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MEAN VALUES DEPENDING ON SOME GENERAL FUNCTIONS

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1. INTRODUCTION

In Orlicz space theory, (see for example [2] and [4]), an important role is played by functions $\phi:R^+ \to R^+$ such that

$$(\Delta_2) \qquad \phi(2u) \leq M\phi(u) \quad \text{for all} \quad u \in R^+$$

for some fixed $M > 0$. This condition is known as the Birnbaum and Orlicz condition. The function $u \to u^r$ is such a function. In this particular example, $M = 2^r$ and equality holds in (Δ_2). In this note, we introduce a larger class of functions which also satisfy condition (Δ_2) with equality. Furthermore, we show that the functions introduced satisfy generalizations of some classical inqualities.

Let $u \in]0,\infty[$ and $n = [\log_2 u]$ where $[x]$ stands for the greatest integer smaller or equal to x. For $r > 0$ and $M > 1$, we define

$$(1) \qquad \phi_{r,M}(u) = \begin{cases} 0 & u = 0 \\ M^n((u/2^n)^r - (M-2^r)/(M-1)) & u > 0. \end{cases}$$

It is easy to verify that

$$(2) \qquad \phi_{r,M}(2u) = M\phi_{r,M}(u) \quad \text{for all} \quad u \in R^+$$

and that $\phi_{r,M}$ is continuous and strictly increasing. If $M = 2^r$, then $\phi_{r,M}(u) = u^r$ for all $u \in R^+$.

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In Section 2, we state a few general facts. In Section 3, we prove that a generalization of the first Clarkson inequality holds for $\phi_{r,M}$ if $r \geq 2$ and $M \geq 2^r$. In Section 4, some results on generalized means are proven. In Section 5, a generalization of the Hölder inequality is established. In view of the fact that the Minkowski inequality for the function $u \to u^r$ is often introduced as a consequence of the Hölder inequality, it is interesting to note that the Minkowski inequality does not holds for the function considered in Section 5, except for the particular case $M = 2^r$. Finally, in Section 6, we introduce a generalization of the Jensen inequality.

The properties of the functions introduced in the present note make them useful in a certain number of areas including the construction of Orlicz spaces with particular properties.

2. GENERALITIES

The following lemma will be used frequently.

Lemma (2.1). Let $S,T > 0$ *and let* $f:R^+ \to R^+$ *be a function such that* $f(Sx) = Tf(x)$ *for all* $x \in R^+$. *Then*

(i) *The kth-derivative* $f^{(k)}$ *of* f *exists at a point* x *if and only if it exists at* $S^n x$ *for all* $n = 0,1,-1,\ldots$ *and we have the following equality*

$$f^{(k)}(S^n x) = (T/S^k)^n f^{(k)}(x);$$

(ii) *The function* f *is increasing (resp. decreasing, convex, concave) if and only if it is increasing (resp. decreasing, convex, concave) in a neighborhood of some interval of the form* $[a,Sa[$ $(a > 0)$.

The proof of this lemma is elementary.

We now return to the function $\phi_{r,M}$ defined in Section 1. It is useful to notice that $\phi_{r,M}:R^+ \to R^+$ can be characterized as follows. Let $K = (M-2^r)/(M-1)$. Then $\phi_{r,M}$ is the only continuous real valued function ϕ defined on R^+ such that (i) $\phi(u) = u^r - K$ for $u \in [1,2[$ and (ii) ϕ satisfies the condition

$$\phi(2u) = M\phi(u) \quad \text{for all} \quad u \geq 0. \tag{3}$$

The function $\phi_{r,M}$ is indefinitely differentiable at all points not of the form 2^n $(n = 0,-1,1,\ldots)$. If $u \in]2^n,2^{n+1}[$, then

(4) $$\phi'_{r,M}(u) = r(M/2)^n(u/2^n)^{r-1}$$

and

(5) $$\phi''_{r,M}(u) = (r-1)r(M/4)^n(u/2^n)^{r-2}.$$

Furthermore, we have the following limits

(6) $$\lim_{u\to(2^n)^-} \phi'_{r,M}(u) = r\ 2^{r-1}(M/2)^{n-1}$$

(7) $$\lim_{u\to(2^n)^+} \phi'_{r,M}(u) = r(M/2)^n.$$

From these facts, we deduce that $\phi_{r,M}$ is convex (resp. concave) if $r \geq 1$ and $M \geq 2^r$ (resp. $0 < r < 1$ and $M \leq 2^r$).

Since $\phi_{r,M}$ is strictly increasing, its inverse is defined. If $v \in [M^n(1-k), M^{n+1}(1-k)[$ then

(8) $$\phi^{-1}_{r,M}(v) = 2^n((v/M^n)+K)^{1/r}.$$

The inverse function satisfies the condition

(9) $$\phi^{-1}_{r,M}(Mv) = 2\phi^{-1}_{r,M}(v) \quad \text{for all} \quad v \geq 0.$$

In particular, lemma (2.1) applies to $\phi^{-1}_{r,M}$.

3. THE FIRST CLARKSON INEQUALITY

In this section, we consider a generalization of the first Clarkson inequality. Given a function $\phi: R^+ \to R^+$, consider the following inequality, where $M > 0$ is fixed.

(10) $$\phi(u+v) + \phi(|u-v|) \leq (M/2)(\phi(u)+\phi(v)).$$

If ϕ is the function $u \to u^r$ and $r \geq 2$, then (10) holds with $M = 2^r$ and is known as the first Clarkson inequality [5]. It is used in the study of uniformly convex spaces [1] and in the study of extremal length [5], [6].

<u>Proposition (3.1)</u>. If $r \geq 2$ and $M \geq 2^r$, then the function $\phi = \phi_{r,M}$ satisfies (10).

<u>Proof</u>. Let $\phi = \phi_{r,M}$. Consider the function

(11) $$F(u,v) = (M/2)(\phi(u)+\phi(v)) - \phi(u+v) - \phi(|u-v|).$$

It suffices to show that $F(u,v) \geq 0$. Without loss of generality, we can assume that $u \geq v$. Furthermore, (2) implies that

$$F(u,v) = M^n F(2^{-n}u, 2^{-n}v). \tag{12}$$

Consequently, it is sufficient to prove that (11) holds for $u \in [1,2[$ and $v \in]0,u]$.

Let $D_1 = \{(u,v) | 1 \leq u < 2 \text{ and } 0 < v \leq u\}$. For $(u,v) \in D_1$, let $\ell = [\log_2 v]$, $m = [\log_2(u+v)]$, $n = [\log_2(u-v)]$, $\alpha = M/2^r$ and $K = (M-2^r)/(M-1)$. Then $\ell \leq 0$, $m = 0$ or 1, $n \leq 0$, $\alpha \geq 1$ and $0 \leq K < 1$. With these definitions, (11) can be rewritten as follows:

$$\begin{aligned} F(u,v) = {} & 2^{r-1}\alpha\{(u^r-K)+(\alpha^\ell u^r - M^\ell K)\} \\ & - (\alpha^m(u+v)^\ell - M^m K) \\ & - (\alpha^n(u-v)^r - M^n K). \end{aligned} \tag{13}$$

Notice that the function $F(u,v)$ is continuously differentiable on an open dense subset D_1' of D_1. We will show that $\partial F/\partial v \leq 0$ on D_1'. An easy computation shows that

$$\frac{1}{r(u+v)^{r-1}} \frac{\partial F}{\partial v} = \alpha^{\ell-m+1}\left(\frac{2v}{u+v}\right)^{r-1} - 1 + \alpha^{n-m}\left(\frac{u-v}{u+v}\right)^{r-1}. \tag{14}$$

Notice that $2v/(u+v) \leq 1$ and $(u-v)/(u+v) \leq 1$. Therefore, (14) decreases as r increases (for α fixed). We claim that (14) decreases as α increases (for r fixed). We must examine the various values of ℓ, m, n possible. If $u + v \in [1,2[$, then $v < 1$. Therefore, if $m = 0$, then $\ell \leq -1$. If $u + v \in [2,4[$, then $v \in [1,2[$ and $u - v \in [0,1[$. Therefore, if $m = 1$, then $\ell = 0$ and $n \leq -1$. In either case, it follows easily that the exponents of α on (14) are negative. Since $\alpha \geq 1$, we can deduce that (14) deceases as α increases, as claimed. For $r = 2$ and $\alpha = 1$, it is easily computed that (14) is equal to 0. We conclude that (14) is negative for $r \geq 2$ and $\alpha = 1$. It follows that $\partial F/\partial v \leq 0$ for $(u,v) \in D_i'$.

Since $F(u,u) = 0$, the continuity of F and the fact that $\partial F/\partial v$ exists and in non-positive on D_i' imply that $F(u,v) \geq 0$ for $(u,v) \in D_1$ as required.

4. GENERALIZED MEANS

Let $a = (a_1, a_2, \dots)$ and $q = (q_1, q_2, \dots)$ be finite or infinite sequences such that $a_k \geq 0$, $q_k \geq 0$ and $\Sigma q_k = 1$. The $\phi_{r,M}$-mean of a with respect to the weight q is defined to be

$$M_{r,M}(a;q) = \phi^{-1}_{r,M}(\Sigma q_k \phi_{r,M}(a_k)),$$

if $\Sigma q_k\ _{r,M}(a_k) < \infty$. Otherwise set $M_{r,M}(a;q) = \infty$. We will consider $M_{r,M} = M_{r,M}(a;q)$ as a function of r and M. We have the following proposition.

<u>Proposition (4.1)</u>. If $0 < r \leq r'$ and $M/2^r \leq M'/2^{r'}$, then $M_{r,M} \leq M_{r',M'}$.

<u>Proof</u>. According to Theorem 92 of [3], it is sufficient to show that the function

$$\psi = \phi_{r',M'} \circ \phi^{-1}_{r,M}$$

is convex.

Since

$$\psi(My) = M'\psi(y) \quad \text{for all} \quad y \geq 0$$

we can apply lemma (2.1). Therefore, we will restrict our attention to a neighborhood of $[(1-K), M(1-K)[$ where $K = (M-2^r)/(M-1)$. First, assume that $y \in\ ]1-K, M(1-K)[$. Then

$$\psi(y) = (y+K)^{r'/r} - K'$$

$$\psi'(y) = (r/r')(y+K)^{(r'/r)-1}$$

$$\psi''(y) = ((r'/r)-1)(r'/r)(y+K)^{(r'/r)-2}.$$

By hypothesis, $r \leq r'$. Therefore $\psi''(y) \geq 0$ and ψ is a convex function on $]1-K, M(1-k)[$. We now consider the point $1 - K$. We must compare limits from the left and from the right:

$$L = \lim_{y \to (1-K)^-} \psi'(y) = (M/M')(r'/r)2^{r'-r}$$

and

$$R = \lim_{y \to (1-K)^+} \psi'(y) = r'/r.$$

We obtain immediately that $L \leq R$ if (and only if) $M/2^r \leq M'/2^{r'}$ as required.

5. HÖLDER INEQUALITY

The following proposition is a generalization of the classical Hölder inequality, which is obtained by taking $M = 2^r$ and $M' = 2^{r'}$.

Proposition (5.1.). Let $r > 1$, $r' > 1$, $(r-1)(r'-1) \geq 1$, $M \geq 2^r$, $M' \geq 2^{r'}$ Then

$$\Sigma q_i a_i b_i \leq \mathcal{M}_{r,M}(a;q)\mathcal{M}_{r',M'}(b;q)$$

where $a = (a_1,a_2,\ldots)$, $b = (b_1,b_2,\ldots)$ and $q = (q_1,q_2,\ldots)$ are finite or infinite sequences of positive numbers, and $\Sigma q_i = 1$.

Proof. According to Theorem 100 of [3], it is sufficient to prove that the function

$$F(y,y') = \phi^{-1}_{r,M}(y)\phi^{-1}_{r',M'}(y')$$

is a concave function of (y,y'). Notice that

$$F(M^n y, M'^{n'} y') = 2^{n+n'}F(y,y').$$

This observation together with an easy generalization of Lemma (2.1) implies that it is sufficient to consider the restriction of F to an open neighborhood of the set

$$D_2 = [1-K,M(1-K)[\times[1-K',M'(1-K')[$$

where K, K' are as in Section 4. Furthermore, in order to prove the concavity of F, it is sufficient to show that the function

$$G(t) = F(c+dt,c'+d't)$$

is concave in t for all c, c', d, d', $t \in R^+$ such that $(c+dt,c'+d't)$ is in the domain under consideration.

Let D_2' denote the interior of D_2 and let c, c', d, d', t be such that $(c+dt,c'+d't) \in D_2'$. Then

$$G(t) = (c+dt+K)^{1/r}(c'+d't+K')^{1/r'}.$$

Let

$$A = \frac{d}{c+dt+K} \quad \text{and} \quad A' = \frac{d'}{c'+d't+K'}.$$

After some computations, we obtain that

$$\frac{G'(t)}{G(t)} = \frac{A}{r} + \frac{A'}{r'}$$

and

$$\frac{G''(t)}{G(t)} = \left(\frac{A}{r} + \frac{A'}{r'}\right)^2 - \frac{A^2}{r} - \frac{A'^2}{r'}$$

$$= \frac{2AA'}{rr'} - \left(1-\frac{1}{r}\right)\frac{A^2}{r} - \left(1-\frac{1}{r'}\right)\frac{A'^2}{r'} .$$

Using the assumption that $(1/r) + (1/r') \leq 1$, it is easily seen that the last expression is smaller or equal to $-(1/rr')/(A-A')^2$. Therefore $G''(t)/G(t) \leq 0$. Since $G(t) > 0$, we deduce that $G''(t) \leq 0$.

We must now consider certain boundary points. For instance, assume that $c + dt_0 + K = 1 - K$. After a simple computation, we obtain that

$$\lim_{t\to t_0^+} G'(t) - \lim_{t\to t_0^-} G'(t)$$

$$= d\left\{ \lim_{y\to(1-K)^+} {\phi^{-1}_{r,M}}'(y) - \lim_{y\to(1-K)^-} {\phi^{-1}}'(y)\right\}\phi^{-1}_{r,M}(c'+d't+K')$$

The middle factor of the last expression is non-positive due to the concavity of $\phi^{-1}_{r,M}$ at $y = 1 - K$. The other factors are obviously positive. Therefore, the concavity of G at this point is proven. Other boundary points are treated similarly.

Remark (5.2). It can be shown that the Minkowski inequality does not hold for the function $\phi_{r,M}$ if $r > 1$ and $M > 2^r$. In order to do this, one can study the convexity of the function $\phi^{-1}_{r,M}\{\Sigma p_i\phi(x_i+u_it)\}$ for p_i, x_i, u_i, $t \in R$ such that $x_i + u_i\ t \in R^+$ and $p_i \geq 0$, $\Sigma p_i = 1$ [3]. It is easy to show that this function is neither convex nor concave for r, M as in proposition (5.1).

6. JENSEN-COOPER INEQUALITY

Let $a = (a_1,\ldots,a_\ell)$ be a finite sequence of positive real numbers and let $\phi:R^+\to R^+$ be a monotonic function. Then define

$$S_\phi(a) = \phi^{-1}\left(\sum_{k=1}^{\ell} \phi(a_1)\right).$$

For r, M as in Section 1, we denote

$$S_{r,M}(a) = S_{\phi_{r,M}}(a).$$

The following proposition generalizes an inequality due to Jensen and Cooper (see [3], page 84, footnote b).

Proposition (6.1). Let $0 < r < r'$ and $1 < M < M'$. Assume that $(M-2^r)/(M-1) \leq (M'-2^{r'})/(M'-1)$. Then $S_{r,M}(a) < S_{r',M'}(a)$ for all finite sequence $a = (a_1,\dots,a_\ell)$ of positive numbers).

Proof. According to Theorem 105 of [3], it is sufficient to prove that the quotient function

$$(15) \qquad Q(u) = \phi_{r',M'}(u)/\phi_{r,M}(u)$$

is an increasing function. Since $Q(2u) = (M'/M)Q(u)$, we can apply lemma (2.1). In fact, it is sufficient to show that $Q(u)$ is increasing on the open interval $]1,2[$. The derivative $Q'(u)$ is positive if and only if

$$(16) \qquad f(u) = \phi'_{r',M'}(u)\phi_{r,M}(u) - \phi_{r',M'}(u)\phi'_{r,M}(u) \geq 0.$$

For $u \in]1,2[$,

$$(17) \qquad f(u) = r'u^{r'-1}(u^r-K) - ru^{r-1}(u^{r'}-K').$$

We claim that

$$(18) \qquad \lim_{u\to 2^-} f(u) \geq 0.$$

Notice that

$$\lim_{u\to 2^-} f(u) = r'2^{r'-1}(2^r-K) - r2^{r-1}(2^{r'-K'}) \geq 0$$

$$= \left(\frac{r2^{r-1}}{2^r-1} \right)\left(\frac{M-1}{M} \right) \leq \left(\frac{r'2^{r'-1}}{2^{r'}-1} \right)\left(\frac{M'-1}{M'} \right).$$

It is easy to establish the last inequality by computing the derivatives

$$\frac{d}{dx}\left(\frac{x2^{x-1}}{2^{x}-1} \right) \text{ and } \frac{d}{dy}\left(\frac{y-1}{y} \right).$$

This proves (18).

Finally, we consider the function

$$(19) \qquad g(u) = u^{-r-r'+1}f(u)$$

$$= (r'-r) + (ru^{-r'}K'-r'u^{-r}K)$$

where $K = (M-2^r)/(M-1)$ and $K' = (M-2^{r'})/(M'-1)$. By hypothesis, $K \leq K'$. It follows easily that

(20) $$g'(u) = -rr'u^{-r'-1}K' + r'ru^{-r-1}K \leq 0.$$

Therefore $g(u)$ is decreasing for $u \in \,]1,2[$. We conclude that $f(u)$ is also decreasing on the same interval. Together with (18), this fact implies that $f(u)$ and therefore $Q(u)$ are increasing on $]1,2[$. The proposition follows.

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