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# ON THE BEST CONSTANT IN THE HAUSDORFF-YOUNG INEQUALITY

Andrzej Korzeniowski

ABSTRACT. For a given Banach space $X$ let

$$L^p(X) = \{f:\mathbb{R}\to X\ ;\ \|f\|_p = (\int \|f(x)\|^p dx)^{1/p} < \infty\}$$

and let

$$(\mathbb{F}f)(x) = \int e^{2\pi ixy} f(y)dy$$

be the Fourier transform of $f \in L^p(X)$. It is shown that

$$\|\mathbb{F} f\|_{p'} \leq (p^{1/p}/p'^{1/p'})^{1/2} \|f\|_p, \quad 1/p + 1/p' = 1, \quad 1 \leq p \leq 2$$

for $X = \{L^q;\ p\leq q\leq 2\}$, which generalized the Babenko-Beckner inequality known to be sharp in the case $X = C$.

We keep the notation of [1] and to avoid repetitions we recall only necessary facts and ideas of the proof, whereas for the details we refer the readers to the paper mentioned above. Beckner's proof of the (H-Y) inequality goes as follows: Instead of proving

(H-Y) $$\|Ff\|_{p'} \leq A_p \|f\|_p, \quad A_p = \left( \frac{p^{1/p}}{p'^{1/p'}} \right)^{1/2}$$

one proves

(1) $$\|T_\omega g\|_{L^{p'}(d\mu)} \leq \|g\|_{L^p(d\mu)}$$

where $\mu$ is a standard gaussian measure $d\mu(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$, $\omega = i\sqrt{p-1}$ and $T_\omega$ is a multiplier operator on $L^2(d\mu)$ defined on Hermite polynomials $H_n = \int (x+iy)^m d\mu(y)$ by $T_\omega : H_m \to \omega^m H_m$. $T_\omega$ is an integral operator with the Mehler kernel

$$T_\omega(x,y) = (1-\omega^2)^{-1/2} \exp\left\{ -\frac{\omega^2(x^2+y^2)}{2(1-\omega^2)} + \frac{\omega xy}{1-\omega^2} \right\}$$

i.e. $(T_\omega g)(x) = \int T(x,y)g(y)d\mu(y)$.

Since (1) can be written as

$$\{\int|\int e^{2\pi inv} g(\sqrt{2\pi p}v)e^{-\pi v^2}dv|^{p'}du\}^{1/p'} \leq A_p\{\int|g(\sqrt{2\pi p}\,u)e^{-\pi u^2}|^p du\}^{1/p}$$

therefore it is sufficient to have (1) for polynomials

$$g(x) = \sum_{\ell=1}^{M} b_\ell H_\ell$$

this in turn amounts to the following inequality

(2) $$\|K_n g_n\|_{L^{p'}[X_n]} \leq \|g_n\|_{L^p[X_n]}$$

where $g_n(x_1,\ldots,x_n) = \sum_{\ell=1}^{M} b_\ell \phi_{n,\ell}\ (x_1,\ldots,x_n) \in X_n$ - the space of polynomials in discrete variables $x_1,\ldots,x_n$ over the product measure space $d\nu(\sqrt{n}\,x_1)\ldots d\nu(\sqrt{n}\,x_n)$ $X_n$ with $\nu(1) = \nu(-1) = \frac{1}{2}$ and $\{\phi_{n,\ell}\}$ forming an orthogonal basis in $L^2[X_n]$, whereas the operator $K_n = C_{n,1}\ldots C_{n,n}$ is a product of operators

$$C_{n,k}: a + bx_k \to a + i\sqrt{p-1}\,bx_k$$

where $a$, $b$ are functions of the remaining $n - 1$ variables. Then by the central limit theorem arguments one gets

$$\|K_n g_n\|_{L^{p'}[X_n]} \to \|T_\omega g\|_{L^{p'}(d\mu)}$$

and

$$\|g_n\|_{L^p[X_n]} \to \|g\|_{L^p(d\mu)}$$

concluding (1). A crucial point in the proof of (2) is so called "2-point" inequality

(B) $$\left(\frac{|a+i\sqrt{p-1}\,b|^{p'}+|a-i\sqrt{p-1}\,b|^{p'}}{2}\right)^{1/p'} \leq \left(\frac{|a+b|^p+|a-b|^p}{2}\right)^{1/p}$$

for all complex numbers $a$, $b$.

The above says that the operator

$$C: a + bx \to a + i\sqrt{p-1}\,bx$$

with the kernel $K(x,y) = 1 + i\sqrt{p-1}\,xy$ on $L^p(d\nu)$ to $L^{p'}(d\nu)$ is of norm one. It can easily be checked that the product of two operators of this type on $L^p(d\nu \times d\nu)$ to $L^{p'}(d\nu \times d\nu)$ is also of norm one which justifies (2).

Observe that if for some Banach space $X$, (B) with $\|\cdot\|$ replacing $|\cdot|$ holds for all $a, b \in X$ then (H-Y) carries over to $L^p(X)$ because $g(x) = \sum_{\ell=1}^{M} b_\ell H_\ell(x)$, $b_\ell \in X$ are dense in $L^p(X)$.

We start by recalling that for any $x_1, x_2$ in an arbitrary Banach space the function

$$\left( \frac{\|x_1 + \frac{x_2}{\sqrt{p-1}}\|^p + \|x_1 - \frac{x_2}{\sqrt{p-1}}\|^p}{2} \right)^{1/p} \tag{3}$$

is decreasing in $1 < p < \infty$ (see [3] p. 76].

*Lemma.* (B) *holds true for either* $L^2(C)$ *or* $L^q(\mathbb{R})$ *over some* $\sigma$-*finite measure space* $(S, \Sigma, m)$.

Proof. Notice that in the first case we may assume that $m$ is a finite measure because otherwise we take $f 1\!\!1_{A_n}$ where $A_1 \subset A_2 \subset \ldots, m(A_n) < \infty$, $\cup A_n = S$ and pass with $n$ to $\infty$. Furthermore one can assume that $|f| > 0$ $m$ almost everywhere because otherwise we take $f 1\!\!1_{\{|f| \geq \varepsilon\}} + \varepsilon 1\!\!1_{\{|f| > \varepsilon\}}$ and pass with $\varepsilon$ to $0$. Moreover under the above assumptions $|f+g| = ||f| + \frac{|f|}{f} g|$ and therefore it is sufficient to show (B) for real-valued $f$.

Now for such $f$ and $g = g_1 + ig_2$ the square of the left-hand side of (B) equals to

$$\left( \frac{\|(f-\sqrt{p-1}\, g_2)^2 + (p-1) g_1^2\|_1^{p'/2} + \|(f+\sqrt{p-1}\, g_2)^2 + (p-1) g_1^2\|_1^{p'/2}}{2} \right)^{2/p'}$$

where $\|\cdot\|_1$ stands for the $L^1$ norm, whence by additivity of $\|\cdot\|_1$ for non-negative functions and the Minkowski inequality is not greater than

$$\left( \frac{\|f_1 + \sqrt{p-1}\, g_2\|_2^{p'} + \|f - \sqrt{p-1}\, g_2\|_2^{p'}}{2} \right)^{2/p'} + (p-1)\|g_1\|_2^2$$

so by (3) and $(p-1)(p'-1) = 1$ does not exceed

$$\|f_1\|_2^2 + \|g_2\|_2^2 + (p-1)\|g_1\|_2^2$$

Next the square of the right-hand side equals to

$$\left(\frac{\|(f+g_1)^2+g_2^2\|_1^{p/2}+\|(f-g_1)^2+g_2^2\|_1^{p/2}}{2}\right)^{2/p}$$

therefore by the reverse Minkowski inequality $(0 < \frac{p}{2} \leq 1)$ is greater than

$$\left(\frac{\|f+g_1\|_2^p+\|f-g_1\|_2^p}{2}\right)^{2/p} + \|g_2\|_2^2$$

thus by applying (3) majorizes

$$\|f\|_2^2 + (p-1)\|g_1\|_2^2 + \|g_2\|_2^2$$

and concludes the first case.

In the second case the left-hand side reduces to $\|f+i\sqrt{p-1}\,g\|_q$ i.e.

$$(\int(f^2+(p-1)g^2)^{q/2})^{1/q}$$

whereas the right hand side equals to

$$\left(\frac{\|f+g\|_q^p+\|f-g\|_q^p}{2}\right)^{1/p}$$

and by (3) it is greater than

$$\left(\frac{\|f+\frac{\sqrt{p-1}}{\sqrt{q-1}}\,g\|_q^q+\|f-\frac{\sqrt{p-1}}{\sqrt{q-1}}\,g\|_q^q}{2}\right)^{1/q}$$

whence it suffices to check

$$\int(f^2+(p-1)g^2)^{q/2} \leq \int \frac{|f+\frac{\sqrt{p-1}}{\sqrt{q-1}}\,g|^q+|f-\frac{\sqrt{p-1}}{\sqrt{q-1}}\,g|^q}{2}$$

what amounts to the inequality

$$(f^2+(p-1)g^2)^{1/2} \leq \left(\frac{|f+\frac{\sqrt{p-1}}{\sqrt{q-1}}|^q+|f-\frac{\sqrt{p-1}}{\sqrt{q-1}}|^q}{2}\right)^{1/q}, \quad m \text{ a.e.}$$

but this follows by applying (3) for $\mathbb{R}$.

Consequently we get the following

*Theorem. Let* $f \in L^p(L^2(C))$ *or* $f \in L^p(L^q(\mathbb{R}))$ *for some* $p \leq q \leq 2$ *then*

$$\|Ff\|_{p'} \leq A_p \|f\|_p \quad 1 < p \leq 2, \quad \frac{1}{p} + \frac{1}{p'} = 1$$

where

$$A_p = \left( \frac{p^{1/p}}{p'^{1/p'}} \right)^{1/2} .$$

Recently Bourgain [2] obtained a vector version of (H-Y) for characters of groups in the sense that for B-convex spaces (H-Y) inequality constant, say C, is finite.

REFERENCES

[1] Beckner, W. "Inequalities in Fourier analysis", Annals of Math., 102 (1975), 159-182.

[2] Bourgain, J. "A Hansdorff-Young inequality for B-convex Banach spaces", Pacific J. of Math., 101 (1982), 255-262.

[3] Lindenstrauss, J., and Tzafriri, L. "Classical Banach spaces II", *Ergebnisse der Math.*, 97, Springer, 1979.