FREQUENCY DOMAIN CRITERIA FOR NUCLEAR REACTOR STABILITY

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In [7], A. Halanay and V. Rasvan investigated recently a class of integro-differential systems occurring in nuclear reactor dynamics. Their systems represent special cases of the following one:

$$\begin{cases} \dot{x}(t) = (Ax)(t) + (b\rho)(t), \\ \dot{\rho}(t) = -\sum_{k=1}^{M} \beta_k \Lambda^{-1}[\rho(t) - \eta_k(t)] - P\Lambda^{-1}[1 + \rho(t)]\nu(t), \\ \dot{\eta}_k(t) = \lambda_k[\rho(t) - \eta_k(t)], \quad k = 1,2,\dots,M, \\ \nu(t) = (c^*x)(t) + (\alpha\rho)(t), \end{cases} \tag{1}$$

where A, b, c^* and α stand for certain difference-integral operators. More precisely, we assume that these operators are formally given by

$$(Ax)(t) = A_0x(t) + \sum_{j=1}^{\infty} A_jx(t-t_j) + \int_0^t B(t-s)x(s)ds, \tag{2}$$

$$(b\xi)(t) = b_0\xi(t) + \sum_{j=1}^{\infty} b_j\xi(t-t_j) + \int_0^t \beta(t-s)\xi(s)ds, \tag{3}$$

$$(c^*x)(t) = c_0^*x(t) + \sum_{j=1}^{\infty} c_j^*x(t-t_j) + \int_0^t d^*(t-s)x(s)ds, \tag{4}$$

$$(\alpha\rho)(t) = \alpha_0\ (t) + \sum_{j=1}^{\infty} \alpha_j\rho(t-t_j) + \int_{-\infty}^t \gamma(t-s)\rho(s)ds, \tag{5}$$

*Research partially supported by U.S. Army Research Grant DAAG29-80-C-0060.

with $t_j > 0$, $j = 1,2,\ldots,$ and such that the following conditions hold true:

$$\begin{cases} \sum_{j=0}^{\infty} \|A_j\| < +\infty, & \int_0^{\infty} \|B(t)\| dt < +\infty, \\ \sum_{j=0}^{\infty} \|b_j\| < +\infty, & \int_0^{\infty} \|\beta(t)\| dt < +\infty, \\ \sum_{j=0}^{\infty} \|c_j^*\| < +\infty, & \int_0^{\infty} \|d^*(t)\| dt < +\infty, \\ \sum_{j=0}^{\infty} |\alpha_j| < +\infty, & \int_0^{\infty} |\gamma(t)| dt < +\infty. \end{cases} \tag{6}$$

The constants β_k, λ_k, $k = 1,2,\ldots,M$, and Λ, P are assumed to be positive. Their physical meaning has been discussed in [7], as well as in the papers quoted there.

One more remark is necessary before proceeding further in investigating the system (1). The fact that the delays t_j are the same for each operator A, b, c^* or α does not cause any loss of generality. Indeed, if different sequences of delays are considered for the above operators, then the union of these sequences is again countable and, therefore, it can be conveniently denoted by $\{t_j\}$. We have to add then some null coefficients A_j, b_j, c_j^* and α_j in the representation of the operators A, b, c^* and α, such that they take the forms (2)-(5). It is worth to point out the fact that conditions (6) keep their validity.

Further assumptions will be made on the system (1), which obviously constitutes an integro-differential system with infinite delay. The nonlinear part of (1) occurs only in the right hand side of the second equation and it is of quadratic type.

In order to determine a unique solution of (1), it is necessary to prescribe some initial data. Taking into account the form of these equations, there results that

$$x(t) = h(t), \quad \rho(t) = \lambda(t), \quad t < 0, \tag{7}$$

and

$$x(0+) = x^0, \quad \rho(0+) = \rho_0, \quad \eta_k(0+) = \eta_k^0, \quad k = 1,2,\dots,M,$$

constitute the appropriate kind of initial conditions for (1).

We usually assume that $\|h(t)\|, |\lambda(t)| \in L(R_-,R)$.

Besides (1), we shall consider the linear system with a real parameter h:

$$\begin{cases} \dot{y}(t) = (Ay)(t) + (b\xi)(t), \\ \dot{\xi}(t) = -\sum_{k=1}^{M} \beta_k \Lambda^{-1}[\xi(t) - \zeta_k(t)] - P\Lambda^{-1}h[(c^*y)(t) + (\alpha\xi)(t)], \\ \dot{\zeta}_k(t) = \lambda_k[\xi(t) - \zeta_k(t)], \quad k = 1,2,\dots,M, \end{cases} \tag{9}$$

on which certain assumptions will be made in the sequel.

Associated to (9) is the linear block (i.e., the linear control system with $\mu(t)$ as input),

$$\begin{cases} \dot{y}(t) = (Ay)(t) + (b\xi)(t), \\ \dot{\xi}(t) = -\sum_{k=1}^{M} \beta_k \Lambda^{-1}[\xi(t) - \zeta_k(t)] - P\Lambda^{-1}h[(c^*y)(t) + (\alpha\xi)(t)] + \mu(t), \\ \dot{\zeta}_k(t) = \lambda_k[\xi(t) - \zeta_k(t)], \quad k = 1,2,\dots,M, \\ \psi(t) = (c^*y)(t) + (\alpha\xi)(t). \end{cases} \tag{10}$$

A unique solution for (9) or (10) is determined if initial conditions of the form (7), (8) are given:

$$y(t) = \ell(t), \quad \xi(t) = \phi(t), \quad t < 0, \tag{11}$$

and

$$y(0+) = y^0, \quad \xi(0+) = \xi_0, \quad \zeta_k(0+) = \zeta_k^0, \quad k = 1,2,\dots,M. \tag{12}$$

Let us define now the following matrices:

$$A_{h0} = \begin{bmatrix} A_0 & b_0 & 0 & 0 & \cdots & 0 \\ -P\Lambda^{-1}hc_0^* & -\Lambda^{-1}(Ph\alpha_0+\sum_1^M \beta_k) & \Lambda^{-1}\beta_1 & \Lambda^{-1}\beta_2 & \cdots & \Lambda^{-1}\beta_M \\ & \lambda_1 & -\lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & -\lambda_2 & \cdots & 0 \\ & \vdots & \vdots & \vdots & & \\ & \lambda_M & 0 & 0 & \cdots & -\lambda_M \end{bmatrix}, \quad (13)$$

$$A_{hj} = \begin{bmatrix} A_j & b_j & 0 \\ -P\Lambda^{-1}hc_j^* & -P\Lambda^{-1}h\alpha_j & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad j = 1,2,\ldots, \quad (14)$$

and

$$B_h(t) = \begin{bmatrix} B(t) & \beta(t) & 0 \\ -P\Lambda^{-1}hd^*(t) & -P\Lambda^{-1}h\gamma(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (15)$$

If we consider the vector $z = \mathrm{col}(y,\xi,\zeta_1,\ldots,\zeta_M)$, then system (9) becomes

$$\dot{z}(t) = A_{h0}z(t) + \sum_{j=1}^{\infty} A_{hj}z(t-t_j) + \int_0^t B_h(t-s)z(s)ds + u(t), \quad (16)$$

with

$$u(t) = \mathrm{col}(\underbrace{0,\ldots,0}_{n},\ -P\Lambda^{-1}h\int_{-\infty}^{0} \gamma(t-s)\phi(s)ds, \underbrace{0,\ldots,0}_{M}).$$

The linear block (10) leads to

$$\dot{z}(t) = A_{h0}z(t) + \sum_{j=1}^{\infty} A_{hj}z(t-t_j) + \int_0^t B_h(t-s)z(s)ds + \tilde{\mu}(t), \quad (17)$$

with

$$\tilde{\mu}(t) = \mu(t)\mathrm{col}(\underbrace{0,\dots,0}_{n},1,\underbrace{0,\dots,0}_{M}) + u(t).$$

Accordingly, the nonlinear system (1) can be rewritten in the form

$$\dot{w}(t) = A_{10}w(t) + \sum_{j=1}^{\infty} A_{1j}w(t-t_j) + \int_0^t B_1(t-s)w(s)ds + f(t;w), \qquad (18)$$

where $w = \mathrm{col}(x,\rho,\eta_1,\dots,\eta_M)$ and $f(t;w)$ is given by

$$f(t;w) = \mathrm{col}(\underbrace{0,\dots,0}_{n},-P\Lambda^{-1}[\int_{-\infty}^{0}\gamma(t-s)\phi(s)ds + \rho(t)\nu(t)],\underbrace{0,\dots,0}_{M}). \qquad (19)$$

It is now obvious that systems (16) and (17) are linear systems of the form we dealt with in [2], [4], [6], while (18) constitutes a nonlinear perturbed system. Taking into account conditions (6) from the definition of the matrices A_{hj} and B_n, there results

$$\sum_{j=0}^{\infty} \| A_{hj} \| < +\infty, \quad \int_0^{\infty} \| B_h(t) \| dt < +\infty, \qquad (20)$$

for any real h.

Therefore, we can use the techniques developed in the papers [2], [4], [6], in order to investigate the stability properties of the systems (16), (17) or (18).

Let us consider now the system (16) for $h = h_2$, and the linear block (17) for $h = h_1$. We shall prove that their solutions coincide on the positive half-axis, provided initial data and input function $\mu(t)$ are chosen in a convenient manner.

More precisely, let us denote by $x(t)$, $\rho(t)$, $\eta_k(t)$, $k = 1, 2,\dots,M$, a solution of (1), with initial data (7), (8). We assume that this solution is defined on a certain interval $[0,T]$, $T > 0$.

On behalf of the above solution of (1) or (18), let us construct the solution $\bar{y}(t)$, $\bar{\xi}(t)$, $\bar{\zeta}_k(t)$, $k = 1,2,\dots,M$, of (9) or (16), for $t \geq T$, corresponding to the following initial data:

$$\bar{y}(t) = \begin{cases} h(t), & t < 0, \\ x(t), & 0 \le t \le T, \end{cases} \qquad \bar{\xi}(t) = \begin{cases} \lambda(t), & t < 0, \\ \rho(t), & 0 \le t \le T, \end{cases} \tag{21}$$

and $\bar{\zeta}_k(T) = \eta_k(T)$, $k = 1,2,\dots,M$. The existence and uniqueness of the solution $\bar{y}(t)$, $\bar{\xi}(t)$, $\bar{\zeta}_k(t)$, $k = 1,2,\dots,M$, are guaranteed by the results given in [2], [4]. As said above, (9) is considered for $h = h_2$.

Concerning the linear system (block) (10) or (17), we are interested in the solution $y(t)$, $\xi(t)$, $\zeta_k(t)$, $k = 1,2,\dots,N$, corresponding to the initial data $h(t)$, x^0, $\lambda(t)$, ρ_0, η_k^0, $k = 1,2,\dots,M$, and to the value h_1 of the parameter h. Moreover, we assume that the input $\mu(t)$ is given by

$$\mu_T(t) = \begin{cases} -P\Lambda^{-1}[1 + \rho(t) - h_1]\nu(t), & 0 \le t \le T, \\ -P\Lambda^{-1}(h_2 - h_1)\bar{\nu}(t), & t > T, \end{cases} \tag{22}$$

with

$$\bar{\nu}(t) = (c^*\bar{y})(t) + (\alpha\bar{\xi})(t), \quad t > T. \tag{23}$$

Lemma 1. With the above mentioned notations, the following equalities hold true:

$$y(t) = x(t),\ \xi(t) = \rho(t),\ \zeta_k(t) = \eta_k(t),\ k = 1,2,\dots,M,\ 0 \le t \le T, \tag{24}$$

and

$$y(t) = \bar{y}(t),\ \xi(t) = \bar{\xi}(t),\ \zeta_k(t) = \bar{\zeta}_k(t),\ k = 1,2,\dots,M,\ t > T. \tag{25}$$

Proof. If we denote $u = y - x$, $v = \xi - \rho$, $w_k = \zeta_k - \eta_k$, $k = 1,2,\dots,M$, on $[0,T)$ and $u = y - \bar{y}$, $v = \xi - \bar{\xi}$, $w_k = \zeta_k - \bar{\zeta}_k$, $k = 1,2,\dots,M$, on $[T,\infty)$, then we find from (1), (9), (10) and (22):

$$\begin{cases} \dot{u}(t) = (Au)(t) + (bv)(t), \\ \dot{v}(t) = -\sum_{k=1}^{M} \beta_k \Lambda^{-1}[v(t) - w_k(t)], \\ \dot{w}_k(t) = \lambda_k[v(t) - w_k(t)], \quad k = 1,2,\ldots,M. \end{cases} \tag{26}$$

The equations in (26) hold true on R_+. First, we restrict our considerations to the interval $[0,T]$. According to our assumptions in constructing x, ρ, η_k and y, ξ, ζ_k, there results that $u(t) \equiv 0$, $v(t) \equiv 0$, $w_k(t) \equiv 0$, $k = 1,2,\ldots,M$, on $[0,T]$. Indeed, u, v, w_k satisfy a homogeneous system and the corresponding initial data are all zero. Then, let us consider (26) as a system with initial data on $(-\infty,T]$. According to (21) and to the above established equalities on $[0,T]$, the solution $y(t)$, $\xi(t)$, $\zeta_k(t)$ of (10), with the control function given by (22), must coincide with the solution $\overline{y}(t)$, $\overline{\xi}(t)$, $\overline{\zeta}_k(t)$ of (9), i.e., (25) hold true.

This ends the proof of Lemma 1.

<u>Remark</u>. One obtains from Lemma 1 that $\psi(t)$ from (10) can be represented as

$$\psi(t) = \begin{cases} \nu(t), & 0 \le t \le T, \\ \overline{\nu}(t), & t > T. \end{cases} \tag{27}$$

Next lemma is a direct consequence of a result established in [5]. We denote by $\hat{h}(t)$ the map reducing to $h(t)$ for $t < 0$, such that $h(t) = 0$ for $t \geq 0$. $\hat{\lambda}(t)$ is defined similarly.

<u>Lemma 2</u>. Let $y(t)$, $\xi(t)$, $\zeta_k(t)$ be the solution of (10), corresponding to the control function $\mu_T(t)$ given by (22), the initial data $h(t)$, x^0, $\lambda(t)$, ρ_0, η_k^0, $k = 1,2,\ldots,M$, and $h = h_1$. Then, this solution is identical with that of the linear system

$$\begin{cases} \dot{y}_T(t) = (Ay_T)(t) + (b\xi_T)(t) + \sum_{j=1}^{\infty} A_j\hat{h}(t-t_j) + \sum_{j=1}^{\infty} b_j\hat{\lambda}(t-t_j), \\ \dot{\xi}_T(t) = -\sum_{k=1}^{M} \beta_k\Lambda^{-1}[\xi_T(t) - \zeta_{kT}(t)] - P\Lambda^{-1}h_1[(c^*y_T)(t) \\ \qquad + \sum_{j=0}^{\infty} \alpha_j\xi_T(t-t_j) + \int_0^t \gamma(t-s)\xi_T(s)ds] - P\Lambda^{-1}h_1[(c^*\hat{h})(t) \\ \qquad + \sum_{j=1}^{\infty} \alpha_j\hat{\lambda}(t-t_j) + \int_{-\infty}^0 \gamma(t-s)\lambda(s)ds] + \mu_T(t), \\ \dot{\zeta}_{kT}(t) = \lambda_k[\xi_T(t) - \zeta_{kT}(t)], \quad k = 1,2,\dots,M, \end{cases} \tag{28}$$

with zero initial data on $(-\infty,0)$ and the data x^0, ρ_0, η_k^0 at $t = 0$.

Indeed, if (10) is considered instead of (1), then (28) represents nothing else but the system given by formula (11) in [5]. Of course, (10) is to be considered for $h = h_1$, and with $\mu(t) = \mu_T(t)$, given by (22).

Before proceeding further with the investigation, let us point out that denoting

$$\begin{cases} \psi_T(t) = (c^*y_T)(t) + \sum_{j=0}^{\infty} \alpha_j\xi_T(t-t_j) + \int_0^t \gamma(t-s)\xi_T(s)ds, \quad t > 0, \\ \psi_0(t) = (c^*\hat{h})(t) + \sum_{j=1}^{\infty} \alpha_j\hat{\lambda}(t-t_j) + \int_{-\infty}^0 \gamma(t-s)\lambda(s)ds, \quad t > 0, \end{cases} \tag{29}$$

and taking into account the relations

$$y(t) = y_T(t) + \hat{h}(t), \quad \xi(t) = \xi_T(t) + \hat{\lambda}(t), \quad \zeta(t) = \zeta_{kT}(t), \tag{30}$$

and (27), we obtain

$$\begin{cases} \psi(t) = \nu(t) = \psi_T(t) + \psi_0(t), \quad 0 \le t \le T, \\ \psi(t) = \bar{\nu}(t) = \psi_T(t) + \psi_0(t), \quad t > T. \end{cases} \tag{31}$$

Lemma 3. Any solution of system (1), that satisfies the initial constraints $1 + \rho(0) > 0$, $1 + \eta_i(0) > 0$, $i = 1,2,\ldots,M$, satisfies also

$$1 + \rho(t) > 0, \quad 1 + \eta_i(t) > 0, \quad i = 1,2,\ldots,M,$$

for $t > 0$.

The proof is elementary and can be found in [11]. It is useful to point out that the equations for $\eta_i(t)$ can be written in the form

$$\frac{d}{dt}[1 + \eta_i(t)] + \lambda_i[1 + \eta_i(t)] = \lambda_i[1 + \rho(t)].$$

The following notations are necessary in order to state the main stability results of this paper. First, we consider the rational function

$$R(s) = s^{-1}(1+\Lambda^{-1}\sum_{k=1}^{M} \frac{\beta_k}{s+\lambda_k})^{-1}, \tag{32}$$

which represents the transfer function corresponding to the effect of the delayed neutrons. Next, let us consider the transfer function

$$\tilde{k}(s) = \tilde{c}^*(s)[sI - A(s)]^{-1}\tilde{b}(s), \tag{33}$$

where $A(s)$ is the symbol of the operator A given by (2) $\tilde{b}(s)$, $\tilde{c}^*(s)$ have similar definitions (see [2]). Further, let us assume $\gamma_0(s) = \tilde{k}(s) + \tilde{\alpha}(s)$ and

$$\gamma_1(s) = R(s)[1 + P\Lambda^{-1}h_1R(s)\gamma_0(s)]^{-1}, \tag{34}$$

$$\gamma_2(s) = P\Lambda^{-1}h_1\gamma_0(s)\gamma_1(s), \quad h_1 > 0. \tag{35}$$

Theorem. Consider the system (1), with A, b, c^* *and* α *given by (2)-(5). The conditions (6) are supposed to hold true, while constants* β_k, λ_k, $k = 1,2,\ldots,M$, Λ *and* P *are assumed positive. Assume further that:*

(a) $\det[sI - A(s)] \neq 0$ *for* $\operatorname{Re} s \geq 0$, *i.e., the linear system* $\dot{x}(t) = (Ax)(t)$ *is asymptotically stable;*

(b) There exist some numbers h_1, h_2, δ_0, δ_1, δ_2, *such that* $0 < h_1 < 1 < h_2$, $\delta_0 \geq 0$, $\delta_2 \geq \delta_1 \geq 0$, *with* $\delta_1 + \delta_2 > 0$, *and*

(1°) *the linear system (9) is asymptotically stable for* $h = h_1$ *and* $h = h_2$;

(2°) *the numbers* $\gamma_0 \in (0, 1-h_1)$ *and* $\bar{\gamma}_0 \in (0, \sqrt{h_2}-1)$ *are so chosen that for the real function*

$$\Phi(\xi) = \xi - \ell n(1+\xi) - (2h_2)^{-1}\xi^2$$

one has

$$\Phi(\xi) \leq \phi(\sqrt{h_2}-1)\min\{1, \beta_k\lambda_k^{-1}\Lambda^{-1};\quad k = 1,2,\dots,M\},$$

as soon as $\xi \in [-\gamma_0, \bar{\gamma}_0]$;

(3°) *if*

$$H(s) = \delta_0[(h_2-h_1)^{-1} + h_1^{-1}\gamma_2(s)] + \delta_1\gamma_1(s) + \delta_2 P\Lambda^{-1}(h_2-h_1)|\gamma_1(s)|^2\gamma_0(s), \tag{36}$$

then

$$\text{Re } H(i\omega) > 0, \quad \omega \in R, \tag{37}$$

holds true when $\delta_0 > 0$, *and besides (37),*

$$\delta_1\Lambda^{-1}\sum_{k=1}^{M}\lambda_k + [\delta_1 h_1 + \delta_2(h_2-h_1)](\alpha_0 - \sum_{j=1}^{\infty}|\alpha_j|) > 0 \tag{38}$$

holds true when $\delta_0 = 0$.

Under the above assumptions, each solution $x(t)$, $\rho(t)$, $\eta_k(t)$, $k = 1,2,\dots,M$, *of the system (1), with*

$$\|h(t)\|, |\lambda(t)| \in L^1(R_-,R) \cap L^2(R_-,R), \tag{39}$$

is defined on the positive half-axis and tends to zero at infinity

$$\lim(\|x(t)\| + |\rho(t)| + \sum_{k=1}^{M}|\eta_k(t)|) = 0, \quad \text{as} \quad t \to +\infty,$$

provided certain initial constraints are imposed (see [6]).

Proof. First, we consider the linear block (10), for $h = h_1$ and $\mu(t) = \mu_T(t)$ as given by (22). It is assumed that $T > 0$ is chosen such that the solution $x(t)$, $\rho(t)$, $\eta_k(t)$, $k = 1,2,\dots,M$, be defined on $[0,T]$. Let us associate to (10) the integral index

$$\chi(T) = \delta_0\int_0^T \mu_T(t)[(h_2-h_1)^{-1}\mu_T(t) + P\Lambda^{-1}\psi(t)]dt$$
$$+ \int_0^t \xi(t)[\delta_1\mu_T(t) + \delta_2 P\Lambda^{-1}(h_2-h_1)\psi(t)]dt. \qquad (41)$$

The initial data for (10) are the same as for (1). Taking into account (24) from Lemma 1 and (22), one obtains by means of elementary calculations

$$\chi(T) = \delta_0(P\Lambda^{-1})^2(h_2-h_1)^{-1}\int_0^T [h_1 - 1 - \rho(t)][h_2 - 1 - \rho(t)]\nu^2(t)dt$$
$$- \delta_1 h_2[\Omega(T) - \Omega(0)]$$
$$- \delta_1\Lambda^{-1}\sum_{k=1}^{M}\beta_k\int_0^T [\rho(t) - \eta_k(t)]^2\left\{h_2[1+\rho(t)]^{-1}[1+\eta_k(t)]^{-1} - 1\right\}dt$$
$$- (\delta_2-\delta_1)(h_2-h_1)\Big\{\Omega_1(T) - \Omega_1(0) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\int_0^T [\rho(t) - \eta_k(t)]^2$$
$$[1+\rho(t)]^{-1}[1+\eta_k(t)]^{-1}dt\Big\}, \qquad (42)$$

with

$$\Omega(t) = \Phi(\rho(t)) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}\Phi(\eta_k(t)),$$

$$\Omega_1(t) = \phi(\rho(t)) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}\phi(\eta_k(t)),$$

$$\phi(\xi) = \xi - \ell n(1+\xi),$$

where $\Phi(\xi)$ is the function defined in the statement of the Therem. Indeed, one gets from (22), (24) and (27)

$$\int_0^T \mu_T(t)[(h_2-h_1)^{-1}\mu_T(t) + P\Lambda^{-1}\psi(t)]dt$$

$$= (P\Lambda^{-1})^2(h_2-h_1)^{-1}\int_0^T [1+\rho(t)-h_1]^2\nu^2(t)dt$$

$$- (P\Lambda^{-1})^2\int_0^T [1+\rho(t)-h_1]\nu^2(t)dt$$

$$= (P\Lambda^{-1})^2(h_2-h_1)^{-1}\int_0^T [h_1-1-\rho(t)][h_2-1-\rho(t)]\nu^2(t)dt$$

$$+ (P\Lambda^{-1})^2\int_0^T [1+\rho(t)-h_1]\nu^2(t)dt - (P\Lambda^{-1})^2\int_0^T [1+\rho(t)-h_1]\nu^2(t)dt$$

$$= (P\Lambda^{-1})^2(h_2-h_1)^{-1}\int_0^T [h_1-1-\rho(t)][h_2-1-\rho(t)]\nu^2(t)dt.$$

We have further

$$\int_0^T \xi(t)\mu_T(t)dt = \int_0^T \rho(t)\{\dot\rho(t) + \Lambda^{-1}\sum_{k=1}^M \beta_k[\rho(t)-\eta_k(t)] + P\Lambda^{-1}h_1\nu(t)\}dt$$

$$= \int_0^T \{\rho(t)\dot\rho(t) + \Lambda^{-1}\sum_{k=1}^M \beta_k[\rho(t)-\eta_k(t)]^2$$

$$+ \Lambda^{-1}\sum_{k=1}^M \beta_k\eta_k(t)[\rho(t)-\eta_k(t)]\}dt + P\Lambda^{-1}h_1\int_0^T \rho(t)\nu(t)dt$$

$$= \int_0^T \{\rho(t)\dot\rho(t) + \Lambda^{-1}\sum_{k=1}^M \beta_k\lambda_k^{-1}\eta_k(t)\dot\eta_k(t)\}dt$$

$$+ \Lambda^{-1}\sum_{k=1}^M \beta_k\int_0^T [\rho(t)-\eta_k(t)]^2dt + P\Lambda^{-1}h_1\int_0^T \rho(t)\nu(t)dt$$

$$= \frac{1}{2}[\rho^2(t) + \Lambda^{-1}\sum_{k=1}^M \beta_k\lambda_k^{-1}\eta_k^2(t)]\Big|_0^T$$

$$+ \Lambda^{-1}\sum_{k=1}^M \beta_k\int_0^T [\rho(t)-\eta_k(t)]^2dt + P\Lambda^{-1}h_1\int_0^T \rho(t)\nu(t)dt$$

and

$$\int_0^T \xi(t)\psi(t)dt = \int_0^T \rho(t)\nu(t)dt$$

$$= -P^{-1}\Lambda\int_0^T [\dot{\rho}(t) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}\dot{\eta}_k(t) + P\Lambda^{-1}\nu(t)]dt$$

$$= -P^{-1}\Lambda[\rho(t) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}\eta_k(t)]\Big|_0^T - \int_0^T \nu(t)dt.$$

In order to evaluate the last integral, we shall use again the second equation of (1). According to Lemma 3, we can divide both members of the second equation of (1) by $1 + \rho(t)$, provided ρ_0 is such that $1 + \rho_0 > 0$. Therefore,

$$[1+\rho(t)]^{-1}\dot{\rho}(t) = -\Lambda^{-1}\sum_{k=1}^{M}\beta_k[1+\rho(t)]^{-1}[\rho(t) - \eta_k(t)] - P\Lambda^{-1}\nu(t),$$

which leads to

$$[1+\rho(t)]^{-1}\dot{\rho}(t) = -\Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}[1+\rho(t)]^{-1}\dot{\eta}_k(t) - P\Lambda^{-1}\nu(t),$$

if one considers also the third equation of (1). But

$$[1+\rho(t)]^{-1}\dot{\eta}_k(t) = [1+\eta_k(t)]^{-1}\dot{\eta}_k(t)$$
$$- [1+\rho(t)]^{-1}[1+\eta_k(t)]^{-1}[\rho(t) - \eta_k(t)]\dot{\eta}_k(t),$$

which allows us to write

$$[1+\rho(t)]^{-1}\dot{\rho}(t) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}[1+\eta_k(t)]^{-1}\dot{\eta}_k(t)$$

$$= \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}[1+\rho(t)]^{-1}[1+\eta_k(t)]^{-1}[\rho(t) - \eta_k(t)]\dot{\eta}_k(t) - P\Lambda^{-1}\nu(t)$$

$$= \Lambda^{-1}\sum_{k=1}^{M}\beta_k[1+\rho(t)]^{-1}[1+\eta_k(t)]^{-1}[\rho(t) - \eta_k(t)]^2 - P\Lambda^{-1}\nu(t).$$

Hence

$$\{\ell n[1+\rho(t)] + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}\ell n[1+\eta_k(t)]\}\Big|_0^T$$

$$= \Lambda^{-1}\sum_{k=1}^{M}\beta_k\int_0^T[\rho(t)-\eta_k(t)]^2[1+\rho(t)]^{-1}[1+\eta_k(t)]^{-1}dt - P\Lambda^{-1}\int_0^T\nu(t)dt.$$

Summing up the above considerations, we get the formula (42) for $\chi(T)$.

We shall write now $\chi(T)$ in another form. Namely

$$\chi(T) = \delta_0\int_0^\infty\mu_T(t)\{(h_2-h_1)^{-1}\mu_T(t) + P\Lambda^{-1}[\psi_T(t) + \psi_0(t)]\}dt$$

$$+ \int_0^\infty\xi_T(t)\{\delta_1\mu_T(t) + \delta_2(h_2-h_1)P\Lambda^{-1}[\psi_T(t) + \psi_0(t)]\}dt$$

$$- (\delta_2-\delta_1)(h_2-h_1)P\Lambda^{-1}\int_T^\infty\xi_T(t)[\psi_T(t) + \psi_0(t)]dt. \qquad (43)$$

Indeed, taking into account (22), (27) and (31), one sees that the integrand is zero, for $t > T$, in the first integral from (43). Concerning the second integral in (43), we have

$$\int_T^\infty\xi_T(t)\{\delta_1\mu_T(t) + \delta_2(h_2-h_1)P\Lambda^{-1}[\psi_T(t) + \psi_0(t)]\}dt$$

$$= \delta_1\int_T^\infty\xi_T(t)\mu_T(t)dt + \delta_2(h_2-h_1)P\Lambda^{-1}\int_T^\infty\xi_T(t)[\psi_T(t) + \psi_0(t)]dt$$

$$= -\delta_1(h_2-h_1)P\Lambda^{-1}\int_T^\infty\xi_T(t)[\psi_T(t) + \psi_0(t)]dt$$

$$+ \delta_2(h_2-h_1)P\Lambda^{-1}\int_T^\infty\xi_T(t)[\psi_T(t) + \psi_0(t)]dt$$

$$= (\delta_2-\delta_1)(h_2-h_1)P\Lambda^{-1}\int_T^\infty\xi_T(t)[\psi_T(t) + \psi_0(t)]dt.$$

which proves the validity of (43).

Let us point out that, under our assumptions, the second integral in (43) is convergent. Indeed, taking (39) into account, there results $\psi_0(t) \in L^1 \cap L^2$. From condition (b) in the statement of the Theorem, one derives $\mu_T(t), \xi_T(t) \in L^p$, $1 \leq p \leq \infty$, using again the results in [4]. Consequently, the integrand belongs to L^1.

Before using Parseval's formula to find a new form for $\chi(T)$, we shall express conveniently certain Laplace transforms of some functions we deal with. First, let us compute the Laplace transform $\tilde{\xi}_T(s)$ from the system (28). Taking the Laplace transform of both sides of each equation and substituting the values of $\tilde{y}_T(s)$ and $\tilde{\zeta}_{kT}(s)$ in the second equation, there results

$$\tilde{\xi}_T(s) = \gamma_1(s)\tilde{\mu}_T(s) + \gamma_1(s)[\rho_0 + N(s)], \tag{44}$$

with $\gamma_1(s)$ given by (34) and

$$\begin{cases} N(s) = \Lambda^{-1} \sum_{k=1}^{M} \beta_k \eta_k^0 (s+\lambda_k)^{-1} - P\Lambda^{-1} H_1 M(s), \\ M(s) = \tilde{\psi}_0(s) + \tilde{c}^*(s)[sI - A(s)]^{-1} m(s), \end{cases} \tag{45}$$

where

$$m(s) = x^0 + \sum_{j=1}^{\infty} A_j e^{-st_j} \int_{-t_j}^{0} h(t) e^{-st} dt + \sum_{j=1}^{\infty} b_j e^{-st_j} \int_{-t_j}^{0} \lambda(t) d^{-st} dt. \tag{46}$$

From the definition of $\psi_T(t)$ (see (29)), one finds

$$\begin{aligned} \tilde{\psi}_T(s) &= \tilde{c}^*(s)\tilde{y}_T(s) + \tilde{\alpha}(s)\tilde{\xi}_T(s) \\ &= \gamma_0(s)\tilde{\xi}_T(s) + \tilde{c}^*(s)[sI - A(s)]^{-1} m(s) \\ &= \gamma_0(s)\gamma_1(s)\tilde{\mu}_T(s) + \gamma_0(s)\gamma_1(s)[\rho_0 + N(s)] \\ &\quad + \tilde{c}^*(s)[sI - A(s)]^{-1} m(s). \end{aligned} \tag{46}$$

The proof has to be continued in accordance to the following feature: the parameter $\delta_0 \neq 0$, or $\delta_0 = 0$.

Case 1 ($\delta_0 \neq 0$). In this case, we shall apply a lemma due to Popov [10], based on Parseval's formula, in order to find another form for $\chi(T)$. We start from (43) and denote:

$$\begin{cases} f_1(t) = -\xi_T(t), \\ f_2(t) = \delta_1\mu_T(t) + \delta_2(h_2-h_1)P\Lambda^{-1}[\psi_T(t) + \psi_0(t)], \\ f_3(t) = \delta_0(h_2-h_1)^{-1}\mu_T(t) + \delta_0 P\Lambda^{-1}[\psi_T(t) + \psi_0(t)], \\ f_4(t) = -\mu_T(t). \end{cases} \tag{48}$$

Taking into account (44) and (47), we find out the following formulas for the Fourier transforms of the functions defined by (48):

$$\tilde{f}_j(i\omega) = U_j(i\omega)\tilde{f}_4(i\omega) + V_j(i\omega), \quad j = 1,2,3. \tag{49}$$

In (49), the functions U_j and V_j, $j = 1,2,3$, are given by

$$\begin{cases} U_1(i\omega) = \gamma_1(i\omega), \\ -U_2(i\omega) = \delta_1 + \delta_2(h_2-h_1)P\Lambda^{-1}\gamma_0(i\omega)\gamma_1(i\omega), \\ -U_3(i\omega) = \delta_0(h_2-h_1)^{-1} + \delta_0 h_1^{-1}\gamma_2(i\omega), \\ V_1(i\omega) = -\gamma_1(i\omega)[\rho_0 + N(i\omega)], \\ V_2(i\omega) = \delta_2(h_2-h_1)P\Lambda^{-1}\{\gamma_0(i\omega)\gamma_1(i\omega)[\rho_0 + N(i\omega)] + M(i\omega)\}, \\ V_3(i\omega) = \delta_0 P\Lambda^{-1}\{\gamma_0(i\omega)\gamma_1(i\omega)[\rho_0 + N(i\omega)] + M(i\omega)\}. \end{cases} \tag{50}$$

Popov's lemma states that

$$\int_0^\infty f_1(t)f_2(t)dt + \int_0^\infty f_3(t)f_4(t)dt \leq \frac{1}{8\pi}\int_{-\infty}^{\infty} \frac{|W(i\omega)|^2 d\omega}{\text{Re } H(i\omega)} + \frac{1}{4\pi}\int_{-\infty}^{\infty}[V_1(i\omega)V_2(-i\omega) + V_1(-i\omega)V_2(i\omega)]d\omega, \tag{51}$$

with $H(s)$ given by (36), and

$$W(i\omega) = V_1(i\omega)U_2(-i\omega) + V_2(i\omega)U_1(-i\omega) + V_3(i\omega)$$
$$= [\delta_0 h_1^{-1}\gamma_2(i\omega) + \delta_1\gamma_1(i\omega)$$
$$+ 2\delta_2 P\Lambda^{-1}|\gamma_1(i\omega)|^2 \mathrm{Re}\gamma_0(i\omega)[\rho_0 + N(i\omega)]$$
$$+ P\Lambda^{-1}[\delta_0 + \delta_2(h_2-h_1)\gamma_1(-i\omega)]M(i\omega). \tag{52}$$

From (36) there follows

$$\lim \mathrm{Re}\, H(i\omega) = \delta_0(h_2-h_1)^{-1} > 0, \quad \text{as} \quad |\omega| \to \infty. \tag{53}$$

Consequently, one finds $\varepsilon_0 > 0$, such that

$$\mathrm{Re}\, H(i\omega) \geq \varepsilon_0, \quad \omega \in R. \tag{54}$$

Taking into account (54), it suffices to show that $W(i\omega)$, $V_1(i\omega)$, $V_2(i\omega) \in L^2(R,C)$, in order to make sure that the integrals in the right hand side of (51) are convergent. Indeed, $R(i\omega)$ given by (32) behaves at infinity like $|\omega|^{-1}$, while $\gamma_0(i\omega)$ is bounded on R. The boundedness of $\gamma_0(i\omega)$ is a consequence of conditions (6), of the fact that $\tilde{X}(s) = [sI - A(s)]^{-1}$, and of condition (a) from the Theorem (see also [4]). Moreover, $\gamma_1(i\omega)$ and $\gamma_2(i\omega)$ also behave at infinity like $|\omega|^{-1}$. It is an elementary matter to show that: $m(s)$ is bounded for $\mathrm{Re} s = 0$; $M(i\omega)$ is a function in L^2; $N(i\omega)$ also belongs to L^2. Summing up the above discussion, one finds out that $W(i\omega)$, $V_1(i\omega)$ and $V_2(i\omega)$ are in L^2. Therefore, the right hand side of (51) is finite.

Substituting in (51) the functions $f_j(t)$, $j = 1,2,3,4$, by their values given by (48), and taking into account (43), one obtains

$$-\chi(T) \leq \frac{1}{8\pi}\int_{-\infty}^{\infty} \frac{|W(i\omega)|^2}{\mathrm{Re}\, H(i\omega)}\, d\omega + \frac{1}{4\pi}\int_{-\infty}^{\infty} [V_1(i\omega)V_2(-i\omega)+V_1(-i\omega)V_2(i\omega)]d\omega$$
$$+ (\delta_2-\delta_1)(h_2-h_1)P\Lambda^{-1}\int_T^{\infty} \xi_T(t)[\psi_T(t) + \psi_0(t)]dt. \tag{55}$$

We want to transform now the last integral occurring in the right hand side of (55). We have to repeat practically some computations encountered in deriving formula (42). Taking into account that (9) is asymptotically stable for $h = h_2$, there results

$$P\Lambda^{-1}\int_T^\infty \xi_T(t)[\psi_T(t) + \psi_0(t)]dt = P\Lambda^{-1}\int_T^\infty \overline{\xi}(t)\overline{\nu}(t)dt$$

$$= -h_2^{-1}\int_T^\infty \overline{\xi}(t)[\dot{\overline{\xi}}(t) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k(\overline{\xi}(t)-\overline{\zeta}_k(t))]dt$$

$$= \frac{1}{2h_2}[\rho^2(T) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}\eta_k^2(T)] - (\Lambda h_2)^{-1}\sum_{k=1}^{M}\beta_k\int_T^\infty[\overline{\xi}(t) - \overline{\zeta}_k(t)]^2dt.$$

Hence, (55) becomes now

$$-\chi(T) \leq \frac{1}{8\pi}\int_{-\infty}^{\infty}\frac{|W(i\omega)|^2d\omega}{\text{Re } H(i\omega)} + \frac{1}{4\pi}\int_{-\infty}^{\infty}[V_1(i\omega)V_2(-i\omega)+V_1(-i\omega)V_2(i\omega)]d\omega$$

$$+ \frac{1}{2h_2}(\delta_2-\delta_1)(h_2-h_1)[\rho^2(T) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}\eta_k^2(T)]$$

$$- (\Lambda h_2)^{-1}(\delta_2-\delta_1)(h_2-h_1)\sum_{k=1}^{M}\beta_k\int_T^\infty[\overline{\xi}(t) - \overline{\zeta}_k(t)]^2dt. \tag{56}$$

Let us compare now (42) and (56). One obtains, after performing elementary operations and neglecting certain terms to strengthen the inequality:

$$\Omega(T) + \delta_1[\delta_1h_1 + \delta_2(h_2-h_1)]^{-1}\Lambda^{-1}\sum_{k=1}^{M}\beta_k\int_0^T[\rho(t) - \eta_k(t)]^2$$

$$\times \{h_2[1 + \rho(t)]^{-1}[1 + \eta_k(t)]^{-1} - 1\}dt$$

$$+ \delta_0(h_2-h_1)^{-1}[\delta_1h_1 + \delta_2(h_2-h_1)]^{-1}(P\Lambda^{-1})^2\int_0^T[1 + \rho(t) - h_1]$$

$$\times [h_2 - 1 - \rho(t)]\nu^2(t)dt$$

$$\leq \Omega(0) + (\delta_2-\delta_1)(h_2-h_1)\{2h_2[\delta_1 h_1 + \delta_2(h_2-h_1)]\}^{-1}$$

$$\times [\rho^2(0) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}\eta_k^2(0)]$$

$$+ \{4\pi[\delta_1 h_1 + \delta_2(h_2-h_1)]\}^{-1}\{\int_{-\infty}^{\infty}\frac{|W(i\omega)|^2 d\omega}{2\mathrm{Re}\, H(i\omega)}$$

$$+ |\int_{-\infty}^{\infty}[V_1(i\omega)V_2(-i\omega) + V_1(-i\omega)V_2(i\omega)]d\omega|\}. \tag{57}$$

Using (57), it is possible to prove that the solution of (1) exists on the whole positive half-axis, provided the initial data satisfy adequate conditions.

Let us consider the function of $M + 1$ real variables

$$\Omega(\rho,\eta_1,\ldots,\eta_M) = \Phi(\rho) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}\Phi(\eta_k), \tag{58}$$

defined on the set

$$0 < 1 + \rho \leq \sqrt{h_2},\quad 0 < 1 + \eta_k \leq \sqrt{h_2},\quad k = 1,2,\ldots,M. \tag{59}$$

The set (59) contains the origin, i.e., $\rho = \eta_1 = \cdots = \eta_M = 0$. From the definition of $\Phi(\xi)$ there results that $\Omega(0,0,\ldots,0) = 0$. But $\Phi(\xi) \geq 0$ for $-1 < \xi < h_2 - 1$, as it can be readily seen from its definition, the equality taking place only for $\xi = 0$. Therefore, the minimum value that $\Omega(\rho,\eta_1,\ldots,\eta_M)$ can take on the boundary of the polyhedron (59) is

$$\ell = \Phi(\sqrt{h_2} - 1)\min\{1,\beta_k\lambda_k^{-1}\Lambda^{-1};\ k = 1,2,\ldots,M\}. \tag{60}$$

If γ_0 and $\overline{\gamma}_0$ are chosen as mentioned in the statement of the Theorem, i.e. such that $\Phi(\xi) \leq \ell$ for $\xi \in [-\gamma_0,\overline{\gamma}_0]$, then we impose the following restrictions to the initial values:

$$-1 < -\gamma_0 < \rho(0) < \overline{\gamma}_0 < \sqrt{h_2} - 1,$$

$$-1 < -\gamma_0 < \eta_k(0) < \overline{\gamma}_0 < \sqrt{h_2} - 1,\quad k = 1,2,\ldots,M. \tag{61}$$

It is assumed that $[-\gamma_0, \overline{\gamma}_0]$ is the maximal interval on which $\Phi(\xi) \leq \ell$.

We are going to show that the inequalities

$$-\gamma_0 < \rho(t) < \overline{\gamma}_0, \quad -\gamma_0 < \eta_k(t) < \overline{\gamma}_0, \quad k = 1,2,\ldots,M, \tag{62}$$

hold ture on the positive half-axis. Indeed, for continuity reasons, the inequalities (62) must be satisfied on a certain interval $[0,T_0)$, $T_0 > 0$. Again we assume that this is the maximal interval. Therefore, at least one among the inequalities (62) becomes an equality for $t = T_0$, if T_0 is finite. But (61) and (62) imply

$$[1 + \rho(t)][1 + \eta_k(t)] < h_2, \quad k = 1,2,\ldots,M, \quad t \in [0,T_0), \tag{63}$$

$$[1 + \rho(t) - h_1][1 + \rho(t) - h_2] \leq 0, \quad t \in [0,T_0), \tag{64}$$

the last one being a consequence of $h_1 \leq 1 - \gamma_0$. From (57), (63) and (64) we get

$$\begin{aligned}\Omega(T_0) \leq \Omega(0) &+ (\delta_2-\delta_1)(h_2-h_1)\{2h_2[\delta_1 h_1 + \delta_2(h_2-h_1)]\}^{-1} \\ &\times [\rho_0^2 + \Lambda^{-1}\sum_{k=1}^{M} \beta_k \lambda_k^{-1} \eta_k^2(0)] \\ &+ \{4\pi[\delta_1 h_1 + \delta_2(h_2-h_1)]\}^{-1}\{\int_{-\infty}^{\infty} \frac{|W(i\omega)|^2 d\omega}{2\mathrm{Re}\, H(i\omega)} \\ &+ |\int_{-\infty}^{\infty} [V_1(i\omega)V_2(i\omega) + V_1(-i\omega)V_2(i\omega)]d\omega|\}.\end{aligned} \tag{65}$$

Taking into account (34), (35) and (50), one sees that the right hand side of (65) can be done arbitrarily small, provided $|\rho(0)|$, $|\eta_k(0)|$, $\|x^0\|$, $|h|_L$ and $|\lambda|_L$, are chosen small enough. If we choose the initial data such that the right hand side in (65) be smaller than ℓ given by (60), there results $\Omega(T_0) < \ell$. But this does not agree with the fact that T_0 is maximal, which implies $\Omega(T_0) = \ell$. Hence, (62) hold true for $t \in R_+$.

From (57) one derives

$$\int_0^T [1 + \rho(t) - h_1][h_2 - 1 - \rho(t)]\nu^2(t)dt \leq \text{const.} \tag{66}$$

the constant in the right hand side being independent of T. Hence, the integrand belongs to $L^1(R_+,R)$. Since $\rho(t)$ is bounded and $1 + \rho(t) - h_1 > 1 - \gamma_0 - h_1 > 0$, $h_2 - 1 - \rho(t) > h_2 - 1 - \bar{\gamma}_0 > 0$, one obtains $\nu(t) \in L^2(R_+,R)$. Furthermore, we find that $[1 + \rho(t)]\nu(t) \in L^2(R_+,R)$. From the last $M + 1$ equations of the system (1) there results

$$\lim \rho(t) = \lim \eta_k(t) = 0 \quad \text{as} \quad t \to \infty, \quad k = 1,2,\dots,M. \tag{67}$$

Now let us consider the variable $x(t)$ of the system (1). Using the variation of constants formula [2], we find

$$x(t) = X(t)x^0 + \int_0^t X(t-s)(b\rho)(s)ds + (Yh)(t), \quad t \in R_+, \tag{68}$$

with

$$(Yh)(t) = \sum_{j=1}^{\infty} \int_{-t_j}^{0} X(t-t_j-u)A_j h(u)du, \quad t \in R_+. \tag{69}$$

Consequently, $x(t)$ is also definite on R_+ and, due to the fact that $\|X(t)\| \to 0$, $\|(Yh)(t)\| \to 0$, $|\rho(t)| \to 0$ as $t \to \infty$, one obtains

$$\lim \|x(t)\| = 0 \quad \text{as} \quad t \to \infty. \tag{70}$$

<u>Case 2</u> ($\delta_0 = 0$). If $\delta_0 = 0$, then $\chi(T)$ given by (43) becomes

$$\chi(T) = -\frac{1}{2}\delta_1\rho^2(0) + \int_0^\infty \xi_T(t)\{\delta_1[\mu_T(t) - \dot{\xi}_T(t)] + \delta_2(h_2-h_1)P\Lambda^{-1}[\psi_T(t)$$

$$+ \psi_0(t)]\}dt - (\delta_2-\delta_1)(h_2-h_1)P\Lambda^{-1}\int_T^\infty \bar{\xi}(t)\bar{\nu}(t)dt. \tag{71}$$

Indeed, the integral of $\xi_T(t)\dot{\xi}_T(t)$ is $\frac{1}{2}\xi_T^2(t)$ and

$\xi_T(0) = \rho(0)$, $\xi_T(\infty) = 0$. The last equality is a consequence of the fact that the stability of the system (9) takes place for $h = h_2$, and of Lemmas 1 and 2.

From (44) we can find $\tilde{\mu}_T(i\omega)$ and applying Parseval's formula to the first integral occurring in (71) one obtains

$$\int_0^\infty \xi_T(t)\{\delta_1[\mu_T(t) - \dot{\xi}_T(t)] + \delta_2(h_2-h_1)P\Lambda^{-1}[\psi_T(t) + \psi_0(t)]\}dt$$

$$= \frac{1}{2\pi}\mathrm{Re}\int_{-\infty}^{\infty} \tilde{\xi}_T(-i\omega)\{\delta_1[(\gamma_1(i\omega))^{-1}\tilde{\xi}_T(i\omega) - N(i\omega)]$$

$$+ \delta_2(h_2-h_1)P\Lambda^{-1}[\gamma_0(i\omega)\tilde{\xi}_T(i\omega) + M(i\omega)]\}d\omega. \quad (72)$$

If we denote

$$V(i\omega) = -\delta_1 N(i\omega) + \delta_2(h_2-h_1)P\Lambda^{-1}M(i\omega) \quad (73)$$

and

$$G(i\omega) = \delta_1(\gamma_1(i\omega))^{-1} + \delta_2(h_2-h_1)P\Lambda^{-1}\gamma_0(i\omega), \quad (74)$$

then (71) and (72) lead to

$$\chi(T) = -\frac{1}{2}\delta_1\rho^2(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\ G(i\omega)|\tilde{\xi}_T(i\omega)|^2 d\omega$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\ \tilde{\xi}_T(-i\omega)V(i\omega)d\omega - (\delta_2-\delta_1)(h_2-h_1)P\Lambda^{-1}\int_T^\infty \overline{\xi}(t)\overline{\nu}(t)dt. \quad (75)$$

Let us remark further that (36) and (74) give (for $\delta_0 = 0$)

$$|\gamma_1(i\omega)|^2\mathrm{Re}\ G(i\omega) = \mathrm{Re}\ H(i\omega) > 0, \quad (76)$$

if (37) is also considered. Moreover, one derives from (74), by elementary manipulations

$$\lim_{|\omega|\to\infty}\mathrm{Re}\ G(i\omega) \geq \delta_1\Lambda^{-1}\sum_{k=1}^{M}\beta_k + [\delta_1 h_1 + \delta_2(h_2-h_1)](\alpha_0 - \sum_{j=1}^{\infty}|\alpha_j|) > 0, \quad (77)$$

taking also into account (38). From (76), (77) there results the existence of $\varepsilon_0 > 0$, such that

$$\mathrm{Re}\ G(i\omega) \geq \varepsilon_0, \quad \omega \in R. \tag{78}$$

Therefore, if ε is such that $0 < 2\varepsilon < \varepsilon_0$, one derives from (75)

$$\chi(T) = -\frac{1}{2}\delta_1\rho^2(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}[\mathrm{Re}\ G(i\omega) - \varepsilon]|\tilde{\xi}_T(i\omega)|^2 d\omega$$
$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{Re}\ \tilde{\xi}_T(-i\omega)V(i\omega)d\omega$$
$$- (\delta_2-\delta_1)(h_2-h_1)P\Lambda^{-1}\int_T^{\infty}\overline{\xi}(t)\overline{\nu}(t)dt + \varepsilon\int_0^{\infty}\xi_T^2(t)dt. \tag{79}$$

Simple transformations in (79) lead to

$$\chi(T) = -\frac{1}{2}\delta_1\rho^2(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\sqrt{\mathrm{Re}\ G(i\omega) - \varepsilon}\ \tilde{\xi}_T(i\omega) + \frac{1}{2}\frac{V(i\omega)}{\sqrt{\mathrm{Re}\ G(i\omega)-\varepsilon}}\right|^2 d\omega$$
$$- \frac{1}{8\pi}\int_{-\infty}^{\infty}\frac{|V(i\omega)|^2}{\mathrm{Re}\ G(i\varepsilon)-\varepsilon}d\omega - (\delta_2-\delta_1)(h_2-h_1)P\Lambda^{-1}\int_T^{\infty}\overline{\xi}(t)\overline{\nu}(t)dt$$
$$+ \varepsilon\int_0^T\rho^2(t)dt + \varepsilon\int_T^{\infty}\overline{\xi}^2(t)dt, \tag{80}$$

if one takes into account Lemma 2.

According to the formula preceding (56), (80) can be rewritten in the form

$$\chi(T) = -\frac{1}{2}\delta_1\rho^2(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}|\cdot|^2 d\omega - \frac{1}{8\pi}\int_{-\infty}^{\infty}|\cdot|d\omega$$
$$- (2h_2)^{-1}(\delta_2-\delta_1)(h_2-h_1)[\rho^2(T) + \Lambda^{-1}\sum_{k=1}^{M}\beta_k\lambda_k^{-1}\eta_k^2(T)]$$
$$+ (\Lambda h_2)^{-1}(\delta_2-\delta_1)(h_2-h_1)\sum_{k=1}^{M}\beta_k\int_T^{\infty}|\overline{\xi}(t) - \overline{\zeta}_k(t)|^2 dt$$
$$+ \varepsilon\int_0^T\rho^2(t)dt + \varepsilon\int_T^{\infty}\overline{\xi}^2(t)dt. \tag{81}$$

Another form for $\chi(T)$ has been obtained above, valid for any δ_0. Now taking $\delta_0 = 0$ in (42) one has

$$\chi(T) = -\delta_1 h_2[\Omega(T) - \Omega(0)]$$
$$- \delta_1 \Lambda^{-1} \sum_{k=1}^{M} \beta_k \int_0^T [\rho(t) - \eta_k(t)]^2 \{h_2[1 + \rho(t)]^{-1}[1 + \eta_k(t)]^{-1} - 1\} dt$$
$$- (\delta_2 - \delta_1)(h_2 - h_1)\{\Omega_1(T) - \Omega_1(0) + \Lambda^{-1} \sum_{k=1}^{M} \beta_k \int_0^T [\rho(t) - \eta_k(t)]^2$$
$$\times [1 + \rho(t)]^{-1}[1 + \eta_k(t)]^{-1} dt\}. \qquad (82)$$

If we equate now the values of $\chi(T)$ given by (81) and (82), one obtains after elementary operations an inequality like (57):

$$\Omega(T) + \delta_1 \Lambda^{-1}[\delta_1 h_1 + \delta_2(h_2 - h_1)]^{-1} \sum_{k=1}^{M} \beta_k \int_0^T [\rho(t) - \eta_k(t)]^2 \{h_2[1$$
$$+ \rho(t)]^{-1}[1 + \eta_k(t)]^{-1} - 1\} dt + \varepsilon[\delta_1 h_1 + \delta_2(h_2 - h_1)]^{-1} \int_0^T \rho^2(t) dt$$
$$\leq \Omega(0) + [\delta_1 h_1 + \delta_2(h_2 - h_1)]^{-1}\{\frac{1}{8\pi} \int_{-\infty}^{\infty} \frac{|V(i\omega)|^2}{\operatorname{Re} G(i\omega) - \varepsilon} d\omega$$
$$+ \frac{1}{2}\delta_1 \rho^2(0) + \frac{1}{2h_2}(\delta_2 - \delta_1)(h_2 - h_1)[\rho^2(0) + \Lambda^{-1} \sum_{k=0}^{M} \beta_k \lambda_k^{-1} \eta_k^2(0)]\}. \qquad (83)$$

The inequality (83) can be used in order to prove that any solution of (1), such that (61) hold true, is defined on R_+ and satisfies also (62). Moreover, (83) implies $\rho(t) \in L^2(R_+, R)$ because the right hand side in (83) does not depend on T, and therefore, $\int_0^T \rho^2(t) dt$ is bounded above by a fixed number (for each set of initial data).

From the first equation of (1), and $\rho(t) \in L^2(R_+, R)$, one obtains (70), taking into account that $(b\rho)(t) \in L^2(R_+, R^n)$ and the results in [4]. Furthermore, the equations in η_k and $\rho(t) \in L^2(R_+, R)$ imply $\eta_k(t) \to 0$ as $t \to +\infty$. Finally, $\rho(t) \to 0$ as $t \to \infty$ because it is bounded on R_+ and this property implies the boundedness of $\dot{\rho}(t)$ on R_+ (hence, $\rho(t)$ is uniformly continuous on R_+). Consequently, (67) are also verified in Case 2, and together with (70) yield (40).

This ends the proof of the theorem stated above.

Remark. In order to carry out the proof of existence of solutions on the half-axis R_+, such that (62) be verified, a local existence result for (1) is necessary. Such a result is easily obtainable by means of the contraction mapping theorem.

For further results concerning the stability of nuclear reactor systems, the reader is sent to [1], [3], [8], [9], [11]. For new results concerning integral equations that might be used in investigating more sophisticated nuclear reactor systems, the paper [12] offers a valuable source.

The author expresses his appreciation to Vl. Răsvan for helpful discussions during the period this research has been started.

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