ON THE CODIMENSION n MINIMAL SURFACES CARRYING A COVARIANT DECOMPOSABLE TANGENTIAL VECTOR FIELD IN A RIEMANNIAN OR PSEUDO-RIEMANNIAN SPACE FORM

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INTRODUCTION

Let $x : M \to \tilde{M}(k)$ be the immersion of a codimension n surface in a Riemannian space form $\tilde{M}(k)$. If $T(M)$ and $T^{\perp}(M)$ are the vector tangent bundle and the vector normal bundle on M, then the tangent bundle of M restricted to M, is the direct sum $T(\tilde{M})/_M = T(M) \oplus T^{\perp}(M)$.

Let then $X \in T(M)$ and ∇ be a tangent vector on M, and the covariant differential operator on M, respectively. We say that X is <u>covariant decomposable</u> if it satisfies $X = u \oplus T + v \oplus N$, where $T \in T_p^{\perp}(M)$, $N \in T_p(M)$ $(p \in M)$, and $u,v \in \Lambda^1(M)$ are two unit 1-forms.

We agree to call T and N the tangential and the normal vector components of ∇X, and u and v the tangential and the normal Pfaffian components of ∇X.

If M is <u>minimal</u> and r is the scalar curvature of M, then one finds $r = -2\,|N|^2/|X|^2 + 2k$. In particular if M is <u>closed</u>, it follows according to Simon's theorem [1], $r = 2k - \frac{2n}{2n-1}$.

Next one considers the case where M is a spatial minimal surface of a <u>hyperbolic</u> space form $M(k)$. If the normal vector component of $\overline{\nabla X}$ is a <u>null</u> real vector field, it follows that the scalar curvature of M is 2k.

Finally, let $\tilde{M}(\tilde{\phi},\tilde{\eta},\tilde{g},\xi,k)$ be a 5-dimensional Sasakian space-form, with structure tensor fields $\tilde{\phi}$, $\tilde{\eta}$, $\tilde{g}$, ξ, and let M be an <u>anti-invariant</u> [2] minimal surface of $\tilde{M}$, (the tangent space of M is mapped by Φ into the normal space). We say that a tangential vector field X is <u>covariant ϕ-decomposable</u> if

$\nabla X = u \otimes T + v \otimes \phi X$ (one has the decomposition: $T_p^{\perp}(M) = \phi T_p(M) \oplus N_p(M)$). In this case, it is proved that the scalar curvature of M is $(k-1)/2$.

SECTION I

Let $x : (M,g) \to (\tilde{M},\tilde{g})$ be the immersion of the C^{∞}-surface M in a Riemannian $(n+2)$-C^{∞}-manifold $\tilde{M}$. Let $F(M)$ and $F(\tilde{M})$ be the bundles of orthonormal frames of M and $\tilde{M}$ respectively: if $p \in M$, let B be the set of elements $b = (p,e_A : A,B \in \{1,\ldots, n+2\})$ such that $(p,e_i : i,j \in \{1,2\}) \in F(M)$ and $(x(p),e_A) \in F(\tilde{M})$ whose orientation is coherent with the one of $\tilde{M}$. Let ω^i be the dual forms of e_i, induced by x, and $\omega_B^A = \gamma_{Bi}^A \omega^i$ ($\gamma_{Bi}^A \in C^{\infty}(M)$) the connection forms induced by x. Then the line element dp (dp is a canonical differential vectorial 1-form on $F(M)$) is expressed by:

$$dp = \omega^i \otimes e_i. \tag{1.1}$$

If ∇ is the covariant differentiation defined by g, the structure equations (E. Cartan) of M are given by:

$$\nabla e_i = \omega_i^A \otimes e_A, \tag{1.2}$$

$$d\omega^i = \omega^j \wedge \omega_j^i, \tag{1.3}$$

$$d\omega_B^A = \omega_B^C \wedge \omega_C^A + \Omega_B^A, \tag{1.4}$$

where Ω_B^A are the curvature 2-form induced by x.

SECTION II

If $T_p(M)$ is the tangent plane at $p \in M$, let

$$X = \textstyle\sum_i x_i e_i \in T_p(M); \quad x_i \in C^{\infty}(M) \tag{2.1}$$

be a tangent vector field on M. If the covariant differential of X may be expressed as:

$$\nabla X = u \otimes T + v \otimes N \tag{2.2}$$

where $u, v \in \Lambda^1(M)$ are two unitary 1-forms, while $T \in T_p(M)$ and $N \in T_p(M)$ are a tangent vector field and a normal vector field, respectively, we say that X is <u>covariant decomposable</u>.

Set

$$T = \Sigma_i t_i e_i; \quad t_i \in C^\infty(M), \tag{2.3}$$

$$N = \Sigma\, \mu_r e_r; \quad r \in \{3, \ldots, n+2\}; \quad \mu_r \in C^\infty(M), \tag{2.4}$$

$$u = u_i \omega^i, \quad v = v_i \omega^i; \quad u_i v_i \in C^\infty(M). \tag{2.5}$$

We agree to call T and N the tangential and the normal vector components of ∇X, and u and v the tangential and the normal Pfaffian components of ∇X. One has:

$$\Sigma_i (u_i)^2 = 1, \quad \Sigma_i (v_i)^2 = 1 \tag{2.6}$$

and making use of (1.2) one gets:

$$dx_i + x_j \omega_j^i = t_i u, \tag{2.7}$$

$$\Sigma_j x_i \gamma_{ij}^r = v_i \mu_r. \tag{2.8}$$

By Cartan's lemma one has:

$$\gamma_{ij}^r = \gamma_{ji}^r. \tag{2.9}$$

As known from [3],

$$h = \Sigma \gamma_{ij}^r \omega^i \otimes \omega^j \otimes e_r \in (T^* \otimes T^*) T^\perp(M) \tag{2.10}$$

and

$$H = \frac{1}{2} \Sigma_r (\gamma_{11}^r + \gamma_{22}^r) e_r \in T_p^\perp(M) \tag{2.11}$$

represent the <u>second fundamental vectorial</u> form [3] and the <u>mean curvature</u> vector associated with x, respectively (h is an

M-morphism of $T_0^2(M)$ in $T(M)$ and is independent of the normal bundle $T(M) - \cup T_p(M)$). In the following, we shall suppose that M is <u>minimal,</u> that is

$$H = 0 \Leftrightarrow \mu^r_{11} + \mu^r_{22} = 0, \quad \text{for all } r \in \{3,\dots,n+2\}, \qquad (2.12)$$

and that $\tilde{M}$ is a space form of curvature k. Then by (2.9), (2.12) and (2.8) a straightforward calculation gives:

$$\begin{aligned} |X|^2\gamma^r_{ii} &= \mu_r(v_ix_i - v_jx_j) \\ |X|^2\gamma^r_{ij} &= \mu^r(v_ix_j + v_jx_i). \end{aligned} \qquad (2.13)$$

Let $\langle h\rangle$ denote the length of the second fundamental form h, that is [4]

$$\langle h\rangle^2 = \Sigma_{r,i,j}(\gamma^r_{ij})^2. \qquad (2.14)$$

Making use of (2.13) and taking account of (2.6) one derives

$$\langle h\rangle^2 = 2\frac{|N|^2}{|X|^2}. \qquad (2.15)$$

On the other hand since $\tilde{M}$ is a <u>space form</u> of curvature k, the scalar curvature r of M satisfies the following general relation [4] (if m is the dimension of M):

$$r = m^2|H|^2 - \langle h\rangle^2 + m(m-1)k. \qquad (2.16)$$

Therefore, in the case under discussion we have:

$$r = -2\frac{|N|^2}{|X|^2} + 2k \qquad (2.17)$$

So, it follows that the necessary and sufficient condition for M to be of constant scalar curvature, is that $|N|/|X| = \text{const}$.

Since M is not totally geodesic it is worth remarking that if M is a unit (n+2)-sphere, and M is <u>closed</u> (there exists no closed minimal submanifold in a space form of nonpositive curvature) then according to Simon's theorem [1] we have $|N|^2/|X|^2 = n/2n-1$ $\Rightarrow r = 2k - 2n/2n-1$. Denote

$$dx_i = x_{i,j}\omega^j, \qquad (2.18)$$

where $x_{i,j}$ means the Pfaffian derivative, and suppose that X is a gradient vector field. This condition is expressed in intrinsic manner by:

$$\langle\nabla_Z X, Z'\rangle - \langle\nabla_{Z'} X, Z\rangle = 0, \quad \text{for all } Z, Z' \in T_p(M). \qquad (2.19)$$

Referring to (2.1) one finds from (2.19) the following relation

$$x_{1,2} - x_{2,1} + x_2\gamma_{22}^1 - x_1\gamma_{11}^2 = 0. \qquad (2.20)$$

So, taking account of (2.7), it readily follows with the help of (2.20)

$$X = \lambda j^{-1}(u), \quad \lambda \in C^\infty(M) \qquad (2.21)$$

where j is the canonical isomorphism defined by g.

Accordingly we have the following result.

Theorem. Let $x : M \to \tilde{M}$ *be the immersion of a minimal surface in an* $(n+2)$*-dimensional space form* $\tilde{M}$ *of curvature* k. *If* M *carries a tangential covariant decomposable vector field* X, *and* N *and* u *are the normal vectorial component of* ∇X *and the tangential Pfaffian component of* ∇X, *respectively, then:*

(i) the scalar curvature of M *is expressed by* $2(k - |N|^2/|X|^2)$;

(ii) if X *is a gradient vector field, then it is colinear to the dual vector field of* u.

SECTION III

Consider now the immersion $x : (M,g) \to (\tilde{M},\tilde{g})$ where M is a hyperbolic space form (with normal signature) of curvature k, and M a spatial surface. We suppose as in Section II, that $\tilde{M}$ is of dimension $n + 2$, and that M is defined by

$$\omega^i = 0. \qquad (3.1)$$

With respect to an orthonormal basis, we denote by $r,s \in \{3,\dots,n+2\}$ the normal indices and by $\alpha,\beta \in \{1,2,\dots,n+1\}$ the spatial indices ($\langle e_{n+2},e_{n+2}\rangle = 1$, $\langle e_\alpha,e_\alpha\rangle = -1$). Then according to [5], the line element dp of M is

$$dp = -\omega^i \otimes e_i, \tag{3.2}$$

and the structure equations are

$$\begin{aligned} \nabla e_\alpha &= \alpha^A_\alpha \otimes e_A; \quad A \in \{1,\dots,2n+2\}, \\ \nabla e_{n+2} &= -\omega_{n+2} \otimes e_\alpha, \end{aligned} \tag{3.3}$$

$$d\omega^i = \omega^j \wedge \omega^i_j, \tag{3.4}$$

$$\begin{aligned} d\omega^\alpha_\beta &= \Omega^\alpha_\beta + \varepsilon_A \omega^A_\beta \otimes \omega^\alpha_A; \quad \varepsilon_\alpha = 1, \quad \varepsilon_{n+2} = -1, \\ d\omega^{n+2}_\alpha &= \Omega^{n+2}_\alpha + \omega^\beta_\alpha \wedge \omega^{n+2}_\beta. \end{aligned} \tag{3.5}$$

If $X \in T_p(M)$ is a tangential vector field we suppose that X satisfies equation (2.2) (i.e. X is covariant decomposable) and that M is minimal. Since the mean curvature vector H, associated with x, is expressed by [5]

$$H = \Sigma_r(\gamma^r_{11}+\gamma^r_{22})e_r \tag{3.6}$$

condition $H = 0$, leads as in [2] to:

$$\langle h\rangle^2 = 2\frac{|N|^2}{|X|^2}, \tag{3.7}$$

where

$$N = \Sigma_r \mu_r e_r. \tag{3.8}$$

Taking account of the signature of $\tilde{g}$, one has

$$|N|^2 = -\mu_3^2 \cdots - \mu_{n+1}^2 + \mu_{n+2}^2, \tag{3.9}$$

and if $|N|^2 = 0$, then N is a null real vector field. On the other hand the general formula 2.16, still holds good for the immersion $x : M \to \tilde{M}$. Therefore if N is the null real vector field, one may have $\langle h\rangle^2 = 0$, without M to be totally geodesic [4]. Consequently we derive:

$$r = 2k, \tag{3.9'}$$

and so M is of constant scalar curvature.

Thus we have the following

Theorem. Let $x : M \to \tilde{M}$ *be the immersion of a minimal spatial surface* M *in a hyperbolic space form* $\tilde{M}$ *of curvature* k. *If* M *carries a tangential covariant decomposable vector field* X, *and the normal vectorial component of* ∇X *is a null real vector field, then the scalar curvature of* M *is* 2k.

SECTION IV

We shall finally consider the immersion $x : M \to \tilde{M}$, where $\tilde{M}(\tilde{\phi},\tilde{\eta},\tilde{g},\xi)$ is a Sasakian space form of curvature k (or of constant ϕ-sectional curvature k) and M is an anti-invariant [2] surface normal to the canonical vector field ξ. We recall that if $\tilde{M}(\tilde{\phi},\tilde{\eta},\tilde{g},\xi)$ is of dimension $2m + 1$, then the structure tensors satisfy:

$$\begin{gathered}\tilde{\phi}^2 Z = -Z + \tilde{\eta}(Z)\xi \quad \tilde{\phi}\xi = 0, \quad \tilde{\eta}(\tilde{\phi}Z) = 0, \quad \tilde{\eta}(\xi) = 1, \\ d(\tilde{\eta}(Z,Z')) = 2\langle\tilde{\phi}Z,Z'\rangle, \quad \tilde{\nabla}_Z\xi = \tilde{\phi}Z, \tilde{\nabla}\xi \quad = \tilde{\phi}d\tilde{p},\end{gathered} \tag{4.1}$$

where $\tilde{\nabla}$ is the operator of covariant differentiation on $\tilde{M}$, $(\tilde{p} \in \tilde{M})$, and Z, Z' are any vector fields on $\tilde{M}$.

One may take on $\tilde{M}$ a field of orthonormal frames $F(\tilde{M})$, such that if $F = \{e_A;\ A = 0,1,\dots,2m\} \in F(\tilde{M})$, is an element of $F(\tilde{M})$, then $e_A = e_0, e_a, e_{a^*}$; $a = 1,\dots,m$; $a^* = a + m$, such that $e_0 = \xi$, $e_{a^*} = \phi e_a$ (a ϕ-vector basis). If $\tilde{\eta} = \tilde{\omega}^o, \tilde{\omega}^a, \tilde{\omega}^{a^*}$ is the dual basis, then the connections form $\tilde{\omega}^A_B = \tilde{\gamma}^A_{BC}\tilde{\omega}^C$ satisfy:

$$\tilde{\omega}^a_b = \tilde{\omega}^{a^*}_{b^*}, \quad \tilde{\omega}^{a^*}_b = \tilde{\omega}^{b^*}_a, \quad \tilde{\omega}^{a^*}_0 = \tilde{\omega}^a, \quad \tilde{\omega}^a_0 = -\tilde{\omega}^{a^*}. \tag{4.2}$$

In the following, we assume that $\tilde{M}(k)$ is a 5-dimensional Sasakian space-form and M is a C-totally real minimal surface of $\tilde{M}(k)$ (in the sense of S. Yamaguchi, M. Kon, and Y. Miyahara [6]).

At each point $p \in M$ one has the decomposition $T_p^{\perp}(M) = \phi T_p(M) \oplus N_p(M)$ where $N_p(M)$ is the orthogonal complement of $\phi T_p(M)$ in the normal space $T_p^{\perp}(M)$.

Let then $X = \sum_i t_i e_i$ be a tangent vector field on M. We say that X is covariant Φ-decomposable if it satisfies:

$$\nabla X = u \otimes T + v \otimes \phi X. \tag{4.3}$$

Since $\phi X = \sum_i t_i e_{i*}$, clearly one has:

$$|X|^2 = |\phi X|^2. \tag{4.4}$$

On the other hand, M being minimal, the scalar curvature r of M is according to [2], expressed by:

$$r = \frac{k+3}{2} - \sum_{r,i,j} (\gamma_{ij}^r)^2; \quad r = 3,4. \tag{4.5}$$

With the aid of $\nabla e_A = \omega_A^B \otimes e_B$, and taking account of (4.4), we get after some calculation from (4.3):

$$\sum_{r,i,j} (\gamma_{ij}^r)^2 = 2\frac{|\phi X|^2}{|X|^2} = 2.$$

Hence $r = k-1/2$, and we may formulate the result:

Theorem. Let $x : M \to \tilde{M}(k)$ *be the immersion of a* C-*totally minimal surface* M *in a 5-dimensional Sasakian space form* $\tilde{M}(k)$. *If* M *carries a covariant* ϕ*-decomposable tangential vector field, then the scalar curvature of* M *is* $(k-1)/2$.

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