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ULTIMATE BEHAVIOR OF SOLUTIONS TO SOME NONLINEAR INTEGRODIFFERENTIAL EQUATIONS

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J. Moser [4] has investigated the ultimate behavior of solutions to the integrodifferential equation

(1) $$\dot{y}(t) + \int_0^t k(t-s)\dot{y}(s)ds + f(y(t)) = g(t) ,$$

which provides a mathematical model for the description of certain electrical networks .

In our view , what really makes the results in Moser's paper very interesting is the fact that the ultimate behavior of solutions of equation (1) , under adequate conditions , is completely determined by the ultimate behavior of the solutions to the ordinary differential system

(2) $$\dot{z}(t) + f(z(t)) = 0 .$$

The problem of ultimate behavior of solutions of the system (2) is , of course , much better investigated in the mathematical literature , than it is for integrodifferential systems such as (1) .

The aim of this paper is to generalize some of the results in Moser's paper [4] to the case of nonconvolution kernels (we have in mind the kernel k involved in equation (1)) . It turns out that the transform technique used by Moser can be substituted by rather simple monotonicity arguments (monotone integral operators , or positive definite kernels) , the results obtained on this way being still simple enough , and easily applicable .

Let us notice from the beginning that the limit points of trajectories are defined in the same way as in case of ordinary differential systems , because the trajectories of (1) and (2) belong to the same space R^n . An alternate method is to use a function space as the phase space for the integrodifferential equation , a procedure that is mainly encountered with delay equations [2], [3] .

We shall consider now the nonconvolution type integrodifferential equation

(E) $$\dot{y}(t) + \int_0^t k(t,s)\dot{y}(s)ds + f(y(t)) = g(t) ,$$

where y takes values in the Euclidean n-dimensional space R^n and represents the unknown function , while $k(t,s) = (k_{ij}(t,s))$ is an n by n matrix kernel whose entries are measurable on $\Delta = \{ (t,s) ; o \leq s \leq t < \infty \}$, $f(y) = \text{grad } U(y)$ with $U \in C^{(1)}(R^n,R)$, and $g(t) \in C_o(R_+ , R^n) \cap L^2(R_+ , R^n)$. We shall preserve throughout this paper the notations used in our book [1 ; Chapter 3] . Precise conditions for the equation (E) will be stated bellow , in order to assure the ultimate behavior described above .

Associated with the equation (E) , there will be an initial condition of the form

(IC) $$y(0) = y^o \in R^n .$$

The existence of solutions to the problem (E) , (IC) can be discussed in the following manner . Denote

$$(Lx)(t) = x(t) + \int_0^t k(t,s)x(s)ds , t \in R_+ ,$$

and assume the opertor L is invertible in some function space consisting of map - pings from R_+ into R^n . If a resolvent kernel does exist for the kernel -k (and it is known that such kernels do exist for Volterra operators like L , under fairly general assumptions) , say $\tilde{k}(t,s)$, then the equation (E) is equivalent to a Volterra integrodifferential equation of the form

(3) $$\dot{y}(t) = - f(y(t)) + g(t) + \int_0^t \tilde{k}(t,s) [-f(y(s)) + g(s)]ds .$$

For the equation (3) , with initial condition (C) , the local existence can be obtained by standard methods (successive approximations , Peano's type existence , or even Caratheodory's type existence when k(t,s) is not necessarily continuous) .

We are not going to enter into details concerning local existence for the equation (E) . The condition we shall impose on this equation are of such a nature , that they will assure global existence (on R_+) for all solutions of (E) , as well as the boundedness (in R^n) of the trajectories . This last property will enable us to conclude that the limit set of an arbitrary solution of (E) , (IC) is not empty , and it agrees with the limit set of a convenient solution of the ordinary system (2) .

Before we state and prove the main results of this paper , the following (admissibility) lemma will be established .

<u>*Lemma 1*</u> . *Let the matrix valued kernel k be measurable from*

$$\Delta = \{ (t,s) \ ; \ 0 \leq s \leq t < \infty \}$$

into $L(R^n, R^n)$, *and such that the following conditions hold true :*

a) one has for every nonnegative t

$$\lim_{h \to 0} \left(\int_t^{t+h} \| k(t+h,s) \|^2 ds + \int_0^t \| k(t+h,s) - k(t,s) \|^2 ds \right) = 0 \ ; \tag{4}$$

b) for every positive T

$$\int_0^T \| k(t,s) \|^2 ds \in C_0(R_+ , R^n) \ ; \tag{5}$$

c) there exists $A > 0$, *with the property*

$$\int_0^t \| k(t,s) \|^2 ds \leq A^2 \ , \quad t \in R_+ \ . \tag{6}$$

Then , the Volterra integral operator

$$(Kx)(t) = \int_0^t k(t,s)x(s)ds \ , \ t \in R_+ \ , \tag{7}$$

satisfies the condition

$$KL^2(R_+ , R^n) \subset C_0(R_+, R^n) \cap L^2(R_+ , R^n) \ . \tag{8}$$

<u>Proof</u> . Let us notice first that the matrix norm used in (4) - (6) is the Euclidean norm , or an equivalent norm (for instance , the operator norm) .

Assume now that $x \in L^2(R_+ , R^n)$, and let us prove that

$$\lim_{t \to \infty} \left\| \int_0^t k(t,s)x(s)ds \right\| = 0 \ . \tag{9}$$

Given an arbitrary $\varepsilon > 0$, we shall choose $T = T(\varepsilon)$ large enough , such that

$$\int_T^\infty \| x(t) \|^2 dt < (\varepsilon A^{-1})^2 \ . \tag{10}$$

Therefore , for $t > T$ one can write the inequality

$$\left\| \int_0^t k(t,s)x(s)ds \right\| < \int_0^T \| k(t,s) \| \, \| x(s) \| ds + \left(\int_T^t \| k(t,s) \|^2 ds \right)^{\frac{1}{2}} \left(\int_T^t \| x(s) \|^2 ds \right)^{\frac{1}{2}} \ . \tag{11}$$

From (6) and (10) one derives easily the inequality

$$(12)\qquad \left(\int_T^t \|k(t,s)\|^2 ds \right)^{\frac{1}{2}} \left(\int_T^t \|x(s)\|^2 ds \right)^{\frac{1}{2}} < \varepsilon ,$$

for any $t > T$. On the other hand we have

$$(13)\qquad \int_0^T \| k(t,s)\| \| x(s)\| ds < \left(\int_0^T \| k(t,s)\|^2 ds\right)^{\frac{1}{2}} \| x\|_{L^2} .$$

On behalf of (5) and (13) , one can assert

$$(14)\qquad \int_0^T \|k(t,s)\| \ \|x(s)\|ds < \varepsilon \quad , t > T_1(\varepsilon) ,$$

provided T_1 is chosen large enough , such that

$$(15)\qquad \int_0^T \|k(t,s)\|^2 ds < (\varepsilon \|x\|_{L^2}^{-1})^2 \quad , \text{ for } \ t > T_1 .$$

Combining the inequalities (11) to (15) , one obtains

$$(16)\qquad \left\|\int_0^t k(t,s)x(s)ds\right\| < 2\varepsilon \quad , \ t > \max(T , T_1) .$$

Consequenly , (9) is proven , which ends the proof of the Lemma 1 .

Let us point out that condition (4) is assuring the continuity of $(Kx)(t)$ on R_+ , for each $x \in L^2(R_+ , R^n)$, and has not been used in deriving (9) . The inclusion

$$(17)\qquad KL^2(R_+, R^n) \subset L^2(R_+, R^n) ,$$

follows immediately from condition (6) . Of course (9) and (17) imply (8) .

Remark . If one assumes $k(t,s) = k(t-s)$, $0 \leq s < t \leq \infty$, where k is an L^2 - kernel , then all three conditions (4) , (5) , and (6) are automatically satisfied . While checking (5) and (6) constitutes elementary matter , we point out that the validity of (4) is known from the L^2 - space theory . Consequently , our result related to the equation (E) will be applicable to the case of the convolution equation (1) , in which the kernel k is an L^2 - kernel .

We are now able to formulate our basic result regarding the ultimate behavior of the solutions of equation (E) .

Theorem 1 . Assume that the matrix kernel $k(t,s)$ of the equation (E) satisfies the conditions a) , b) and c) of Lemma 1 . Moreover , let the following *monotonicity condition hold for the operator K :*

$$(18) \qquad \int_0^t \langle (Kx)(s) + \eta\, x(s) , x(s) \rangle ds \geq 0 ,$$

for any $t \geq 0$ and $x \in L^2(R_+, R^n)$, where η is a real number such that $\eta < 1$. Assume further that

$$(19) \qquad f(y) = \text{grad } U(y) , \text{ where } U \in C^{(1)}(R^n , R) ,$$

satisfies

$$(19') \qquad \lim U(y) = \infty \text{ as } \| y \| \to \infty ,$$

and finally

$$(20) \qquad g \in C_o(R_+ , R^n) \cap L^2(R_+ , R^n) .$$

Then any solution of the equation (E) is defined on R_+ , it remains bounded there , and its limit set agrees with that of a convenient solution of the ordinary differential equation (2) .

Proof . Indeed , let $y = y(t)$ be a solution of (E) , satisfying the initial condition (IC) : $y(0) = y^o \in R^n$. Such a solution does exists in some right neighborhood of the origin ($t = 0$) . We shall prove first that $y(t)$ is actually defined and bounded on R_+ .

Let us multiply scalarly both members of (E) by $\dot{y}(t)$. One obtains after integrating both sides of (E) from 0 to t ($t > 0$ in the neighborhood where $y(t)$ is defined) :

$$(21) \qquad \int_0^t \langle \dot{y}(s) + (K\dot{y})(s) , \dot{y}(s) \rangle ds + U(y(t)) = U(y^o) + \int_0^t \langle g(s) , \dot{y}(s) \rangle ds .$$

One has to take into account that

$$\int_0^t \langle f(y(s)) , \dot{y}(s) \rangle ds = U(y(t)) - U(y^o) ,$$

which is a consequence of assumptions (19) on $f(y)$.

Let us take now into consideration the condition (18) on K . One obatins from (21) after elementary transformations :

$$(22)\qquad (1-\eta)\int_0^t \|\dot{y}(s)\|^2 ds + U(y(t)) \le U(y^o) + \int_0^t \langle g(s), \dot{y}(s)\rangle ds,$$

where $\delta = 1 - \eta > 0$. Since

$$(23)\qquad \int_0^t \langle g(s), \dot{y}(s)\rangle ds \le 2^{-1}\delta \int_0^t \|\dot{y}(s)\|^2 ds + 2\delta^{-1}\int_0^t \|g(s)\|^2 ds,$$

by combining (22) and (23) , one obtains

$$(24)\qquad 2^{-1}\delta \int_0^t \|\dot{y}(s)\|^2 ds + U(y(t)) \le U(y^o) + 2\,\delta^{-1}\int_0^\infty \|g(s)\|^2 ds .$$

From (24) one can see that no solution of the equation (E) can have a finite escape time . Indeed , the right hand side of (24) is finite , independent of t , while the existence of a solution with finite escape time woulf imply the fact that the left hand side in (24) must be unbounded above (see condition (19') in Theorem 1) . Therefore , any solution of (E) must be defined on the whole R_+ , and must be bounded on that halfaxis . In other words , there exists a positive number M , such that

$$(25)\qquad \| y(t)\| \le M , \; t \in R_+ .$$

The number M in (25) depends only upon U , g , and y^o .

Another conclusion one can draw from (24) is $\dot{y} \in L^2(R_+ , R^n)$, for any solution of (E) . Indeed , since any solution y(t) remains bounded on R_+ , there follows that U(y(t)) is also bounded there . Consequently , (24) implies the boundedness of the integral in the left hand side , which is really what we claim :

$$(26)\qquad \int_0^\infty \|\dot{y}(s)\|^2 ds = A < +\infty .$$

In order to prove the second statement of the theorem , namely , that the limit set of y(t) agrees with that of a convenient solution of the system (2) , one has to follow the same procedure as in the paper [4]. See also [1 ; Ch. 3] . The family of functions $\{ y(t+h) \; ; h \in R_+\}$ is uniformly bounded on R_+ . Moreover , from (26) one easily obtains the estimate

$$(27)\qquad \| y(t) - y(s) \| \le \left|\int_s^t \|\dot{y}(u)\| du\right| \le A^{\frac{1}{2}} |t-s|^{\frac{1}{2}} ,$$

where A is as in (25) , and $t , s \in R_+$ are arbitrary . From (27) one sees that the family $\{ y(t+h) \; ; h \in R_+ \}$ is equicontinuous on R_+ , and consequently , there exists a sequence $\{h_p\}$, $h_p \to \infty$ as $p \to \infty$, such that

(28) $$\lim_{p \to \infty} y(t + h_p) = z(t) , \quad t \in R_+ ,$$

uniformly on any compact interval of R_+ .

We shall prove now that the function $z(t)$ in (28) is a solution of (2) on R_+ , while its limit set coincides with that of $y(t)$.

Let us fix an interval $(t , t+h)$ in R_+ . Simple calculations on (E) lead to the following relationship :

(29) $$\int_t^{t+h} \{ \dot{y}(s+h_p) + (K\dot{y})(s+h_p)\} ds + \int_t^{t+h} f(y(s+h_p))ds = \int_t^{t+h} g(s+h_p)ds .$$

Since $y(t+h_p)$ converges uniformly to $z(t)$ on each compact interval of R_+ ,the Lemma 1 and condition (20) lead to the following consequence of (29) :

(30) $$z(t+h) - z(t) + \int_t^{t+h} f(z(s))ds = 0 .$$

But (30) is equivalent to (2) , due to the fact h is an arbitrary number .

Let us denote now by Y the limit set of the solution $y(t)$ of (E) , and by Z that of the solution $z(t)$ of the system (2) . First of all , we notice that $Z \subset Y$. Indeed , if $z(t_k) \to \bar{z} \in Z$, taking into account that $z(t_k) = \lim y(t_k + t'_m)$ as $m \to \infty$, we obtain for some subsequence $\{t'_{m_k}\} \subset \{t'_m\}$,

$$\bar{z} = \lim_{k \to \infty} y(t_k + t'_{m_k}) ,$$

with $t_k + t'_{m_k} \to \infty$ as $k \to \infty$. This shows that $\bar{z} \in Y$.

Let us assume now $\bar{y} \in Y$. Therefore , $y(t_k) \to \bar{y}$ as $k \to \infty$, for some sequence $\{t_k\}$, with $t_k \to \infty$ as $k \to \infty$. Denote by $z(t)$ the solution of (2) with $z(0) = \bar{y}$. We can prove easily show that $y(t + t_k)$ converges to $z(t)$ as $k \to \infty$, uniformly on any compact interval of R_+ . Choose now a subsequence $\{t_{k_m}\} \subset \{t_k\}$, such that $t_{k_m} > 2t_m$. Then $\bar{t}_m = t_{k_m} - t_m \to \infty$ as $m \to \infty$, and

$$\| y(\bar{t}_m + t_m) - z(\bar{t}_m) \| = \| y(t_{k_m}) - z(t_{k_m} - t_m)\| \to 0$$

as $m \to \infty$. Hence , $\bar{y} = \lim_{m \to \infty} y(t_{k_m}) = \lim_{m \to \infty} z(t_{k_m} - t_m) \in Z$.

Theorem 1 is now completely proven .

Corollary . If we preserve the conditions of Theorem 1 , and assume further that the function $U(y)$ has only a finite number of critical points (i.e. , where $f(y) = 0$) , then any solution of (E) approaches one of those critical points as $t \to \infty$. This follows easily from Theorem 1 , if we take into account the fact that $\dot{z}(t) \to 0$ as $t \to \infty$, and therefore $f(z(t)) \to 0$ as $t \to \infty$. See [4] or [1] for details .

Remark . The positivity (monotonicity) condition (18) , involving the operator $K + \eta I$, is the usual positivity condition on the space $L^2(R_+ , R^n)$. Of course , it can be rewritten introducing the kernel $k(t,s)$. In case of convolution kernels , an equivalent formulation of the positivity condition can be obtained by means of the Fourier transform of the kernel (see again [1] and [4]).

It is interesting to investigate whether the approach used above , for establishing the ultimate behavior of solutions to integrodifferential equations can be extended to higher order systems . Without aiming at the greatest generality , we shall consider the second order integrodifferential system

(31) $$\ddot{y}(t) + A\dot{y}(t) + \int_0^t k(t,s)\dot{y}(s)ds + f(y(t)) = g(t) \quad , t \in R_+ ,$$

where k , f , and g verify the same kind of conditions as in Theorem 1 , while A is a constant n by n matrix with real entries , such that

(32) $$\langle Ay , y \rangle \geq m \|y\|^2 \quad , m > 0 \quad , y \in R^n .$$

Let us multiply (scalarly) both sides of (31) by $\dot{y}(t)$. One obtains

(33) $$\frac{1}{2}\frac{d}{dt}\|\dot{y}(t)\|^2 + \langle A\dot{y}(t) , \dot{y}(t) \rangle + \langle K\dot{y}(t) , \dot{y}(t) \rangle + \langle f(y(t)) , \dot{y}(t) \rangle = \langle g(t) , \dot{y}(t) \rangle ,$$

and integrating both sides from 0 to t ($t > 0$) there follows

(34) $$\frac{1}{2}\|\dot{y}(t)\|^2 + \int_0^t \langle A\dot{y}(s) , \dot{y}(s) \rangle ds + \int_0^t \langle K\dot{y}(s) , \dot{y}(s) \rangle ds + U(y(t)) = \frac{1}{2}\|\dot{y}(0)\|^2 + U(y(0)) + \int_0^t \langle g(s), y(s)\rangle ds.$$

Taking (32) into account , (34) leads to

(35) $$\frac{1}{2}\|\dot{y}(t)\|^2 + m\int_0^t \|\dot{y}(s)\|^2 ds + \int_0^t \langle K\dot{y}(s) , \dot{y}(s)\rangle ds + U(y(t)) \leq \frac{1}{2}\|\dot{y}(0)\|^2 + U(y(0)) + \int_0^t \langle g(s),\dot{y}(s) \rangle ds .$$

Using the same kind of estimate as above for the integral of the scalar product in the right hand side of (35) , one obtains from that inequality

$$(36)\qquad \frac{1}{2}\|\dot{y}(t)\|^2 + \frac{m}{2}\int_0^t \|\dot{y}(s)\|^2 + \int_0^t \langle K\dot{y}(s), \dot{y}(s)\rangle ds + U(y(t)) < \frac{1}{2}\|\dot{y}(0)\|^2 + U(y(0)) + 2m^{-1}\int_0^\infty \|g(s)\|^2 ds .$$

Combining (18) and (36) one obtains

$$(37)\qquad \frac{1}{2}\|\dot{y}(t)\|^2 + \left(\frac{m}{2} - \eta\right)\int_0^t \|\dot{y}(s)\|^2 ds + U(y(t)) \le \frac{1}{2}\|\dot{y}(0)\|^2 + U(y(0)) + 2m^{-1}\int_0^\infty \|g(s)\|^2 ds ,$$

for any t positive , for which the solution is defined . But (37) leads easily to the conclusion that any solution of (31) is defined on the whole halfaxis R_+ , and it is bouded there together with its first derivative (which belongs to the space L^2) , provided

$$(38)\qquad m - 2\eta > 0 .$$

Considering equation (31) , one sees that $\ddot{y}(t)$ is represented (on behalf of our hypotheses and on what we have found above) as a sum of bounded functions . Hence , $\dot{y}(t)$ is also uniformly continuous on R_+ . By repeating the same kind of arguments as in the proof of Theorem 1 , one finds out that for any solution y(t) of (31) there exists a solution z(t) of the system

$$(39)\qquad \ddot{z}(t) + A\dot{z}(t) + f(z(t)) = 0 ,$$

such that $\lim_{k\to\infty} y(t + t_k) = z(t)$, and $\lim_{k\to\infty} \dot{y}(t + t_k) = \dot{z}(t)$, uniformly on any compact interval of R_+ . The sequence $\{t_k\}$ satisfies $t_k \to \infty$ as $k \to \infty$. Furthermore , in the (phase) space of 2n dimensions , where the couple $(y(t),\dot{y}(t))$ belongs for all $t \in R_+$, the limit sets for both $(y(t),\dot{y}(t))$ and $(z(t),\dot{z}(t))$ are the same .

Summarizing the discussion conducted above in regard to the system (31) , we can state the following result .

Theorem 2 . Consider the integrodifferential system (31) , and assume that k , f , and g verify the conditions of the Theorem 1 . If the matrix A is such that (32) and (38) hold true , then any solution of (31) is defined and bounded *on R_+ , together with its first derivative , and the limit set of any solution of the system (31) agrees with that of convenient solution of the ordinary differential system (39) .*

The procedure used in proving Theorems 1 and 2 can be slightly modified , and applied to many other classes of integrodifferential equations . Again without main concern for the greatest possible generality , we shall consider the first order integrodifferential equation

$$(40) \qquad \dot{y}(t) + \int_0^t k(t,s)y(s)ds + f(y(t)) = g(t) \quad , \quad t \in R_+ ,$$

under the initial condition (IC) .

The hypotheses we shall make on k , f , and g will guarantee an ultimate behavior for the solutions of (40) of the same kind as in case of the "reduced" equationn (2) : $\dot{y}(t) + f(y(t)) = 0$.

Local existence for (40) , (IC) is relatively an elementary matter . We shall be concerned only with the ultimate behavior of solutions . It is worth to point out that we shall drop the condition (19) imposed on f(y) in the first two theorems of this paper .

<u>Theorem 3</u> . *Consider the integrodifferential system (40) under the following assumptions :*

1) the matrix-valued kernel $k(t,s)$ is continuous from Δ into $L(R^n, R^n)$, and such that

$$(41) \qquad \lim_{t \to \infty} \int_0^t \| k(t,s) \| ds = 0 \quad ,$$

and

$$(42) \qquad \int_0^t \langle Ky(s) , y(s) \rangle ds \geq 0 \, ,$$

for any $t \geq 0$, and any $y \in L^2(R_+ , R^n)$;

2) $f : R^n \to R^n$ is a continuous map , and verifies

$$(43) \qquad \langle f(y) , y \rangle \geq 0 \, , \quad y \in R^n \, ;$$

3) g is such that

$$g \in C_0(R_+ , R^n) \cap L^1(R_+ , R^n) \, .$$

Then any solution of the system (40) is defined on R_+ , is bounded there , and its limit set agrees with that of a solution of the system (2) .

Proof . Let us prove first that any solution of the system (40) does exist on R_+ , and is bounded there .

Multiplying (scalarly) by y(t) both sides of (40) , one derives after an integration is performed

(44) $$\frac{1}{2}\left(\|y(t)\|^2 - \|y^o\|^2 \right) + \int_o^t \langle Ky(s) , y(s) \rangle ds + \int_o^t \langle f(y(s)),y(s) \rangle ds = \int_o^t \langle g(s),y(s) \rangle ds .$$

Since the second and third terms in the left hand side of (44) are nonnegative , according to (42) and (43) , from (44) one derives

(45) $$\|y(t)\|^2 \leq \|y^o\|^2 + 2 \int_o^t |\langle g(s),y(s)\rangle| \, ds ,$$

for any t > 0 , for which y(t) is defined . Since the right hand side of (45) is a nondecreasing function , (45) and Cauchy's inequality imply

(46) $$Y^2(t) \leq \|y^o\|^2 + 2Y(t) \int_o^\infty \|g(s)\|ds ,$$

where

(47) $$Y(t) = \sup \{ \|y(s)\| ; 0 \leq s \leq t\} .$$

The inequality (46) implies obviously

(48) $$Y(t) \leq \{\|y^o\|^2 + (\int_o^\infty \|g(s)\|ds)^2\}^{1/2} + \int_o^\infty \|g(s)\|ds ,$$

for any t > 0 , which shows that y(t) must remain bounded in the future . Of course , this means that y(t) can be continued to infinity , while the right hand side of (48) provides an upper estimate for $\|y(t)\|$.

Next , the boundedness of y(t) on R_+ implies the boundedness of $\dot{y}(t)$. This is clear from (40) , if one takes into account our hypotheses on k , f , and g . Therefore , y(t) is uniformly continuous on R_+ , which means that the family $\{ y(t+h) ; h \in R_+\}$ is compact in the sense of uniform convergence on any compact interval of R_+ . The proof of Theorem 3 can be further conducted exactly as the proof of Theorem 1 , provided we can show

(49) $$\lim_{t \to \infty} \|\int_o^t k(t,s)y(s)ds \| = 0 .$$

The conditions on k(t,s) have been deliberately chosen in such a manner that (49) hold true for any y(t) bounded on R_+ , which is immediate on behalf of (41)

Theorem 3 is thereby proven .

Remark 1 . An inspection of the proof shows that conditions (42) and (43) can be replaced by a unique condition of positivness , namely

$$\int_0^t \langle Ky(s) + f(y(s)) , y(s) \rangle ds \geq 0 , \quad (50)$$

for any $t > 0$, and any $y(t)$ bounded and continuous on R_+ .

Remark 2 . The procedure used above in investigating the ultimate behavior of solutions of integrodifferential equations , can be adapted to the study of similar problems for equations with infinite delay of the form

$$\dot{y}(t) + \int_{-\infty}^t k(t,s)y(s)ds + f(y(t)) = g(t) , \quad (51)$$

which can be reduced to the form (40) if an initial condition of the form $y(s) = y_o(s)$ is assigned on the negative halfaxis .

R E F E R E N C E S

[1] Corduneanu , C. , *Integral Equations and Stability of Feedback Systems* . Academic Press , New York , 1973 .

[2] Hale , J. K. and Kato , J. , Phase space for retarded differential equations with infinite delay . Funkcialaj Ekvacioj 21 (1978) , 11-41.

[3] Marcus , Moshe and Mizel, Victor , Limiting equations for problems involving long range memory . Memoirs A.M.S. , Vol. 43 , No. 278 , 1983 .

[4] Moser , J. , On nonoscillating networks . Quarterly of Applied Mathematics , XXV (1967) , 1 - 9 .