ which is the Binet formula for balancing numbers, that is, $B_n=\frac{\gamma^n-\delta^n}{\gamma-\delta}$. Since $\alpha^2=\gamma$ and $\beta^2=\delta$, he conclude that the Binet formula for balancing numbers is

$$B_n=\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}}.$$

Similarly

$$b_n=\frac{\alpha^{2n-1}-\beta^{2n-1}}{4\sqrt{2}}-\frac{1}{2}, C_n=\frac{\alpha^{2n}+\beta^{2n}}{2} \quad \text{and} \quad c_n=\frac{\alpha^{2n-1}+\beta^{2n-1}}{2}$$

for $n\geq 1$ (see also [4, 10, 11, 15]).

Balancing numbers and their generalizations have been investigated by several authors from many aspects. In [7], Liptai proved that there is no Fibonacci balancing number except 1 and in [8] he proved that there is no Lucas balancing number. In [19], Szalay considered the same problem and obtained some nice results by a different method. In [5], Kovács, Liptai and Olajos extended the concept of balancing numbers to the (a, b)-balancing numbers defined as follows: Let $a > 0$ and $b \geq 0$ be coprime integers. If

$$(a+b)+\cdots+(a(n-1)+b)=(a(n+1)+b)+\cdots+(a(n+r)+b)$$

for some integers $n, r \geq 1$, then $an + b$ is an (a, b)-balancing number. The sequence of (a, b)-balancing numbers is denoted by $B_m^{(a,b)}$ for $m \geq 1$. In [6], Liptai, Luca, Pintér and Szalay generalized the notion of balancing numbers to numbers defined as follows: Let $y, k, l \in \mathbb{Z}^+$ such that $y \geq 4$. Then a positive integer x with $x \leq y-2$ is called a (k, l)-power numerical center for y if $1^k+\cdots+(x-1)^k = (x+1)^l+\cdots+(y-1)^l$. They studied the number of solutions of the equation above and proved several effective and ineffective finiteness results for (k, l)-power numerical centers. For integers $k, x \geq 1$, let

$$\Pi_k(x) = x(x+1)\dots(x+k-1).$$

Then it was proved in [5] that the equation $B_m = \Pi_k(x)$ for fixed integer $k \geq 2$ has only infinitely many solutions and for $k \in \{2, 3, 4\}$ all solutions were determined. In [21] Tengely, considered the case $k = 5$, that is, $B_m = x(x+1)(x+2)(x+3)(x+4)$ and proved that this Diophantine equation has no solution for $m \geq 0$ and $x \in \mathbb{Z}$. In [14], Panda, Komatsu and Davala considered the reciprocal sums of sequences involving balancing and Lucas-balancing numbers. In [16], Patel, Irmak and Ray considered incomplete balancing and Lucas-balancing numbers and in [18], Ray considered the sums of balancing and Lucas-balancing numbers by matrix methods.

Recently, almost balancing numbers first defined by Panda and Panda in [13]. A natural number n is called an almost balancing number if the Diophantine equation

$$|[(n+1)+(n+2)+\cdots+(n+r)]-[1+2+\cdots+(n-1)]| = 1$$

holds for some positive integer r which is called the almost balancer. In [20], the first author derived some new results on almost balancing numbers, triangular numbers and square triangular numbers.

2 t-Balancing numbers.

In this section we try to determine the general terms of all t-balancers, t-balancing numbers and Lucas t-balancing numbers.

Let $t \geq 1$ be an integer. Then by considering (1.1), a positive integer n is called a t-balancing number if the Diophantine equation

$$1+2+\cdots+n-1=(n+1+t)+(n+2+t)+\cdots+(n+r+t) \tag{2.1}$$

holds for some positive integer r which is called t-balancer corresponding to n.

Let B_n^t denote the n^{th} t-balancing number and let R_n^t denote the n^{th} t-balancer. Then from (2.1), we get

$$R_n^t = \frac{-2B_n^t - 2t - 1 + \sqrt{8(B_n^t)^2 + 8tB_n^t + (2t+1)^2}}{2} \quad \text{and} \tag{2.2}$$

$$B_n^t = \frac{2R_n^t + 1 + \sqrt{8(R_n^t)^2 + 8(t+1)R_n^t + 1}}{2}. \tag{2.3}$$

From (2.2), we note that B_n^t is a t-balancing number if and only if $8(B_n^t)^2 + 8tB_n^t + (2t+1)^2$ is a perfect square. Thus

$$C_n^t = \sqrt{8(B_n^t)^2 + 8tB_n^t + (2t+1)^2} \tag{2.4}$$

is an integer which is called Lucas t-balancing number.

In order to determine the general terms of t-balancers, t-balancing numbers and Lucas t-balancing numbers, we have to determine the set of all (positive) integer solutions of the Pell equation

$$2x^2 - y^2 = 2t^2 + 4t + 1. \tag{2.5}$$

We see from (2.3) that R_n^t is a t-balancer if and only if $8(R_n^t)^2 + 8(t+1)R_n^t + 1$ is a perfect square. So we set

$$8(R_n^t)^2 + 8(t+1)R_n^t + 1 = y^2 \tag{2.6}$$

for some integer $y \geq 1$. Then $2(2R_n^t + t + 1)^2 - y^2 = 2t^2 + 4t + 1$ and putting

$$x = 2R_n^t + t + 1, \tag{2.7}$$

we get the Pell equation defined in (2.5).

Now let Δ be a non-square discriminant. Then the Δ-order O_Δ is defined to be the ring $O_\Delta = \{x + y\rho_\Delta : x, y \in \mathbb{Z}\}$, where $\rho_\Delta = \sqrt{\frac{\Delta}{4}}$ if $\Delta \equiv 0 (\text{mod } 4)$ or $\frac{1+\sqrt{\Delta}}{2}$ if $\Delta \equiv 1 (\text{mod } 4)$. So O_Δ is a subring of $\mathbb{Q}(\sqrt{\Delta}) = \{x + y\sqrt{\Delta} : x, y \in \mathbb{Q}\}$. The unit group O_Δ^u is defined to be the group of units of the ring O_Δ.

Let $F(x,y)=ax^2+bxy+cy^2$ be an indefinite integral quadratic form ([3]) of discriminant $\Delta=b^2-4ac$. Then we can rewrite $F(x,y)=((xa+y\frac{b+\sqrt{\Delta}}{2})(xa+y\frac{b-\sqrt{\Delta}}{2}))/a$. So the module M_F of F is

$$M_F=\{xa+y\frac{b+\sqrt{\Delta}}{2}:x,y\in\mathbb{Z}\}\subset\mathbb{Q}(\sqrt{\Delta}).$$

Therefore we get $(u+v\rho_\Delta)(xa+y\frac{b+\sqrt{\Delta}}{2})=x'a+y'\frac{b+\sqrt{\Delta}}{2}$, where

$$[x' \quad y']=\begin{cases}[x \quad y]\begin{bmatrix}u-\frac{b}{2}v & av\\ -cv & u+\frac{b}{2}v\end{bmatrix} & \text{if } \Delta\equiv 0(\text{mod } 4)\\ [x \quad y]\begin{bmatrix}u+\frac{1-b}{2}v & av\\ -cv & u+\frac{1+b}{2}v\end{bmatrix} & \text{if } \Delta\equiv 1(\text{mod } 4).\end{cases} \tag{2.8}$$

Let m be any integer and let Ω denote the set of all integer solutions of $F(x,y)=m$, that is, $\Omega=\{(x,y):F(x,y)=m\}$. Then there is a bijection

$$\Psi:\Omega\to\{\gamma\in M_F:N(\gamma)=am\}.$$

The action of $O^u_{\Delta,1}=\{\alpha\in O^u_\Delta:N(\alpha)=1\}$ on the set Ω is most interesting when Δ is a positive non-square since $O^u_{\Delta,1}$ is infinite. Therefore the orbit of each solution will be infinite and so the set Ω is either empty or infinite. Since $O^u_{\Delta,1}$ can be explicitly determined, the set Ω is satisfactorily described by the representation of such a list, called a set of representatives of the orbits. Let ε_Δ be the smallest unit of O_Δ that is greater than 1 and let $\tau_\Delta=\varepsilon_\Delta$ if $N(\varepsilon_\Delta)=1$ or ε^2_Δ if $N(\varepsilon_\Delta)=-1$. Then every $O^u_{\Delta,1}$ orbit of integral solutions of $F(x,y)=m$ contains a solution $(x,y)\in\mathbb{Z}\times\mathbb{Z}$ such that $0\leq y\leq U$, where $U=\left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}}(1-\frac{1}{\tau_\Delta})$ if $am>0$ or $U=\left|\frac{am\tau_\Delta}{\Delta}\right|^{\frac{1}{2}}(1+\frac{1}{\tau_\Delta})$ if $am<0$. So for finding a set of representatives of the $O^u_{\Delta,1}$ orbits of integral solutions of $F(x,y)=m$, we must find for each integer y_0 in the range $0\leq y_0\leq U$, whether Δy_0^2+4am is a perfect square or not since

$$ax_0^2+bx_0y_0+cy_0^2=m\Leftrightarrow\Delta y_0^2+4am=(2ax_0+by_0)^2. \tag{2.9}$$

If Δy_0^2+4am is a perfect square, then from (2.9) we get

$$x_0=\frac{-by_0\pm\sqrt{\Delta y_0^2+4am}}{2a}.$$

So there is a set of representatives Rep $=\{[x_0 \quad y_0]\}$. Consequently for the matrix M defined in (2.8), the set of all integer solutions of $F(x,y)=m$ is $\Omega=\{\pm(x,y):[x \quad y]=[x_0 \quad y_0]M^n,n\in\mathbb{Z}\}$. If Δy_0^2+4am is not a perfect square, then there are no integer solutions.

For the set of all integer solutions of (2.5), the indefinite form is $F(x,y)=2x^2-y^2$ of discriminant $\Delta=8$. So $\tau_8=3+2\sqrt{2}$ and

$$M=\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \tag{2.10}$$

from (2.8). Here we have two cases: $2t^2+4t+1$ is a perfect square or not for $t\geq 1$.

2.1 Case 1: $2t^2+4t+1$ is a perfect square.

In this case, we can give the following theorem first.

Theorem 2.1. *The quadratic Diophantine equation* $2t^2+4t+1=h^2$ *is satisfied for* $(t,h)=(P_{2n-1}-1,c_n)$ *for* $n\geq 2$.

Proof. Let $2t^2+4t+1=h^2$ for some integer $h\geq 1$. Then $2(t+1)^2-h^2=1$ and taking $t+1=w$, we get the Pell equation $2w^2-h^2=1$. The set of representatives is $\text{Rep}=\{[\pm 1\ \ 1]\}$ and in this case $[1\ \ -1]M^n$ generates all integer solutions (w_n,h_n) for $n\geq 1$ for $M=\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$. It can be easily seen that $M^n=\begin{bmatrix} C_n & 4B_n \\ 2B_n & C_n \end{bmatrix}$ for $n\geq 1$. So the set of all integer solutions of $2w^2-h^2=1$ is $\{(-2B_n+C_n,4B_n-C_n):n\geq 1\}$. But we notice that $-2B_n+C_n=P_{2n-1}$ and $4B_n-C_n=c_n$. So the quadratic equation $2t^2+4t+1=h^2$ is satisfied for $(t,h)=(P_{2n-1}-1,c_n)$. □

For the set of all integer solutions of (2.5) and the general terms of all t-balancers, t-balancing numbers and Lucas t-balancing numbers, we have two cases: #Rep $=4$ or #Rep >4.

Theorem 2.2. *If* #*Rep* $=4$, *then*

1. *the set of all integer solutions is* $\Omega=\{(x_{3n+1},y_{3n+1}):n\geq 0\}\cup\{(x_{3n-1},y_{3n-1}),(x_{3n},y_{3n}):n\geq 1\}$, *where*

$$\begin{aligned}
(x_{3n+1},y_{3n+1})&=(2B_n+(t+1)C_n,(4t+4)B_n+C_n)\\
(x_{3n-1},y_{3n-1})&=(-2hB_n+hC_n,4hB_n-hC_n)\\
(x_{3n},y_{3n})&=(-2B_n+(t+1)C_n,(4t+4)B_n-C_n).
\end{aligned}$$

2. *the general terms of* t*-balancers,* t*-balancing numbers and Lucas* t*-balancing numbers are*

$$\begin{aligned}
R^t_{3n}&=\frac{2B_n+(t+1)C_n-t-1}{2}\\
R^t_{3n-1}&=\frac{-2B_n+(t+1)C_n-t-1}{2}\\
R^t_{3n-2}&=\frac{-2hB_n+hC_n-t-1}{2}
\end{aligned}$$

$$\begin{aligned}
B_{3n}^t &= \frac{(4t+6)B_n+(t+2)C_n-t}{2} \\
B_{3n-1}^t &= \frac{(4t+2)B_n+tC_n-t}{2} \\
B_{3n-2}^t &= \frac{2hB_n-t}{2} \\
C_{3n}^t &= \sqrt{8(B_{3n}^t)^2+8tB_{3n}^t+(2t+1)^2} \\
C_{3n-1}^t &= \sqrt{8(B_{3n-1}^t)^2+8tB_{3n-1}^t+(2t+1)^2} \\
C_{3n-2}^t &= \sqrt{8(B_{3n-2}^t)^2+8tB_{3n-2}^t+(2t+1)^2}
\end{aligned}$$

for $n \geq 1$.

Proof. **(1)** Let #Rep = 4. Then the set of representations is

$$\text{Rep} = \{[\pm(t+1) \quad 1], [\pm h \quad h]\},$$

and in this case

1. $[t+1 \quad 1]M^n$ generates all integer solutions (x_{3n+1}, y_{3n+1}) for $n \geq 0$,
2. $[t+1 \quad -1]M^n$ generates all integer solutions (x_{3n}, y_{3n}) for $n \geq 1$,
3. $[h \quad -h]M^n$ generates all integer solutions (x_{3n-1}, y_{3n-1}) for $n \geq 1$.

Thus the set of all integer solutions is $\Omega = \{(2B_n+(t+1)C_n, (4t+4)B_n+C_n) : n \geq 0\} \cup \{(-2hB_n+hC_n, 4hB_n-hC_n), (-2B_n+(t+1)C_n, (4t+4)B_n-C_n) : n \geq 1\}$.

(2) Note that $x = 2R_n^t + t + 1$ from (2.7). So

$$R_{3n}^t = \frac{2B_n+(t+1)C_n-t-1}{2}$$

and from (2.3) and (2.6), we observe that

$$\begin{aligned}
B_{3n}^t &= \frac{2R_{3n}^t+1+\sqrt{8(R_{3n}^t)^2+8(t+1)R_{3n}^t+1}}{2} \\
&= \frac{2B_n+(t+1)C_n-t-1+1+(4t+4)B_n+C_n}{2} \\
&= \frac{(4t+6)B_n+(t+2)C_n-t}{2}
\end{aligned}$$

Thus

$$C_{3n}^t = \sqrt{8(B_{3n}^t)^2+8tB_{3n}^t+(2t+1)^2}$$

by (2.4). The other cases can be proved similarly. □

Theorem 2.3. *If* $\#Rep = 2k > 4$, *then*

1. *the set of all integer solutions is*

$$\begin{aligned}\Omega = \{&(x_{(2k-1)n+1}, y_{(2k-1)n+1}), (x_{(2k-1)n+i+1}, y_{(2k-1)n+i+1}),\\ &(x_{(2k-1)n+k}, y_{(2k-1)n+k}) : n \geq 0\} \cup\\ &\{(x_{(2k-1)n}, y_{(2k-1)n}), (x_{(2k-1)n-i}, y_{(2k-1)n-i}) : n \geq 1\},\end{aligned}$$

where

$$\begin{aligned}(x_{(2k-1)n+1}, y_{(2k-1)n+1}) &= (2B_n + (t+1)C_n, (4t+4)B_n + C_n)\\ (x_{(2k-1)n+i+1}, y_{(2k-1)n+i+1}) &= (2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n + t_{2i}C_n)\\ (x_{(2k-1)n+k}, y_{(2k-1)n+k}) &= (2hB_n + hC_n, 4hB_n + hC_n)\\ (x_{(2k-1)n}, y_{(2k-1)n}) &= (-2B_n + (t+1)C_n, (4t+4)B_n - C_n)\\ (x_{(2k-1)n-i}, y_{(2k-1)n-i}) &= (-2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n - t_{2i}C_n).\end{aligned}$$

2. *the general terms of t-balancers, t-balancing numbers and Lucas t-balancing numbers are*

$$\begin{aligned}R^t_{(2k-1)n} &= \frac{2B_n + (t+1)C_n - t - 1}{2}\\ R^t_{(2k-1)n-1} &= \frac{-2B_n + (t+1)C_n - t - 1}{2}\\ R^t_{(2k-1)n-i-1} &= \frac{-2t_{2i}B_n + t_{2i-1}C_n - t - 1}{2}\\ B^t_{(2k-1)n} &= \frac{(4t+6)B_n + (t+2)C_n - t}{2}\\ B^t_{(2k-1)n-1} &= \frac{(4t+2)B_n + tC_n - t}{2}\\ B^t_{(2k-1)n-i-1} &= \frac{(-2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} - t_{2i})C_n - t}{2}\\ C^t_{(2k-1)n} &= \sqrt{8(B^t_{(2k-1)n})^2 + 8tB^t_{(2k-1)n} + (2t+1)^2}\\ C^t_{(2k-1)n-1} &= \sqrt{8(B^t_{(2k-1)n-1})^2 + 8tB^t_{(2k-1)n-1} + (2t+1)^2}\\ C^t_{(2k-1)n-i-1} &= \sqrt{8(B^t_{(2k-1)n-i-1})^2 + 8tB^t_{(2k-1)n-i-1} + (2t+1)^2}\end{aligned}$$

for $n \geq 1$ *and*

$$\begin{aligned}R^t_{(2k-1)n+i} &= \frac{2t_{2i}B_n + t_{2i-1}C_n - t - 1}{2}\\ R^t_{(2k-1)n+k-1} &= \frac{2hB_n + hC_n - t - 1}{2}\end{aligned}$$

$$\begin{aligned}
B^t_{(2k-1)n+i} &= \frac{(2t_{2i}+4t_{2i-1})B_n+(t_{2i-1}+t_{2i})C_n-t}{2}\\
B^t_{(2k-1)n+k-1} &= \frac{6hB_n+2hC_n-t}{2}\\
C^t_{(2k-1)n+i} &= \sqrt{8(B^t_{(2k-1)n+i})^2+8tB^t_{(2k-1)n+i}+(2t+1)^2}\\
C^t_{(2k-1)n+k-1} &= \sqrt{8(B^t_{(2k-1)n+k-1})^2+8tB^t_{(2k-1)n+k-1}+(2t+1)^2}
\end{aligned}$$

for $n \geq 0$,

where t_{2i-1} *and* t_{2i} *are positive integers such that* $2t^2_{2i-1}-t^2_{2i}=2t^2+4t+1$ *for* $1 \leq i \leq k-2, t+1<t_1<t_3<\cdots<t_{2k-5}<h$ *and* $1<t_2<t_4<\cdots<t_{2k-4}<h$.

Proof. **(1)** Let #Rep > 4. Then the set of representations is

$$\text{Rep} = \{[\pm(t+1) \quad 1], [\pm t_{2i-1} \quad t_{2i}], [\pm h \quad h]\},$$

where t_{2i-1} and t_{2i} are positive integers such that $2t^2_{2i-1}-t^2_{2i}=2t^2+4t+1$ for $1 \leq i \leq k-2, t+1<t_1<t_3<\cdots<t_{2k-5}<h$ and $1<t_2<t_4<\cdots<t_{2k-4}<h$. Here

1. $[t+1 \quad 1]M^n$ generates all integer solutions $(x_{(2k-1)n+1}, y_{(2k-1)n+1})$ for $n \geq 0$,
2. $[t_{2i-1} \quad t_{2i}]M^n$ generates all integer solutions $(x_{(2k-1)n+i+1}, y_{(2k-1)n+i+1})$ for $n \geq 0$,
3. $[h \quad h]M^n$ generates all integer solutions $(x_{(2k-1)n+k}, y_{(2k-1)n+k})$ for $n \geq 0$,
4. $[t+1 \quad -1]M^n$ generates all integer solutions $(x_{(2k-1)n}, y_{(2k-1)n})$ for $n \geq 1$,
5. $[t_{2i-1} \quad -t_{2i}]M^n$ generates all integer solutions $(x_{(2k-1)n-i}, y_{(2k-1)n-i})$ for $n \geq 1$.

Thus the set of all integer solutions is $\Omega = \{(2B_n+(t+1)C_n, (4t+4)B_n+C_n), (2t_{2i}B_n+t_{2i-1}C_n, 4t_{2i-1}B_n+t_{2i}C_n), (2hB_n+hC_n, 4hB_n+hC_n) : n \geq 0\} \cup \{(-2B_n+(t+1)C_n, (4t+4)B_n-C_n), (-2t_{2i}B_n+t_{2i-1}C_n, 4t_{2i-1}B_n-t_{2i}C_n) : n \geq 1\}$.

(2) It can be proved as in the same way that Theorem 2.2 was proved. □

When #Rep $= 2k > 4$, it is impossible to determine the set of representatives and #Rep in terms of t. For example in Table 1, the set of representatives is given for some values of t. That is why we assume that the set of representatives is Rep $= \{[\pm(t+1) \quad 1], [\pm t_{2i-1} \quad t_{2i}], [\pm h \quad h]\}$, where t_{2i-1} and t_{2i} are positive integers such that $2t^2_{2i-1}-t^2_{2i}=2t^2+4t+1$ for $1 \leq i \leq k-2, t+1<t_1<t_3<\cdots<t_{2k-5}<h$ and $1<t_2<t_4<\cdots<t_{2k-4}<h$.

Table 1.

t	set of representatives
984	$\{[\pm 985 \quad 1], [\pm 995 \quad 199], [\pm 1025 \quad 401], [\pm 1267 \quad 1127], [\pm 1393 \quad 1393]\}$
5740	$\{[\pm 5741 \quad 1], [\pm 6001 \quad 2471], [\pm 6739 \quad 4991], [\pm 6805 \quad 5167], [\pm 8119 \quad 8119]\}$
33460	$\{[\pm 33461 \quad 1], [\pm 35155 \quad 15247], [\pm 38935 \quad 28153], [\pm 40409 \quad 32039], [\pm 47321 \quad 47321]\}$
195024	$\{[\pm 195025 \quad 1], [\pm 195083 \quad 6767], [\pm 195257 \quad 13457], [\pm 197005 \quad 39401], [\pm 197743 \quad 46207], [\pm 199547 \quad 59737], [\pm 202985 \quad 79601], [\pm 205933 \quad 93527], [\pm 205973 \quad 93703], [\pm 207607 \quad 100657], [\pm 209405 \quad 107849], [\pm 211327 \quad 115103], [\pm 219883 \quad 143623], [\pm 222425 \quad 151249], [\pm 227837 \quad 166583], [\pm 236623 \quad 189503], [\pm 243355 \quad 205849], [\pm 243443 \quad 206057], [\pm 246977 \quad 214303], [\pm 250747 \quad 222887], [\pm 254665 \quad 231601], [\pm 271133 \quad 266377], [\pm 275807 \quad 275807]\}$

2.2 Case 2: $2t^2 + 4t + 1$ is not a perfect square.

In this case we again two cases: $\#\mathrm{Rep} = 2$ or $\#\mathrm{Rep} > 2$.

Theorem 2.4. *If $\#Rep = 2$, then*

1. *the set of all integer solutions is $\Omega = \{(x_{2n+1}, y_{2n+1}) : n \geq 0\} \cup \{(x_{2n}, y_{2n}) : n \geq 1\}$, where*

$$\begin{aligned}(x_{2n+1}, y_{2n+1}) &= (2B_n + (t+1)C_n, (4t+4)B_n + C_n)\\ (x_{2n}, y_{2n}) &= (-2B_n + (t+1)C_n, (4t+4)B_n - C_n).\end{aligned}$$

2. *the general terms of t-balancers, t-balancing numbers and Lucas t-balancing numbers are*

$$\begin{aligned}R^t_{2n} &= \frac{2B_n + (t+1)C_n - t - 1}{2}\\ R^t_{2n-1} &= \frac{-2B_n + (t+1)C_n - t - 1}{2}\\ B^t_{2n} &= \frac{t(c_{n+1} - 1) + 2B_{n+1}}{2}\end{aligned}$$

$$\begin{aligned} B^t_{2n-1} &= \frac{t(c_{n+1}-1)+2B_n}{2} \\ C^t_{2n} &= \sqrt{8(B^t_{2n})^2+8tB^t_{2n}+(2t+1)^2} \\ C^t_{2n-1} &= \sqrt{8(B^t_{2n-1})^2+8tB^t_{2n-1}+(2t+1)^2} \end{aligned}$$

for $n \geq 1$.

Proof. **(1)** Let #Rep $= 2$. Then the set of representatives is

$$\text{Rep} = \{[\pm(t+1) \quad 1]\}.$$

In this case $[t+1 \quad 1]M^n$ generates all integer solutions (x_{2n+1}, y_{2n+1}) for $n \geq 0$ and $[t+1 \quad -1]M^n$ generates all integer solutions (x_{2n}, y_{2n}) for $n \geq 1$. Thus the set of all integer solutions is $\Omega = \{(2B_n+(t+1)C_n, (4t+4)B_n+C_n) : n \geq 0\} \cup \{(-2B_n+(t+1)C_n, (4t+4)B_n - C_n) : n \geq 1\}$.

(2) From (1), we observe that

$$R^t_{2n} = \frac{2B_n+(t+1)C_n-t-1}{2}.$$

Hence from (2.3) and (2.6), we get

$$\begin{aligned} B^t_{2n} &= \frac{2R^t_{2n}+1+\sqrt{8(R^t_{2n})^2+8(t+1)R^t_{2n}+1}}{2} \\ &= \frac{2B_n+(t+1)C_n-t-1+1+(4t+4)B_n+C_n}{2} \\ &= \frac{t(4B_n+C_n-1)+6B_n+2C_n}{2} \\ &= \frac{t\left(4(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}})+\frac{\alpha^{2n}+\beta^{2n}}{2}-1\right)+6(\frac{\alpha^{2n}-\beta^{2n}}{4\sqrt{2}})+2(\frac{\alpha^{2n}+\beta^{2n}}{2})}{2} \\ &= \frac{t\left(\alpha^{2n}(\frac{1}{\sqrt{2}}+\frac{1}{2})+\beta^{2n}(\frac{-1}{\sqrt{2}}+\frac{1}{2})-1\right)+\alpha^{2n}(\frac{3}{2\sqrt{2}}+1)+\beta^{2n}(\frac{-3}{2\sqrt{2}}+1)}{2} \\ &= \frac{t(\frac{\alpha^{2n+1}+\beta^{2n+1}}{2}-1)+2(\frac{\alpha^{2n+2}-\beta^{2n+2}}{4\sqrt{2}})}{2} \\ &= \frac{t(c_{n+1}-1)+2B_{n+1}}{2}. \end{aligned}$$

Thus

$$C^t_{2n} = \sqrt{8(B^t_{2n})^2+8tB^t_{2n}+(2t+1)^2}$$

by (2.4). The others can be proved similarly. □

Theorem 2.5. *If* $\#Rep = 2k > 2$*, then*

1. *the set of all integer solutions is* $\Omega = \{(x_{2kn+1}, y_{2kn+1}), (x_{2kn+i+1}, y_{2kn+i+1}) : n \geq 0\} \cup \{(x_{2kn}, y_{2kn}), (x_{2kn-i}, y_{2kn-i}) : n \geq 1\}$*, where*

$$\begin{aligned}(x_{2kn+1}, y_{2kn+1}) &= (2B_n + (t+1)C_n, (4t+4)B_n + C_n)\\(x_{2kn+i+1}, y_{2kn+i+1}) &= (2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n + t_{2i}C_n)\\(x_{2kn}, y_{2kn}) &= (-2B_n + (t+1)C_n, (4t+4)B_n - C_n)\\(x_{2kn-i}, y_{2kn-i}) &= (-2t_{2i}B_n + t_{2i-1}C_n, 4t_{2i-1}B_n - t_{2i}C_n).\end{aligned}$$

2. *the general terms of t-balancers, t-balancing numbers and Lucas t-balancing numbers are*

$$\begin{aligned}R^t_{2kn} &= \frac{2B_n + (t+1)C_n - t - 1}{2}\\R^t_{2kn-1} &= \frac{-2B_n + (t+1)C_n - t - 1}{2}\\R^t_{2kn-i-1} &= \frac{-2t_{2i}B_n + t_{2i-1}C_n - t - 1}{2}\\B^t_{2kn} &= \frac{t(c_{n+1} - 1) + 2B_{n+1}}{2}\\B^t_{2kn-1} &= \frac{t(c_{n+1} - 1) + 2B_n}{2}\\B^t_{2kn-i-1} &= \frac{(-2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} - t_{2i})C_n - t}{2}\\C^t_{2kn} &= \sqrt{8(B^t_{2kn})^2 + 8tB^t_{2kn} + (2t+1)^2}\\C^t_{2kn-1} &= \sqrt{8(B^t_{2kn-1})^2 + 8tB^t_{2kn-1} + (2t+1)^2}\\C^t_{2kn-i-1} &= \sqrt{8(B^t_{2kn-i-1})^2 + 8tB^t_{2kn-i-1} + (2t+1)^2}\end{aligned}$$

for $n \geq 1$ *and*

$$\begin{aligned}R^t_{2kn+i} &= \frac{2t_{2i}B_n + t_{2i-1}C_n - t - 1}{2}\\B^t_{2kn+i} &= \frac{(2t_{2i} + 4t_{2i-1})B_n + (t_{2i-1} + t_{2i})C_n - t}{2}\\C^t_{2kn+i} &= \sqrt{8(B^t_{2kn+i})^2 + 8tB^t_{2kn+i} + (2t+1)^2}\end{aligned}$$

for $n \geq 0$,

where t_{2i-1} *and* t_{2i} *are positive integers such that* $2t_{2i-1}^2 - t_{2i}^2 = 2t^2+4t+1$ *for* $1 \leq i \leq k-1, t+1 < t_1 < t_3 < \cdots < t_{2k-3}$ *and* $1 < t_2 < t_4 < \cdots < t_{2k-2}$.

Proof. **(1)** Let $\#\text{Rep} = 2k > 2$. Then the set of representatives is

$$\text{Rep} = \{[\pm(t+1) \quad 1], [\pm t_{2i-1} \quad t_{2i}]\},$$

where t_{2i-1} and t_{2i} are positive integers such that $2t_{2i-1}^2 - t_{2i}^2 = 2t^2+4t+1$ for $1 \leq i \leq k-1, t+1 < t_1 < t_3 < \cdots < t_{2k-3}$ and $1 < t_2 < t_4 < \cdots < t_{2k-2}$. Here

1. $[t+1 \quad 1]M^n$ generates all integer solutions (x_{2kn+1}, y_{2kn+1}) for $n \geq 0$,
2. $[t+1 \quad -1]\ M^n$ generates all integer solutions (x_{2kn}, y_{2kn}) for $n \geq 1$,
3. $[t_{2i-1} \quad t_{2i}]M^n$ generates all integer solutions $(x_{2kn+i+1}, y_{2kn+i+1})$ for $n \geq 0$,
4. $[t_{2i-1} \quad -t_{2i}]M^n$ generates all integer solutions (x_{2kn-i}, y_{2kn-i}) for $n \geq 1$.

Thus the set of all integer solutions is $\Omega = \{(2B_n+(t+1)C_n, (4t+4)B_n+C_n), (2t_{2i}B_n+t_{2i-1}C_n, 4t_{2i-1}B_n+t_{2i}C_n) : n \geq 0\} \cup \{(-2B_n+(t+1)C_n, (4t+4)B_n-C_n), (-2t_{2i}B_n+t_{2i-1}C_n, 4t_{2i-1}B_n - t_{2i}C_n) : n \geq 1\}$.

(2) It can be proved as in the same way that Theorem 2.4 was proved. □

Again when $\#\text{Rep} = 2k > 2$, it is impossible to determine the set of representatives and #Rep in terms of t. For example in Table 2, the set of representatives is given for some values of t.

Table 2.

t	set of representatives
11	$\{[\pm 12 \quad 1], [\pm 16 \quad 15]\}$
28	$\{[\pm 29 \quad 1], [\pm 41 \quad 41]\}$
43	$\{[\pm 44 \quad 1], [\pm 46 \quad 19], [\pm 56 \quad 49]\}$
57	$\{[\pm 58 \quad 1], [\pm 62 \quad 31], [\pm 74 \quad 65]\}$
36	$\{[\pm 37 \quad 1], [\pm 41 \quad 25], [\pm 43 \quad 31], [\pm 47 \quad 41]\}$
53	$\{[\pm 54 \quad 1], [\pm 56 \quad 21], [\pm 60 \quad 37], [\pm 70 \quad 63]\}$

That is why we assume that the set of representatives is $\text{Rep} = \{[\pm(t+1) \quad 1], [\pm t_{2i-1} \quad t_{2i}]\}$, where t_{2i-1} and t_{2i} are positive integers such that $2t_{2i-1}^2 - t_{2i}^2 = 2t^2+4t+1$ for $1 \leq i \leq k-1, t+1 < t_1 < t_3 < \cdots < t_{2k-3}$ and $1 < t_2 < t_4 < \cdots < t_{2k-2}$.

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Fractional differential inclusions with non instantaneous impulses

Mouffak Benchohra and Mehdi Slimane

Abstract: This paper is devoted to study the existence of solutions for a class of fractional differential inclusions with non instantaneous impulses and multivalued jump involving the Caputo fractional derivative in a Banach space. The arguments are based upon Mönch's fixed point theorem and the technique of measures of noncompactness.

Keywords: Differential inclusions, Caputo fractional derivative, impulses, multivalued jump, measure of noncompactness, fixed point, Banach space.

MSC2010: 26A33, 34A37, 34G20, 47H08.

1 Introduction

The theory of fractional differential equations and inclusions is an important branch of differential equation theory, which has an extensive physical, chemical, biological, and engineering background, and hence has been emerging as an important area of investigation in the last few decades; see the monographs of Abbas *et al.* [1, 2, 3], Atangana [11], Hilfer [28], Kilbas *et al.* [29], Podlubny [35], and Zhou [37], and the references therein. On the other hand, the theory of impulsive differential equations has undergone rapid development over the years and played a very important role in modern applied mathematical models of real processes rising in phenomena studied in physics, population dynamics, chemical technology, biotechnology and economics; see for instance the monographs by Bainov and Simeonov [17], Benchohra *et al.* [18], Lakshmikantham *et al.* [25], and Samoilenko and Perestyuk [36] and references therein. Moreover, fractional differential equations and inclusions present a natural framework for mathematical modeling of several real-world problems. The model with multivalued jump sizes arise in a control problem where we want to control the jump sizes in order to achieve given objectives. There are very few results for impulsive differential inclusions with multivalued jump operator see [6, 20, 21, 32]. However, to our knowledge, no papers exist in the literature which are devoted to impulsive fractional differential inclusions with multivalued jump operator.

In pharmacotherapy, instantaneous impulses cannot describe the dynamics of certain evolution processes. For example, when one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are a gradual and continuous process. In [4, 5, 7, 27, 34] the authors initially studied some new classes of abstract fractional differential equations with non instantaneous impulses in Banach spaces.

However, the theory for fractional differential equations in Banach spaces has yet been sufficiently developed. Recently, Benchohra *et al.* [19] applied the measure of noncompactness to a class of Caputo fractional differential equations of order$r \in (0,1]$ in a Banach space. Let E be a separable Banach space with norm $\|\cdot\|$.

In this paper, we study the following fractional differential inclusions with non instantaneous impulses

$$^cD^r y(t) \in F(t, y(t)), \quad \text{a.e. } \ t \in J_k := (s_k, t_{k+1}], \ k = 0, \ldots, m, \ 0 < r \leq 1, \tag{1.1}$$

$$y(t) \in G_k(t, y(t)), \ t \in J'_k := (t_k, s_k], \ k = 1, \ldots, m, \tag{1.2}$$

$$y(0) = y_0, \tag{1.3}$$

where $^cD^r$ is the Caputo fractional derivative, $F : [0,T] \times E \to \mathcal{P}(E)$ is a multivalued map, $G_k : (t_k, s_k] \times E \to \mathcal{P}(E), \ k = 1, \ldots, m$, is a given multivalued map, $y_0 \in E, 0 = s_0 < t_1 < s_1 < \cdots < t_m < s_m < t_{m+1} = T$.

To our knowledge no paper has been considered for impulsive fractional differential inclusions in abstract spaces. This paper fills the gap in the literature. To investigate the existence of solutions of the problem above, we use Mönch's fixed point theorem combined with the technique of measures of noncompactness, which is an important method for seeking solutions of differential equations. See Akhmerov *et al.* [9], Alvz [10], Banaś *et al.* [14, 15, 16], Guo *et al.* [24], Monch [30], Mönch and Von Harten [33].

2 Preliminaries

In this section, we first state the following definitions, lemmas and some notations. Let $J = [0,T], \ T > 0$. By $C(J,E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| : t \in J\}.$$

Let $L^1(J,E)$ be the Banach space of measurable functions $y : J \to E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt.$$

$PC(J,E)=\{y:J\to E: y\in C([0,t_1]\cup(t_k,s_k]\cup(s_k,t_{k+1}],E),\ k=1,\ldots,m$ and there exist $y(t_k^-),\ y(t_k^+), y(s_k^-)$and $y(s_k^+)\ k=1,\ldots,m$ with $y(t_k^-)=y(t_k)$and $y(s_k^-)=y(s_k)\}$.

$PC(J,E)$ is a Banach space with the norm

$$\|y\|_{PC}=\sup_{t\in J}\|y(t)\|.$$

Set

$$J'=[0,T]\setminus\cup_{k=1}^{m}(t_k,s_k].$$

Moreover, for a given set V of functions $v:J\to E$ let us denote by

$$V(t)=\{v(t),\ v\in V\},\ t\in J$$

and

$$V(J)=\{v(t),\ v\in V,\ t\in J\}.$$

$AC^1(J,E)$ is the space of continuously differentiable functions whose first derivative is absolutely continuous. We use the notations: 2^E is the collection of all subsets of E and $\mathcal{P}(E)=2^E\backslash\emptyset$.

$$\mathcal{P}_C\ (E)=\{A\subset E: A \text{ is nonempty, convex}\},$$
$$\mathcal{P}_{KC}\ (E)=\{A\subset E: A \text{ is nonempty, compact, convex}\}.$$

Let X,Y be two sets, $N:X\to Y$ a set-valued map, and $A\to Y$. We define

$$graph(N)=(x,y):x\in X, y\in N(X)\quad \text{(the graph of N)}.$$

Let $R>0$, and let

$$B=\{x\in E:\|x\|\le R\},$$

and

$$U=\{x\in C(J,E):\|x\|_\infty<R\}.$$

Clearly $\bar{U}=C(J,B)$.
For more details on multivalued maps see[12, 13, 22, 23, 31].

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.1. ([14]). Let X be a Banach space and Ω_X the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha:\Omega_X\to[0,\infty]$ defined by

$$\alpha(B)=\inf\{\epsilon>0: B\subseteq\cup_{i=1}^{n}B_i\ and\ diam(B_i)\le\epsilon\}; hereB\in\Omega_X.$$

The Kuratowski measure of noncompactness satisfies the following properties (for more details see [14])

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(b) $\alpha(B) = \alpha(\overline{B})$.

(c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

(e) $\alpha(cB) = |c|\alpha(B); c \in \mathbb{R}$.

(f) $\alpha(convB) = \alpha(B)$.

For completeness we recall the definition of Caputo derivative of fractional order.

Definition 2.2. ([29]). The fractional (arbitrary) order integral of the function $h \in L^1([0, T], E)$ of order $r \in \mathbb{R}_+$ is defined by

$$I^r h(t) = \frac{1}{\Gamma(r)} \int_0^t (t - s)^{r-1} h(s) ds,$$

where Γ is the Euler gamma function defined by $\Gamma(r) = \displaystyle\int_0^\infty t^{r-1} e^{-t} dt, \ r > 0$.

Definition 2.3. ([29]). For a function $h \in AC^n(J, E)$, the Caputo fractional-order derivative of order r of h is defined by

$$({}^c D_0^r h)(t) = \frac{1}{\Gamma(n - r)} \int_0^t (t - s)^{n-r-1} h^{(n)}(s) ds,$$

where $n = [r] + 1$.

We need the following auxiliary lemmas [29].

Lemma 2.4. *Let $r > 0$ and $h \in AC^n(J, E)$. Then the differential equation*

$${}^c D_0^r h(t) = 0, \quad \text{for a.e. } t \in J$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}, \ i = 0, 1, 2, \ldots, n - 1$, $n = [r] + 1$.

Lemma 2.5. *Let $r > 0$ and $h \in AC^n(J, E)$. Then*

$$I^{rc}D_0^r h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}, \quad \textit{for a.e. } t \in J$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [r] + 1$.

Definition 2.6. . A multivalued map $F : J \times E \to \mathcal{P}(E)$ is said to be Carathéodory if

i $t \to F(t, u)$ is measurable for each $u \in E$

ii $u \to F(t, u)$ is upper semicontinuous for a.e. $t \in J$.

For each $y \in C(J, E)$, define the set of selections of F by

$$S_{F,y} = \{f \in L^1(J, E), f(t) \in F(t, y(t)) \ \ a.e.\ t \in J\},$$

and for each $y \in C(J_k^{'}, E)$, define the set of selections of G_k by

$$S_{G_k,y} = \{g_k \in L^1(J_k^{'}, E), g_k(t) \in G_k(t, y(t)) \ \ a.e.\ t \in J_k^{'}\}$$

For our purpose we will only need the following fixed point theorem, and the important Lemma.

Theorem 2.7. *([8, 30]). Let E be a Banach space and $f \in L^1(J, E)$ countable with $|u(t)| \leq h(t)$ for a.e.$t \in J$, and every $u \in C$; where $h \in L^1(J, \mathbb{R}_+)$ then the function $\phi(t) = \alpha(C(t))$ belongs to $L^1(J, \mathbb{R}_+)$ and satisfies*

$$\alpha\left(\left\{\int_0^T u(s)ds : u \in C\right\}\right) \leq 2\int_0^T \alpha(C(s)ds.$$

Theorem 2.8. *([8, 30]) (the set-valued analog of Mönch's fixed point theorem). Let K be a closed, convex subset of a Banach space E ; U a relatively open subset of K, and $N : \bar{U} \to \mathcal{P}_C$ (K) Assume graph (N) is closed, N maps compact sets into relatively compact sets, and that for some $0 \in U$; the following two conditions are satisfied:*

$$\left(\begin{array}{c} M \subset U, M \subset conv(\{0\} \bigcup N(M)) \\ and\ \overline{M} = \overline{C}\ with\ C \subset M\ countable \end{array}\right) \Rightarrow \overline{M} compact. \tag{2.1}$$

$$x \notin \lambda N(x) for\ all\ x \in \overline{U} \backslash U,\ \ \lambda \in (0, 1). \tag{2.2}$$

Then there exists $x \in U$ with $x \in N(x)$.

Lemma 2.9. *[26] Let J be a compact real interval. Let F be a multivalued map satisfying $(H1)$ and let Θ be a linear continuous map from $L^1(J,E) \to C(J,E)$. Then the operator*

$$\Theta \circ S_{F,y} : C(J,E) \to \mathcal{P}_{KC}\ (C(J,E)), y \mapsto (\Theta \circ S_{F,y})(y) = \Theta(S_{F,y})$$

is a closed graph operator in $C(J,E) \times C(J,E)$.

3 Existence of Solutions

First of all, we define what we mean by a solution of the problem (1.1)-(1.3).

Definition 3.1. A function $y \in PC(J,E)$ is said to be a solution of (1.1)-(1.3) if there exists a function $f \in L^1(J,E)$ with $f(t) \in F(t,y(t))$, for a.e. $t \in J$ and a function $g_k \in L^1(J_k,E)$ with $g_k(t) \in G_k(t,y(t))$, for a.e. $t \in J_k^{'}$ such that

$$^cD^r y(t) = f(t), for\ a.e.\ t \in J_k, 0 < r \leq 1,$$

$$y(t) = g_k(t), for\ a.e.\ t \in J_k^{'},$$

and the function y satisfies conditions (1.3).

To prove the existence of solutions to (1.1)-(1.3), we need the following auxiliary lemmas.

Lemma 3.2. *Let $0 < r \leq 1$ and let $h : J \to E$ be measurable. Then linear problem*

$$^cD^r y(t) = h(t), \quad t \in J_k, \quad k = 0, \dots, m, \tag{3.1}$$

$$y(t) = \sigma_k(t), \quad t \in J_k^{'}, \quad k = 1, \dots, m, \tag{3.2}$$

$$y(0) = y_0, \tag{3.3}$$

has a unique solution which is given by :

$$y(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(s) ds & if\ \ t \in [0, t_1], \\ \sigma_k(t), & if\ \ t \in J_k', \ k = 1, \dots, m, \\ \sigma_k(t) + \frac{1}{\Gamma(r)} \int_{t_k}^t (t-s)^{r-1} h(s) ds, & if\ \ t \in J_k, \ k = 1, \dots, m. \end{cases} \tag{3.4}$$

We are now in a position to state and prove our existence result for the problem (1.1)–(1.3) based on Mönch's fixed point. Let us list some conditions on the functions involved in the (1.1)–(1.3).

(H1) $F : J \times E \to \mathcal{P}_{kc}(E)$ is a Carathéodory multivalued map.

(H2) There exists a function $p \in L^{\infty}(J, \mathbb{R}_+)$ such that

$$\|F(t,y))\|_{\mathcal{P}} = sup\{\|v\| : v(t) \in F(t,y)\} \leq p(t).$$

for each $(t,y) \in J \times E$.

(H3) For each bounded set $B \subset PC(J,E)$ and for each $t \in J$, we have

$$\alpha(F(t,B(t)) \leq p(t)\alpha(B(t)),$$

where $B(t) = \{u(t) : u \in B\}$.

(H4) $G_k : J \times E \to \mathcal{P}_{kc}(E)$ and there exists $c_k \in C(J, \mathbb{R}_+)$ such that

$$\|G_k(t,y))\|_{\mathcal{P}} = sup\{|u| : u(t) \in G_k(t,y)\} \leq c_k(t)\|y\|,$$

for each $y \in E$ *and* $t \in J,\ k = 1, \ldots, m.$

(H5) For each constant $d > 0$, the function $\phi \equiv 0$ is the unique solution in $PC(J;E)$ of the inequality

$$\Phi(t) \leq d \int_{s_k}^{t} (t-s)^{r-1}\Phi(s)ds,\ k = 0, \ldots, m.$$

(H6) For each $y \in PC(J;E)$, the selection functions $g_k \in S_{G_k,y}$ are uniformly continuous on J.

(H7) For each bounded set $B \subset E$ we have

$$\alpha(G_k(t,B)) \leq c_k(t)\alpha(B),\ t \in J.$$

Let

$$p^* = esssup_{t\in J} p(t),$$

$$c^* = \max_{k=1,\ldots,m} (\sup_{t \in J}(c_k(t))) < 1. \tag{3.5}$$

Remark 3.3. In (H3) and (H7), α is the Kuratowski measure of noncompactness on the space E.

Theorem 3.4. *Assume that assumptions (H1) - (H7) hold. Then the problem (1.1)–(1.3) has at least one solution J.*

Proof. Transform the problem (1.1)–(1.3) into a fixed point problem. Consider the multi-valued map $N : PC(J,E) \to \mathcal{P}\left(PC(J,E)\right)$ defined by

$$N(y)(t) = \{h \in PC(J,E) : h(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f(s)ds & \text{if } t \in [0,t_1], \\ g_k(t), & \text{if } t \in J_k', \\ g_k(s_k) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f(s))ds, & \text{if } t \in J_k, \end{cases}$$

$$f \in S_{F,y}, g_k \in S_{G_k,y}\}.$$

Clearly, the fixed points of operator N are solutions of problem (1.1)–(1.3). The proof will be given in a couple of steps.

Step 1: *N is convex for each $y \in PC(J,E)$.*
If h_1, h_2 belong to $N(y)$, then there exist $f_1, f_2 \in S_{F,y}$ and $g_{k_1}, g_{k_2} \in S_{G_k,y}$ such that for a.e.$t \in J$ we have

$$h_i(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f_i(s)ds & \text{if } t \in [0,t_1], \\ g_{k_i}(t), & \text{if } t \in J_k', \\ g_{k_i}(s_k) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f_i(s))ds, & \text{if } t \in J_k. \end{cases} \quad i = 1,2$$

Let $0 \leq \lambda \leq 1$. For each $t \in J$, we have

$$(\lambda h_1 + (1-\lambda)h_2)(t) =$$

$$\begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}(\lambda f_1 + (1-\lambda)f_2)(s)ds & \text{if } t \in [0,t_1], \\ (\lambda g_{k_1} + (1-\lambda)g_{k_2})(t), & \text{if } t \in J_k', \\ (\lambda g_{k_1} + (1-\lambda)g_{k_2})(s_k) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1}(\lambda f_1 + (1-\lambda)f_2)(s))ds, & \text{if } t \in J_k. \end{cases}$$

Since $S_{F,y}, S_{G_k,y}$ are convex (because F, G_k has convex values),we have

$$(\lambda h_1 + (1-\lambda)h_2) \in N(y).$$

Step 2: *For each compact $M \in \bar{U}$ $N(M)$ is relatively compact .*
To prove this, let $M \in \bar{U}$ be a compact set and let (h_n) be any sequence of elements $N(M)$. Since $(h_n) \in N(M)$ there exist $(y_n) \in M$, $(f_n) \in S_{F,y_n}$ and $g_{k_n} \in S_{G_k,y_n}$ such that

$$h_n(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f_n(s)ds & \text{if } t \in [0,t_1], \\ g_{k_n}(t), & \text{if } t \in J_k', \\ g_{k_n}(s_k) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f_n(s))ds, & \text{if } t \in J_k. \end{cases}$$

Using Theorem 2.7 and the properties of measure α, we obtain that

$$\alpha(\{h_n(t)\}) = \alpha\left(\begin{cases} y_0 + \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} f_n(s)ds & \text{if } t \in [0, t_1], \\ g_{k_n}(t), & \text{if } t \in J'_k, \\ g_{k_n}(s_k) + \frac{1}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1} f_n(s))ds, & \text{if } t \in J_k. \end{cases}\right)$$

If $t \in [0, t_1]$

$$\begin{aligned} \alpha(\{h_n(t)\}) &\leq \tfrac{2}{\Gamma(r)}\int_0^t \alpha\{(t-s)^{r-1} f_n(s)ds\} \\ &= \tfrac{2}{\Gamma(r)}\int_0^t (t-s)^{r-1}\alpha\{f_n(s)ds\}. \end{aligned} \tag{3.6}$$

If $t \in J_k$, we have

$$\begin{aligned} \alpha(\{h_n(t)\}) &\leq \alpha\{g_{k_n}(s_k)\} + \tfrac{2}{\Gamma(r)}\int_{s_k}^t \alpha\{(t-s)^{r-1} f_n(s)\}ds \\ &\leq \alpha\{G_k(s_k, y_n(s_k))\} + \tfrac{2}{\Gamma(r)}\int_{s_k}^t \alpha\{(t-s)^{r-1} f_n(s)\}ds \\ &\leq c_k(t)\alpha\{y_n(s_k)\} + \tfrac{2}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1}\alpha\{f_n(s)\}ds \\ &\leq c^*\alpha\{y_n(t)\} + \tfrac{2}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1}\alpha\{f_n(s)\}ds, \end{aligned} \tag{3.7}$$

if $t \in J'_k$

$$\begin{aligned} \alpha(\{h_n(t)\}) &= \alpha\{g_{k_n}(t)\} \\ &\leq \alpha\{G_k(t, y_n(t)\} \\ &\leq c_k(t)\alpha\{y_n(t)\} \\ &\leq c^*\alpha\{y_n(t)\}. \end{aligned} \tag{3.8}$$

On the other hand, by (H1)and since M is compact in $\overline{U}$, the sets $\{f_n(s), n \geq 1\}, \{y_n(t), n \geq 1\}$ are compact. Consequently, $\alpha\{f_n(s), n \geq 1\} = 0$ for a.e. $s \in J$ and $\alpha\{y_n(t), n \geq 1\} = 0$ for a.e. $t \in J$. we conclude that $\{h_n(t), n \geq 1\}$ is relatively compact in E , for each $t \in J$. In addition let τ_1 and τ_2 from J , $\tau_1 < \tau_2$. Then, for $\tau_1, \tau_2 \in J_k$, we have

$$\begin{aligned} \|h_n(\tau_2) - |h_n(\tau_1)\| &= \|\tfrac{1}{\Gamma(r)}\int_{\tau_1}^{\tau_2}((\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}) f_n(s)\|ds \\ &\leq \tfrac{1}{\Gamma(r)}\int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| p(s)ds. \end{aligned} \tag{3.9}$$

For $\tau_1, \tau_2 \in [0, t_1]$, we have

$$\begin{aligned} \|h_n(\tau_2) - |h_n(\tau_1)\| &= \|\tfrac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} ((\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}) f_n(s)\| ds. \\ &\leq \tfrac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}| p(s) ds, \end{aligned} \tag{3.10}$$

and for $\tau_1, \tau_2 \in J'_k$, we have

$$\|h_n(\tau_2) - |h_n(\tau_1)\| = \|g_{k_n}(\tau_2) - g_{k_n}(\tau_1)\|. \tag{3.11}$$

By (H6) as $\tau_1 \to \tau_2$, the right hand side of the above inequality tends to zero. This shows that $\{h_n(t), n \geq 1\}$ is equicontinuous. Consequently, $\{h_n, n \geq 1\}$ is relatively compact in $PC(J, E)$.

Step 3: *N has a closed graph.*
Let $(y_n, h_n) \in graph(N), n \geq 1\}$, with $\|y_n - y\|, \|h_n - h\| \to 0$ as $n \to \infty$. We must show that $(y, h) \in graph(N)$. $(y_n, h_n) \in graph(N)$ means that $h_n \in N(y_n)$ which means that there exists $f_n \in S_{F,y_n}$ and $g_{k_n} \in S_{G_k,y_n}$ such that, such that for each $t \in J$,

$$h_n(t) = \begin{cases} y_0 + \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f_n(s) ds & \text{if } t \in [0, t_1], \\ g_{k_n}(t), & \text{if } t \in J'_k, \\ g_{k_n}(s_k) + \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f_n(s)) ds, & \text{if } t \in J_k. \end{cases}$$

Let

$$a_n(t) = \begin{cases} y_0 & \text{if } t \in [0, t_1], \\ g_{k_n}(t), & \text{if } t \in J'_k, \\ g_{k_n}(s_k), & \text{if } t \in J_k, \end{cases}$$

and

$$a(t) = \begin{cases} y_0 & \text{if } t \in [0, t_1], \\ g_k(t), & \text{if } t \in J'_k, \\ g_k(s_k), & \text{if } t \in J_k. \end{cases}$$

We have $\|a_n - a\| \to 0$ as $n \to \infty$.
Consider the continuous linear operator

$$\Theta : L^\infty(J, E) \longrightarrow C(J, E)$$

$$f \longmapsto \Theta(f)(t) = \begin{cases} \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s) ds & \text{if } t \in [0, t_1], \\ 0, & \text{if } t \in J'_k, \\ \frac{1}{\Gamma(r)} \int_{s_k}^t (t-s)^{r-1} f(s)) ds, & \text{if } t \in J_k. \end{cases}$$

We have

$$\begin{aligned}\|(h_n-a_n)(t)-(h-a)(t)\| &= \|(h_n-h)(t)+(a-a_n)(t)\|\\ &\le \|(h_n-h)(t)\|+\|(a-a_n)(t)\|.\end{aligned}$$

Thus

$$\|(h_n-a_n)(t)-(h-a)(t)\| \longrightarrow 0 \text{ as } n\to\infty.$$

From Lemma 2.9 it follows that $\Theta\circ S_F$ is a closed graph operator. Moreover, we have

$$(h_n-a_n)(t)\in\Theta(S_{F,y_n}).$$

Since $y_n\longrightarrow y$, Lemma 2.9 implies that

$$(h_n-a_n)(t)=\begin{cases}\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}f(s)ds & \text{if } t\in[0,t_1],\\ 0, & \text{if } t\in J'_k,\\ \frac{1}{\Gamma(r)}\int_{s_k}^t(t-s)^{r-1}f(s))ds, & \text{if } t\in J_k.\end{cases}$$

for some $f\in S_{F,y}$.

Step 4: *M relatively compact in $PC(J,E)$*

Let$M\subset U$, where M $\subset$conv$(\{0\}\cup N(M))$ and for some countable set $C\subset M$ let $\overline{M}=\overline{C}$. Taking into account (3.9)-(3.11) , it is easily seen that $N(M)$ is equicontinuous. Therefore, M $\subset$conv$(\{0\}\cup N(M))$ implies that M is equicontinuous. It remains to apply the Arzela-Ascoli theorem to show that for each $t\in I$ the set $M(t)$ is relatively compact. By taking into account that C is countable and $C\subset M$ $\subset$conv$(0\cup N(M))$, we can find a countable set $H=\{h_n : n\ge 1\}\subset N(M)$ such that $C\subset$ conv$(\{0\}\cup H)$. Then, there are $y_n\in M$, $g_{k_n}\in S_{G_k,y_n}$ and $f_n\in S_{F,y_n}$ with

$$h_n(t)=\begin{cases}y_0+\frac{1}{\Gamma(r)}\int_0^t(t-s)^{r-1}f_n(s)ds & \text{if } t\in[0,t_1],\\ g_{k_n}(t), & \text{if } t\in J'_k,\\ g_{k_n}(s_k)+\frac{1}{\Gamma(r)}\int_{s_k}^t(t-s)^{r-1}f_n(s))ds, & \text{if } t\in J_k.\end{cases}$$

Taking into account Theorem 2.7 and the fact that $M\subset\overline{C}\subset\overline{conv}(\{0\}\cup H)$, we obtain

$$\alpha(M(t))\le(\alpha(\overline{C}(t))\le\alpha(H(t))=\alpha\{h_n(t):n\le 1\}).$$

Using (3.6)-(3.8), we obtain for $t\in[0,t_1]$

$$\alpha(\{M(t)\})\le\frac{2}{\Gamma(r)}\int_0^t(t-s)^{r-1}\alpha\{f_n(s)ds,n\ge 1\},$$

for $t \in J_k$, we have

$$\alpha(\{M(t)\}) \leq c^*\alpha\{y_n(t)\} + \frac{2}{\Gamma(r)}\int_{s_k}^{t}(t-s)^{r-1}\alpha\{f_n(s), n \geq 1\}ds$$

,

and for $t \in J'_k$

$$\alpha(\{M(t)\}) \leq c^*\alpha\{y_n(t), n \geq 1\}.$$

Also, since $f_n \in S_{F,y_n}$ and $y_n(s) \in M(s)$, then from (H3) we have

$$(t-s)^{r-1}\alpha\{f_n(s)ds, n \geq 1\} = (t-s)^{r-1}p(s)\alpha(M(s))ds.$$

It follows that if $t \in [0, t_1]$, we have

$$\begin{aligned}\alpha(\{M(t)\}) &\leq \frac{2p^*}{\Gamma(r)}\int_0^t (t-s)^{r-1}\alpha(M(s))ds,\\ &\leq \frac{2p^*}{(1-c^*)\Gamma(r)}\int_0^t (t-s)^{r-1}\alpha(M(s))ds,\end{aligned}$$

if $t \in J_k$, we have

$$\alpha(\{M(t)\}) \leq \frac{2p^*}{(1-c^*)\Gamma(r)}\int_{s_k}^t (t-s)^{r-1}\alpha(M(s))ds$$

,

if $t \in J'_k$

$$\alpha(\{M(t)\}) \leq c^*\alpha(M(t)) \Rightarrow (1-c^*)\alpha(M(t)) \leq 0.$$

Consequently, if $t \in [0, t_1] \cup J_k$, we have by (H5), the function Φ given by $\Phi(t) = \alpha(M(t))$ satisfies $\Phi \equiv 0$; that is, $\alpha(M(t)) = 0$ for all $t \in J$. Now, by the Arzela-Ascoli theorem, M is relatively compact in $PC(J, E)$.

Step 5: *A priori estimate.*
Let $y \in PC(J, E)$ be such that $y \in \lambda N(y)$ for some $\lambda \in (0, 1)$. Then for each $t \in J$ we have

$$y(t) = \begin{cases} \lambda y_0 + \frac{\lambda}{\Gamma(r)}\int_0^t (t-s)^{r-1}f(s)ds & \text{if } t \in [0, t_1],\\ \lambda g_{n_k}(t), & \text{if } t \in J'_k,\\ \lambda g_{n_k}(s_k) + \frac{\lambda}{\Gamma(r)}\int_{s_k}^t (t-s)^{r-1}f(s)ds, & \text{if } t \in J_k,\end{cases}$$

for some $f \in S_{F,y}$. On the other hand we have,

$$\begin{aligned}\|y(t)\| &\leq \|g_{n_k}(t)\| + \|y_0\| + \frac{1}{\Gamma(r)}\int_{s_k}^{t_{k+1}} (t-s)^{r-1}\|f(s)\|ds \\ &\leq c_k(t)\|y(t)\| + \|y_0\| + \frac{1}{\Gamma(r)}\int_{s_k}^{t_{k+1}} (t-s)^{r-1}p(s)ds \\ &\leq c_*\|y(t)\| + \|y_0\| + \frac{p^*T^r}{\Gamma(r+1)} \\ &\leq c^*\|y(t)\| + \|y_0\| + \frac{p^*T^r}{\Gamma(r+1)}.\end{aligned}$$

Then

$$\|y\| \leq \frac{1}{1-c^*}\left(\|y_0\| + \frac{p^*T^r}{\Gamma(r+1)}\right) := d.$$

Set

$$U = \{y \in PC(J,E) : \|y\| < d+1\}.$$

Condition (2.2) is satisfied by our choice of the open set U. From Theorem 3.4, we conclude that N has at least one fixed point $y \in PC(J,E)$ being a solution of problem (1.1)-(1.3).

4 An Example

Let us consider the following problem fractional differential inclusions with non instantaneous impulses,

$$^cD^{\frac{1}{2}}y_n(t) \in \frac{1}{(9+n+e^t)(1+\|y(t)\|)}[y_n(t)-1, y_n(t)], \text{ for each } t \in \left(0, \frac{1}{3}\right] \cup \left(\frac{1}{2}, 1\right], \tag{4.1}$$

$$y_n(t) \in \frac{\sin|y_n(t)|}{4+n+e^t}[y_n(t), y_n(t)+1], t \in \left(\frac{1}{3}, \frac{1}{2}\right], \tag{4.2}$$

$$y_n(0) = 0. \tag{4.3}$$

Set

$$E = l^1 = \{y = (y_1, y_2, \ldots, y_n, \ldots,), \sum_{n=1}^{\infty}|y_n| < \infty\}.$$

E is a Banach space with the norm

$$\|y\| = \sum_{n=1}^{\infty}|y_n|.$$

Let

$$F(t,y) = (F_1(t,y), F_2(t,y), \dots, F_n(t,y), \dots),$$

$$F_n(t,y) = \frac{1}{(9+n+e^t)(1+\|y(t)\|)}[y_n(t)-1, y_n(t)],$$

and

$$G_1(t,y) = (G_{11}(t,y), G_{12}(t,y), \dots, G_{1n}(t,y), \dots),$$

$$G_{1n}(t,y) = \frac{\sin|y_n(t)|}{4+n+e^t}[y_n(t), y_n(t)+1].$$

Clearly F is closed and convex valued. For each $y \in E$ and $t \in [0,1]$, we have

$$\|F(t,y))\|_{\mathcal{P}} \leq \frac{1}{9+e^t}.$$

Clearly G is closed and convex valued. For each $y \in E$ and $t \in [0,1]$, we have

$$\|G_1(t,y))\|_{\mathcal{P}} \leq \frac{1}{4+e^t}y$$

Hence, (H2) and (H4) are satisfied with $p(t) = \frac{1}{9+e^t}$ and $c_1(t) = \frac{1}{4+e^t}$.

Therefore all conditions of Theorem 3.4 are satisfied with $p^* = \frac{1}{10}$ and $c^* = \frac{1}{5}$. Hence, the problem (4.1)-(4.3) has at least one solution.

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Noncommutative Perspectives of Operator Monotone Functions in Hilbert Spaces

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Abstract: Assume that the function $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and has the representation

$$f(t)=f(0)+bt+\int_0^\infty\frac{t\lambda}{t+\lambda}dw(\lambda),$$

where $b\geq 0$ and w is a positive measure on $(0,\infty)$. In this paper we obtained among others that

$$\begin{aligned}&\mathcal{P}_f(B,P)-\mathcal{P}_f(A,P)\\&=b(B-A)+\int_0^\infty\lambda^2\left[\int_0^1 P((1-t)A+tB+\lambda P)^{-1}(B-A)\right.\\&\left.\times((1-t)A+tB+\lambda P)^{-1}Pdt\right]dw(\lambda)\end{aligned}$$

for all A, B, $P>0$. Applications for *weighted operator geometric mean* and *relative operator entropy* are also provided.

Keywords: Operator monotone functions, Noncommutative perspectives, Weighted operator geometric mean, Relative operator entropy.

MSC2010: 47A63, 47A30, 15A60, 26D15, 26D10

1 Introduction

Consider a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$. An operator T is said to be positive (denoted by $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$ and also an operator T is said to be *strictly positive* (denoted by $T>0$) if T is positive and invertible. A real valued continuous function f on $(0,\infty)$ is said to be operator monotone if $f(A)\geq f(B)$ holds for any $A\geq B>0$.

We have the following integral representation for the power function when $t>0$, $r\in(0,1]$, see for instance [1, p. 145]

$$t^{r-1}=\frac{\sin(r\pi)}{\pi}\int_0^\infty\frac{\lambda^{r-1}}{\lambda+t}d\lambda. \tag{1.1}$$

Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln \left(\frac{u + t}{u + 1} \right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{1.2}$$

for all $t > 0$.

In 1934, K. Löwner [10] had given a definitive characterization of operator monotone functions as follows, see for instance [1, p. 144-145]:

Theorem 1.1. *A function* $f : [0, \infty) \to \mathbb{R}$ *is operator monotone in* $[0, \infty)$ *if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t + \lambda} dw(\lambda) \tag{1.3}$$

where $b \geq 0$ *and a positive measure* w *on* $(0, \infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1 + \lambda} dw(\lambda) < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : [0, \infty) \to \mathbb{R}$, $f(t) = t^\alpha$ is an operator monotone function for any $\alpha \in [0, 1]$, [9]. The function ln is also operator monotone on $[0, \infty$.

For other examples of operator monotone functions, see [7] and [8].

Let f be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $Sp\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B, A)$ by setting

$$\mathcal{P}_f(B, A) := A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B, A) = Af\left(BA^{-1}\right)$$

provided $Sp\left(BA^{-1}\right)\subset \mathring{I}$.

For any function $f:(0,\infty)\to\mathbb{R}$ the transpose $\tilde{f}$ of f is defined by

$$\tilde{f}\left(x\right)=xf\left(x^{-1}\right),\ x>0.$$

It is well known that (see for instance [12]), if $f:(0,\infty)\to\mathbb{R}$ is continuous on $(0,\infty)$, then for all A, $B>0$,

$$\mathcal{P}_{\tilde{f}}\left(A,B\right)=\mathcal{P}_{f}\left(B,A\right). \tag{1.4}$$

If f is nonnegative and operator monotone on $(0,\infty)$, then $\tilde{f}$ is operator monotone on $(0,\infty)$, see [12].

The following inequality is of interest, see [12]:

Theorem 1.2. *Assume that f is nonnegative and operator monotone on $(0,\infty)$. If $A\geq C>0$ and $B\geq D>0$, then*

$$\mathcal{P}_{f}\left(A,B\right)\geq\mathcal{P}_{f}\left(C,D\right). \tag{1.5}$$

It is well known that (see [3] and [2] or [4]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$\left(B,A\right)\mapsto\mathcal{P}_{f}\left(B,A\right)$$

defined in pairs of positive definite operators, is operator convex.

If $f_{\nu}:[0,\infty)\to[0,\infty)$, $f_{\nu}\left(t\right)=t^{\nu}$, $\nu\in[0,1]$, then

$$P_{f_{\nu}}\left(B,A\right):=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2}=:A\sharp_{\nu}B,$$

is the *weighted operator geometric mean* of the positive invertible operators A and B with the weight ν.

We define the *weighted operator arithmetic mean* by

$$A\nabla_{\nu}B:=\left(1-\nu\right)A+\nu B,\ \nu\in\left[0,1\right].$$

It is well known that the following *Young's type inequality* holds:

$$A\sharp_{\nu}B\leq A\nabla_{\nu}B$$

for any $\nu\in\left[0,1\right]$.

If we take the function $f=\ln$, then

$$P_{\ln}\left(B,A\right):=A^{1/2}\ln\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}=:S\left(A|B\right),$$

is the *relative operator entropy*, for positive invertible operators A and B.

Kamei and Fujii [5], [6] defined the *relative operator entropy* $S\left(A|B\right)$, for positive invertible operators A and B, which is a relative version of the operator entropy considered by Nakamura-Umegaki [11].

2 Main Results

We start to the following identity of interest:

Lemma 2.1. *Assume that the function* $f : [0, \infty) \to \mathbb{R}$ *is operator monotone in* $[0, \infty)$ *and has the representation (1.3). Then for all* U, $V > 0$ *we have*

$$\begin{aligned} f(V) - f(U) = {} & b(V-U) \\ & + \int_0^\infty \lambda^2 \left[\int_0^1 ((1-t)U + tV + \lambda)^{-1} \right. \\ & \left. \times (V-U)((1-t)U + tV + \lambda)^{-1} dt \right] dw(\lambda). \end{aligned} \tag{2.1}$$

Proof. Since the function $f : [0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ and has the representation (1.3), then for U, $V > 0$ we have the representation

$$f(V) - f(U) = b(V-U) + \int_0^\infty \lambda \left[V(V+\lambda)^{-1} - U(U+\lambda)^{-1} \right] dw(\lambda). \tag{2.2}$$

Observe that for $\lambda > 0$

$$\begin{aligned} & V(V+\lambda)^{-1} - U(U+\lambda)^{-1} \\ & = (V+\lambda-\lambda)(V+\lambda)^{-1} - (U+\lambda-\lambda)(U+\lambda)^{-1} \\ & = (V+\lambda)(V+\lambda)^{-1} - \lambda(V+\lambda)^{-1} - (U+\lambda)(U+\lambda)^{-1} + \lambda(U+\lambda)^{-1} \\ & = \lambda\left[(U+\lambda)^{-1} - (V+\lambda)^{-1}\right]. \end{aligned}$$

Therefore, (2.2) becomes, see also [8]

$$f(V) - f(U) = b(V-U) + \int_0^\infty \lambda^2 \left[(U+\lambda)^{-1} - (V+\lambda)^{-1} \right] dw(\lambda). \tag{2.3}$$

Let T, $S > 0$. The function $f(t) = -t^{-1}$ is operator monotonic on $(0, \infty)$, operator Gâteaux differentiable and the Gâteaux derivative is given by

$$\nabla f_T(S) := \lim_{t \to 0} \left[\frac{f(T+tS) - f(T)}{t} \right] = T^{-1}ST^{-1} \tag{2.4}$$

for T, $S > 0$.

Consider the continuous function f defined on an interval I for which the corresponding operator function is Gâteaux differentiable and for C, D selfadjoint operators with spectra in I we consider the auxiliary function defined on $[0, 1]$ by

$$f_{C,D}(t) = f((1-t)C + tD), \ t \in [0, 1].$$

If $f_{C,D}$ is Gâteaux differentiable on the segment $[C,D] := \{(1-t)\,C + tD,\ t \in [0,1]\}$, then we have, by the properties of the Bochner integral, that

$$f(D) - f(C) = \int_0^1 \frac{d}{dt}\left(f_{C,D}(t)\right) dt = \int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt. \tag{2.5}$$

If we write this equality for the function $f(t) = -t^{-1}$ and $C,\ D > 0$, then we get the representation

$$C^{-1} - D^{-1} = \int_0^1 \left((1-t)\,C + tD\right)^{-1}(D-C)\left((1-t)\,C + tD\right)^{-1} dt. \tag{2.6}$$

Now, if we replace in (2.6) $C = U + \lambda$ and $D = V + \lambda$ for $\lambda > 0$, then

$$\begin{aligned} &(U+\lambda)^{-1} - (V+\lambda)^{-1} \\ &= \int_0^1 \left((1-t)\,U + tV + \lambda\right)^{-1}(V-U)\left((1-t)\,U + tV + \lambda\right)^{-1} dt. \end{aligned} \tag{2.7}$$

By the representation (2.3), we derive (2.1). □

Theorem 2.2. *Assume that the function* $f : [0,\infty) \to \mathbb{R}$ *is operator monotone in* $[0,\infty)$ *and has the representation (1.3). Then for all* $A,\ B,\ P > 0$ *we have*

$$\begin{aligned} &\mathcal{P}_f(B,P) - \mathcal{P}_f(A,P) \\ &= b(B-A) + \int_0^\infty \lambda^2 \left[\int_0^1 P\left((1-t)\,A + tB + \lambda P\right)^{-1}(B-A)\right. \\ &\left.\times \left((1-t)\,A + tB + \lambda P\right)^{-1} P dt\right] dw(\lambda). \end{aligned} \tag{2.8}$$

Proof. If we take $V = P^{-1/2}BP^{-1/2}$ and $U = P^{-1/2}AP^{-1/2}$ in (2.1), then we get

$$\begin{aligned} &f\left(P^{-1/2}BP^{-1/2}\right) - f\left(P^{-1/2}AP^{-1/2}\right) \\ &= b\left(P^{-1/2}BP^{-1/2} - P^{-1/2}AP^{-1/2}\right) \\ &+ \int_0^\infty \lambda^2 \left[\int_0^1 \left((1-t)\,P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda\right)^{-1}\right. \\ &\times \left(P^{-1/2}BP^{-1/2} - P^{-1/2}AP^{-1/2}\right) \\ &\left.\times \left((1-t)\,P^{-1/2}AP^{-1/2} + tP^{-1/2}BP^{-1/2} + \lambda\right)^{-1} dt\right] dw(\lambda). \end{aligned} \tag{2.9}$$

Observe that

$$P^{-1/2}BP^{-1/2}-P^{-1/2}AP^{-1/2}=P^{-1/2}\left(B-A\right)P^{-1/2},$$

and

$$\left(1-t\right)P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda=P^{-1/2}\left(\left(1-t\right)A+tB+\lambda P\right)P^{-1/2},$$

which gives

$$\begin{aligned}&\left(\left(1-t\right)P^{-1/2}AP^{-1/2}+tP^{-1/2}BP^{-1/2}+\lambda\right)^{-1}\\&=P^{1/2}\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}P^{1/2}\end{aligned}$$

and by (2.9),

$$\begin{aligned}&f\left(P^{-1/2}BP^{-1/2}\right)-f\left(P^{-1/2}AP^{-1/2}\right)\\&=bP^{-1/2}\left(B-A\right)P^{-1/2}\\&+\int_0^\infty\lambda^2\left[\int_0^1P^{1/2}\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}P^{1/2}P^{-1/2}\left(B-A\right)P^{-1/2}\right.\\&\left.\times P^{1/2}\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}P^{1/2}dt\right]dw\left(\lambda\right)\\&=bP^{-1/2}\left(B-A\right)P^{-1/2}\\&+\int_0^\infty\lambda^2\left[\int_0^1P^{1/2}\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}\left(B-A\right)\right.\\&\left.\times\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}P^{1/2}dt\right]dw\left(\lambda\right).\end{aligned}\tag{2.10}$$

If we multiply both sides of (2.10) by $P^{1/2}$ we obtain the desired identity (2.8). □

Lemma 2.3. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *and has the representation (1.3). Then for all* U, $V>0$ *we have*

$$\begin{aligned}&\tilde{f}\left(V\right)-\tilde{f}\left(U\right)\\&=f\left(0\right)\left(V-U\right)+\int_0^\infty\lambda\left(\int_0^1\left(1+\lambda\left[\left(1-t\right)U+tV\right]\right)^{-1}\right.\\&\left.\times\left(V-U\right)\left(1+\lambda\left[\left(1-t\right)U+tV\right]\right)^{-1}dt\right)dw\left(\lambda\right).\end{aligned}\tag{2.11}$$

Proof. From (1.3) we have

$$f\left(t\right)=f\left(0\right)+bt+t\int_0^\infty\frac{\lambda}{t+\lambda}dw\left(\lambda\right),\ t>0.$$

If we put $\frac{1}{t}$ instead of t we get

$$\begin{aligned} f\left(\frac{1}{t}\right) &= f(0)+b\frac{1}{t}+\frac{1}{t}\int_0^\infty \frac{\lambda}{\frac{1}{t}+\lambda}dw(\lambda) \\ &= f(0)+b\frac{1}{t}+\frac{1}{t}\int_0^\infty \frac{t\lambda}{1+t\lambda}dw(\lambda) \end{aligned}$$

and by multiplication with $t>0$, we get

$$\begin{aligned} \tilde{f}(t) &= b+tf(0)+\int_0^\infty \frac{t\lambda}{1+t\lambda}dw(\lambda) \\ &= b+tf(0)+\int_0^\infty \left(1-\frac{1}{1+t\lambda}\right)dw(\lambda). \end{aligned}$$

Therefore

$$\tilde{f}(V)-\tilde{f}(U) = f(0)(V-U)+\int_0^\infty \left[(1+U\lambda)^{-1}-(1+V\lambda)^{-1}\right]dw(\lambda). \tag{2.12}$$

From (2.6) we get

$$\begin{aligned} &(1+U\lambda)^{-1}-(1+V\lambda)^{-1} \\ &= \int_0^1 ((1-t)(1+U\lambda)+t(1+V\lambda))^{-1}((1+V\lambda)-(1+U\lambda)) \\ &\times((1-t)(1+U\lambda)+t(1+V\lambda))^{-1}dt \\ &= \int_0^1 \lambda(1+\lambda[(1-t)U+tV])^{-1}(V-U)(1+\lambda[(1-t)U+tV])^{-1}dt. \end{aligned} \tag{2.13}$$

Therefore, by (2.12) we get

$$\begin{aligned} &\tilde{f}(V)-\tilde{f}(U) \\ &= f(0)(V-U)+\int_0^\infty \left[(1+U\lambda)^{-1}-(1+V\lambda)^{-1}\right]dw(\lambda) \\ &= f(0)(V-U)+\int_0^\infty \lambda\left(\int_0^1 (1+\lambda[(1-t)U+tV])^{-1}\right. \\ &\left.\times(V-U)(1+\lambda[(1-t)U+tV])^{-1}dt\right)dw(\lambda) \end{aligned} \tag{2.14}$$

and the identity (2.11) is proved. □

Theorem 2.4. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.3). Then for all* C, D, $Q>0$ *we have*

$$\begin{aligned}&\mathcal{P}_{\tilde{f}}(D,Q)-\mathcal{P}_{\tilde{f}}(C,Q) \qquad (2.15)\\ &=a(D-C)+\int_0^\infty\lambda\left(\int_0^1 Q\left[(Q+\lambda[(1-t)C+tD])\right]^{-1}(D-C)\right.\\ &\left.\times\left[(Q+\lambda[(1-t)C+tD])\right]^{-1}Qdt\right)dw(\lambda).\end{aligned}$$

Proof. If we take $V=Q^{-1/2}DQ^{-1/2}$ and $U=Q^{-1/2}CQ^{-1/2}$ in (2.11), then we get

$$\begin{aligned}&\tilde{f}\left(Q^{-1/2}DQ^{-1/2}\right)-\tilde{f}\left(Q^{-1/2}CQ^{-1/2}\right) \qquad (2.16)\\ &=f(0)\left(Q^{-1/2}DQ^{-1/2}-Q^{-1/2}CQ^{-1/2}\right)\\ &+\int_0^\infty\lambda\left(\int_0^1\left(1+\lambda\left[(1-t)Q^{-1/2}CQ^{-1/2}+tQ^{-1/2}DQ^{-1/2}\right]\right)^{-1}\right.\\ &\times\left(Q^{-1/2}DQ^{-1/2}-Q^{-1/2}CQ^{-1/2}\right)\\ &\left.\times\left(1+\lambda\left[(1-t)Q^{-1/2}CQ^{-1/2}+tQ^{-1/2}DQ^{-1/2}\right]\right)^{-1}dt\right)dw(\lambda)\end{aligned}$$

$$\begin{aligned}&=f(0)Q^{-1/2}(D-C)Q^{-1/2}\\ &+\int_0^\infty\lambda\left(\int_0^1\left[Q^{-1/2}(Q+\lambda[(1-t)C+tD])\right]^{-1}\right.\\ &\left.\times Q^{-1/2}(D-C)Q^{-1/2}\left[Q^{-1/2}(Q+\lambda[(1-t)C+tD])Q^{-1/2}\right]^{-1}dt\right)dw(\lambda)\\ &=f(0)Q^{-1/2}(D-C)Q^{-1/2}\\ &+\int_0^\infty\lambda\left(\int_0^1 Q^{1/2}\left[(Q+\lambda[(1-t)C+tD])\right]^{-1}(D-C)\right.\\ &\left.\times\left[(Q+\lambda[(1-t)C+tD])\right]^{-1}Q^{1/2}dt\right)dw(\lambda).\end{aligned}$$

If we multiply both sides by $Q^{1/2}$ we get the desired result (2.15). □

Corollary 2.5. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in*

$[0,\infty)$ *and has the representation (1.3). Then for all* C, D, $Q>0$ *we have*

$$\begin{aligned}&\mathcal{P}_f(Q,D)-\mathcal{P}_f(Q,C)\\&=f(0)(D-C)+\int_0^\infty\lambda\left(\int_0^1 Q\left[(Q+\lambda[(1-t)C+tD])\right]^{-1}(D-C)\right.\\&\left.\times\left[(Q+\lambda[(1-t)C+tD])\right]^{-1}Qdt\right)dw(\lambda).\end{aligned}\tag{2.17}$$

We also have:

Corollary 2.6. *Assume that the function* $f:(0,\infty)\to\mathbb{R}$ *is operator monotone in* $(0,\infty)$ *and has the representation (1.3). Then for all* A, B, C, $D>0$ *we have*

$$\begin{aligned}&\mathcal{P}_f(A,B)-\mathcal{P}_f(C,D)\\&=b(A-C)+f(0)(B-D)\\&+\int_0^\infty\lambda^2\left[\int_0^1 B((1-t)C+tA+\lambda B)^{-1}(A-C)\right.\\&\left.\times((1-t)C+tA+\lambda B)^{-1}Bdt\right]dw(\lambda)\\&+\int_0^\infty\lambda\left(\int_0^1 C\left[(C+\lambda[(1-t)D+tB])\right]^{-1}(B-D)\right.\\&\left.\times\left[(C+\lambda[(1-t)D+tB])\right]^{-1}Cdt\right)dw(\lambda).\end{aligned}\tag{2.18}$$

Proof. Observe that

$$\mathcal{P}_f(A,B)-\mathcal{P}_f(C,D)=\mathcal{P}_f(A,B)-\mathcal{P}_f(C,B)+\mathcal{P}_f(C,B)-\mathcal{P}_f(C,D).\tag{2.19}$$

Since, by (2.8),

$$\begin{aligned}&\mathcal{P}_f(A,B)-\mathcal{P}_f(C,B)\\&=b(A-C)+\int_0^\infty\lambda^2\left[\int_0^1 B((1-t)C+tA+\lambda B)^{-1}(A-C)\right.\\&\left.\times((1-t)C+tA+\lambda B)^{-1}Bdt\right]dw(\lambda)\end{aligned}\tag{2.20}$$

and by (2.17),

$$\begin{aligned}&\mathcal{P}_f(C,B)-\mathcal{P}_f(C,D)\\&=f(0)(B-D)+\int_0^\infty\lambda\left(\int_0^1 C\left[(C+\lambda[(1-t)D+tB])\right]^{-1}(B-D)\right.\\&\left.\times\left[(C+\lambda[(1-t)D+tB])\right]^{-1}Cdt\right)dw(\lambda),\end{aligned}\tag{2.21}$$

hence by (2.19)-(2.21) we obtain (2.18). □

As a natural consequence of the above representations, we derive the following inequalities:

Theorem 2.7. *Assume that the function* $f:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *and has the representation (1.3). If* $B\geq A>0$ *and* $P>0$, *then*

$$\mathcal{P}_f\left(B,P\right)-\mathcal{P}_f\left(A,P\right)\geq b\left(B-A\right)\geq 0 \tag{2.22}$$

and

$$\mathcal{P}_f\left(P,B\right)-\mathcal{P}_f\left(P,A\right)\geq f\left(0\right)\left(B-A\right). \tag{2.23}$$

If $A\geq C>0$ *and* $B\geq D>0$, *then*

$$\mathcal{P}_f\left(A,B\right)-\mathcal{P}_f\left(C,D\right)\geq b\left(A-C\right)+f\left(0\right)\left(B-D\right). \tag{2.24}$$

Proof. If $B-A\geq 0$, then by multiplying both sides by $\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}$ for $t\in[0,1]$ and $\lambda\geq 0$ we get

$$\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}\left(B-A\right)\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}\geq 0.$$

Also by multiplying both sides by $P>0$, we get

$$P\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}\left(B-A\right)\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}P\geq 0,$$

for $t\in[0,1]$ and $\lambda\geq 0$.

If we multiply this inequality by λ^2 integrate over $t\in[0,1]$ and over the measure $w\left(\lambda\right)$ on $[0,\infty)$ we get

$$\begin{aligned}\int_0^\infty \lambda^2\Bigg[\int_0^1 & P\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}\left(B-A\right)\\ &\times\left(\left(1-t\right)A+tB+\lambda P\right)^{-1}Pdt\Bigg]dw\left(\lambda\right)\geq 0\end{aligned}$$

and by representation (2.8) we deduce (2.22).

The inequality (2.23) follows in a similar way by (2.17). The inequality (2.24) follows by the representation (3.2). □

Remark 2.8. If $f:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ and nonnegative, then

$$\mathcal{P}_f\left(P,B\right)-\mathcal{P}_f\left(P,A\right)\geq f\left(0\right)\left(B-A\right)\geq 0, \tag{2.25}$$

if $B\geq A>0$ and $P>0$.

If f is defined on $[0,\infty)$ and nonnegative, then the inequality (2.24) improves (1.5).

3 Some Examples of Interest

We also have identities for the *weighted operator geometric mean:*

Proposition 3.1. *For all A, B, $P > 0$ and $r \in (0,1]$ we have*

$$\begin{aligned} &P\sharp_r B - P\sharp_r A \\ &= \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^{r+1} \left[\int_0^1 P\left((1-t)A + tB + \lambda P\right)^{-1} (B-A) \right. \\ &\left. \times \left((1-t)A + tB + \lambda P\right)^{-1} P dt \right] d\lambda. \end{aligned} \tag{3.1}$$

The proof follows by (2.8) and (1.1) for the measure $dw(\lambda) = \frac{\sin(r\pi)}{\pi}\lambda^{r-1}d\lambda$. The dual case follows by (2.17) and (1.1).

Proposition 3.2. *For all C, D, $Q > 0$ and $r \in (0,1]$ we have*

$$\begin{aligned} &D\sharp_r Q - C\sharp_r Q \\ &= \frac{\sin(r\pi)}{\pi} \int_0^\infty \lambda^r \left(\int_0^1 Q\left[\left(Q + \lambda\left[(1-t)C + tD\right]\right)\right]^{-1} (D-C) \right. \\ &\left. \times \left[\left(Q + \lambda\left[(1-t)C + tD\right]\right)\right]^{-1} Q dt \right) d\lambda. \end{aligned} \tag{3.2}$$

The following identity for the logarithmic function also holds:

Lemma 3.3. *For all U, $V > 0$ we have the identity:*

$$\begin{aligned} &\ln V - \ln U \\ &= \int_0^\infty \left(\int_0^1 \left(\lambda + (1-t)U + tV\right)^{-1} (V-U) \left(\lambda + (1-t)U + tV\right)^{-1} dt \right) d\lambda. \end{aligned} \tag{3.3}$$

Proof. We have from the representation of logarithm (1.2) that

$$\ln V - \ln U = \int_0^\infty \frac{1}{\lambda+1} \left[(V-1)(\lambda+V)^{-1} - (U-1)(\lambda+U)^{-1} \right] d\lambda \tag{3.4}$$

for U, $V > 0$.

Since

$$\begin{aligned} &(V-1)(\lambda+V)^{-1} - (U-1)(\lambda+U)^{-1} \\ &= V(\lambda+V)^{-1} - U(\lambda+U)^{-1} - \left((\lambda+V)^{-1} - (\lambda+U)^{-1}\right) \end{aligned}$$

and

$$\begin{aligned}
&V(\lambda+V)^{-1}-U(\lambda+U)^{-1}\\
&=(V+\lambda-\lambda)(\lambda+V)^{-1}-(U+\lambda-\lambda)(\lambda+U)^{-1}\\
&=1-\lambda(\lambda+V)^{-1}-1+\lambda(\lambda+U)^{-1}=\lambda(\lambda+U)^{-1}-\lambda(\lambda+V)^{-1},
\end{aligned}$$

hence

$$\begin{aligned}
&(V-1)(\lambda+V)^{-1}-(U-1)(\lambda+U)^{-1}\\
&=\lambda(\lambda+U)^{-1}-\lambda(\lambda+V)^{-1}-\left((\lambda+V)^{-1}-(\lambda+U)^{-1}\right)\\
&=(\lambda+1)\left[(\lambda+U)^{-1}-(\lambda+V)^{-1}\right]
\end{aligned}$$

and by (3.4) we get

$$\ln V-\ln U=\int_0^\infty\left[(\lambda+U)^{-1}-(\lambda+V)^{-1}\right]d\lambda. \tag{3.5}$$

Since, by (2.6) we have

$$\begin{aligned}
&(\lambda+U)^{-1}-(\lambda+V)^{-1} \qquad (3.6)\\
&=\int_0^1(\lambda+(1-t)U+tV)^{-1}(V-U)(\lambda+(1-t)U+tV)^{-1}dt,
\end{aligned}$$

for all $\lambda\geq 0$, hence by (3.5) and (3.6) we get (3.3). □

Theorem 3.4. *For all* A, B, $P>0$ *we have*

$$\begin{aligned}
S(P|B)-S(P|A)=\int_0^\infty\Bigg[\int_0^1 &P((1-t)A+tB+\lambda P)^{-1}(B-A) \qquad (3.7)\\
&\times((1-t)A+tB+\lambda P)^{-1}Pdt\Bigg]d\lambda.
\end{aligned}$$

Proof. Follows by Lemma 3.3 by taking $V=P^{-1/2}BP^{-1/2}$ and $U=P^{-1/2}AP^{-1/2}$ and multiplying both sides by $P^{1/2}$. □

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