LIBERTAS MATHEMATICA, VOL. 4, 1984

ON CLARKSON INEQUALITIES

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In the study of the extremal length of a family of measures (see for example [4]), two inequalities due to J. A. Clarkson [2] were very useful. Let us remember these inequalities.

Throughout this paper (X,A,μ) will be a (positive) measure space and integrals of some measurable functions will be taken over all X. If f and g are two such functions, the Clarkson inequalities say that:

$$\int(f+g)^p d\mu + \int|f-g|^p d\mu \leq 2^{p-1}(\int f^p d + g^p d\mu), \quad p \geq 2 \tag{1}$$

$$\begin{gathered}\{\int(f+g)^p d\mu\}^{\frac{1}{p-1}} + \{\int|f-g|^p d\mu\}^{\frac{1}{p-1}} \\ \leq 2\{\int f^p d\mu + \int g^p d\mu\}^{\frac{1}{p-1}}, \quad 1 < p \leq 2\end{gathered} \tag{2}$$

In [1] and [3], we studied the functions $\Phi_{p,M}$ defined on $[0,\infty[$ by:

$$\Phi_{p,M}(x) = \sum_{n=-\infty}^{+\infty} [(\frac{M}{2^p})^n x^p + M^n \frac{2^p - M}{M-1}] \chi_{I_n}(x), \tag{3}$$

where $M > 0$, $M \neq 1$, $p \in R$ and χ_{I_n} representes the characteristic function of the interval

$$I_n = [2^n, 2^{n+1}[.$$

Some classical relations were generalized by means of $\Phi_{p,M}$ and we mention the following:

$$\Phi_{p,M}(f+g) + \Phi_{p,M}(|f-g|) \leq \frac{M}{2} [\Phi_{p,M}(f) + \Phi_{p,M}(g)], \tag{4}$$

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[*] Partially supported by the National Sciences and Engineering Research Council of Canada, Grant A5361.

$$p \geq 2, \quad M \geq 2^p. \tag{4}$$

Integrating side by side in (4), we obtain:

$$\begin{aligned} &\int \Phi_{p,M}(f+g)\,d\mu + \int \Phi_{p,M}(|f-g|)\,d\mu \\ &\qquad \leq \frac{M}{2}\left[\int \Phi_{p,M}(f)\,d\mu + \int \Phi_{p,M}(g)\,d\mu\right], \quad p \geq 2, \quad M \geq 2^p \end{aligned} \tag{5}$$

which becomes (1) when M becomes 2^p.

In this paper we give a generalization of (2) by means of the same functions $\Phi_{p,M}$.

Let $1 < p \leq 2$, $q = \frac{p}{p-1}$ and $M \geq 2^q$. For each couple (u,v), $0 \leq v \leq u$, let us define:

$$H(u,v) = 2^{p-1}(u^p+v^p) - [(u+v)^q+(u-v)^q]^{p-1} \tag{6}$$

$$H_M(u,v) = \frac{M^{p-1}}{2}[\Phi^{p-1}_{q,M}(u)+\Phi^{p-1}_{q,M}(v)] - [\Phi_{q,M}(u+v)+\Phi_{q,M}(u-v)]^{p-1}. \tag{7}$$

It is easily seen that H_M is a continuous function of M and

$$H_{2^q}(u,v) = H(u,v). \tag{8}$$

From the proof of (2) it results that

$$\begin{aligned} &H(u,v) > 0, \quad \text{for} \quad 0 < v < u \\ &H(u,u) = 0, \quad H(u,u) = 0 \end{aligned} \tag{9}$$

We see that $H_M(u,0) = (\frac{M^{p-1}}{2} - 2^{p-1})\Phi_{q,M}(u)$ and hence

$$H_M(u,0) > 0, \quad \text{for} \quad u > 0, \quad M > 2^q. \tag{10}$$

Furthermore, the relation $\Phi_{q,M}(2u) = M\Phi_{q,M}(u)$ implies:

$$H_M(u,u) = M^{p-1}\Phi^{p-1}_{q,M}(u) - \Phi^{p-1}_{q,M}(2u) = 0 \tag{11}$$

Let us find the conditions under which $H_M(u,v) \geq 0$. Since

$$H_M(2^n u, 2^n v) = M^{n(p-1)}H_M(u,v) \tag{12}$$

for each entire n, it is sufficient to restrict our attention to the domain D defined by

$$1 \le u \le 2, \quad 0 \le v \le u.$$

The expansion of $H_M(u,u-t)$ in the neighborhood of $t = 0$ shows that if $t > 0$ is small enough, then $H_M(u,u-t) > 0$.

This fact, the relations (8)-(11) and the continuity of H_M as a function of M allow us to state that for each point $(u_0,v_0) \in D$, there is a neighborhood V_0 of (u_0,v_0) and there is a positive number η_0 such that $H_M(u,v) \ge 0$ for $(u,v) \in V_0 \cap D$ and

$$2^q \le M \le 2^q + \eta_0.$$

Since D is compact, it is covered by a finite number of such neighborhoods, say $V_1, V_2, \ldots, V_n$.

Let $\eta_1, \eta_2, \ldots, \eta_n$ be the corresponding positive numbers such that $H_M(u,v) \ge 0$ for $(u,v) \in V_k \cap D$ as long as $2^q \le M \le 2^q + \eta_k$, $k = 1,2,\ldots,n$.

Let $\eta = \min\{\eta_1, \eta_2, \ldots, \eta_n\}$. Then $2^q \le M \le 2^q + \eta$ implies that

$$H_M(u,v) \ge 0, \quad \text{for} \quad (u,v) \in D \tag{13}$$

and by (12),

$$H_M(u,v) \ge 0, \quad \text{for} \quad 0 \le v \le u. \tag{14}$$

Let us apply the reverse Minkowsky inequality:

$$\left(\int |\Phi|^{p-1} d\mu\right)^{\frac{1}{p-1}} + \left(\int |\Psi|^{p-1} d\mu\right)^{\frac{1}{p-1}} \le \left[\int (|\Phi| + |\Psi|)^{p-1} d\mu\right]^{\frac{1}{p-1}}$$

where $\Phi = \Phi_{q,M}(f+g)$, $\Psi = \Phi_{q,M}(|f-g|)$ and take into account that $H(f,g) \ge 0$. We have:

$$\begin{aligned}
&\left[\int \Phi^{p-1}_{q,M}(f+g)\, d\mu\right]^{\frac{1}{p-1}} + \left[\int \Phi^{p-1}_{q,M}(|f-g|)\, d\mu\right]^{\frac{1}{p-1}} \\
&\le \left\{\int [\Phi_{q,M}(f+g) + \Phi_{q,M}(|f-g|)]^{p-1} d\mu\right\}^{\frac{1}{p-1}} \\
&\le \frac{M^{p-1}}{2} \int [\Phi^{p-1}_{q,M}(f) + \Phi^{p-1}_{q,M}(g)] d\mu\}^{\frac{1}{p-1}} \\
&= \frac{M}{2^{q-1}} \left\{\int [\Phi^{p-1}_{q,M}(f) + \Phi^{p-1}_{q,M}(g)] d\mu\right\}^{\frac{1}{p-1}}
\end{aligned}$$

The following theorem is thus proved:

Theorem. For each $1 < p \leq 2$, *there is a positive number* η *such that* $2^q \leq M \leq 2^q + \eta$ *implies*

$$(16)\qquad \begin{aligned} &[\int \Phi^{p-1}_{q,M}(f+g)d\mu]^{\frac{1}{p-1}} + [\Phi^{p-1}_{q,M}(|f-g|)d\mu]^{\frac{1}{p-1}} \\ &\leq \frac{M}{2^{q-1}} \{\int[\Phi^{p-1}_{q,M}(f)+\Phi^{p-1}_{q,M}(g)]d\mu\}^{\frac{1}{p-1}} \end{aligned}$$

where $q = \frac{p}{p-1}$.

It is easily seen that the inequality (16) becomes (12) when $M = 2^q$.

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