A GENERALIZATION OF THE STABILITY CONCEPT

Christian Constanda

I. INTRODUCTION

In this paper a generalization to abstract sets of the concept of stability is developed. The theory constructed below provides us with a unified language to describe some apparently unrelated mathematical notions like those of compact space, continuity, and dynamic stability. Theorems 1 and 2 constitute an attempt to express the essential features of some types of stability and instability of motion and include certain known results as special cases.

II. GENERAL THEORY

Let us denote by I the identity function and by $f|_D$ the restriction of a function f to a subset D of its domain of definition. Also, if B is a given non-empty set, we denote by $\tilde{B}$ the family of all one-element subsets of B, by $\exp B$ the collection of all the subsets of B, and by B_α the subset, endowed with an equivalence relation $\underset{\alpha}{\sim}$, of all the elements of B satisfying a prescribed property (α). (Throughout the paper Greek letters will be used exclusively with the above meaning; such a subscript occurring in the notation of a certain set will also occur in the notation of the elements of that set.) Further, we denote by $\underset{\omega}{\sim}$ the identity relation, by $\tilde{I}$ the natural bijection from $\tilde{B}$ onto B, and by $\mathcal{F}(B)$ the set of all the filters F_B on B, which is partially ordered: if $F_B^{(1)}$ is finer than $F_B^{(2)}$ we then write $F_B^{(1)} \geq F_B^{(2)}$.

Let two equivalence relations $\underset{\alpha}{\sim}$, $\underset{\beta}{\sim}$ be defined on two sets A, B respectively. A function $f : A \to B$ is said to be an (α,β) - function if $a_1 \underset{\alpha}{\sim} a_2$ implies $f(a_1) \underset{\beta}{\sim} f(a_2)$ for any $a_1, a_2 \in A$.

Let M be a non-empty set and A_0, A two subsets of $\exp M$ such that $A_0 \subseteq A$, $A_0 \neq \phi$, $\phi \notin A$. Generic elements of A_0, A

will be denoted by M_0, M respectively. We consider a function $p : A \times A \to R$ with the property that $p(M,M) = 0$ for any $M \in A$, and define a quasidynamic configuration to be a collection $S(M_0) = \{M \in A : p(M,M_0) \geq 0\}$. ($S(M_0)$ is not empty since it contains at least M_0.) We put $S = \{S(M_0) : M_0 \in A_0\}$ and $\mathfrak{M} = \bigcup_{M_0 \in A_0} S(M_0)$.

For a quasidynamic configuration $S(M_0)$ we can construct collections of the form $F(M_0) = \{F_M\}_{M \in S(M_0)}$, each of them containing one and only one filter F_M on every $M \in S(M_0)$. We denote by $G(M_0)$ the set of all $F(M_0)$ for a given $S(M_0)$ (i.e. for a given $M_0 \in A_0$) and put $G = \bigcup_{M_0 \in A_0} G(M_0)$, $F = \bigcup_{M_0 \in A_0} F(M_0)$. We also introduce a partial ordering on G : $F^{(1)}(M_0^{(1)}) \geq F^{(2)}(M_0^{(2)})$ if $M_0^{(1)} = M_0^{(2)} = M_0$ and $F_M^{(1)} \geq F_M^{(2)}$ for every $M \in S(M_0)$, where $F_M^{(i)}$ are the elements of $F^{(i)}(M_0)$ $(i = 1,2)$.

Let U be a non-empty set, W a non-empty subset of U, and $V \subset \exp U$, $\phi \notin V$, $V \neq \phi$. We consider two functions $h : V \to W$ and $q : U \to R$, where R is a certain partially ordered set. The function h is said to be q-dominant for V in U if $q(h(V))$ is an upper bound of the set $q(V)$ for every $V \in V$.

A function $k : M \to U$, where U is a certain non-empty set, is called a quasidynamic projector if $k|_M : M \to U$ is a surjection for every $M \in \mathfrak{M}$. It is clear that a quasidynamic projector k generates a function which associates with every filter F_M $(M \in \mathfrak{M})$ a filter $k(F_M)$ on U. We will denote this function also by k.

A quintuple $(S;G_\alpha,F_\beta;k,U)$ constructed as above is called a quasidynamic structure on M.

An (α,β)-function $f : G_\alpha \to F_\beta$ is said to be initial if $f(G_\alpha(M_0)) \subseteq F_\beta(M_0)$ for every $M_0 \in A_0$.

<u>Definition 1</u>. A quasidynamic structure $(S;G_\alpha,F_\beta;k,U)$ on M is said to be stable if there is an initial (α,β)-function which is k-dominant for G_α in $\bigcup_{M \in \mathfrak{M}} F(M)$.

<u>Definition 2</u>. A quasidynamic structure $(S;G_\alpha,F_\beta;k,U)$ on M is said to be unstable if it is not stable.

Remark 1. On any non-empty set M we can construct at least one (trivial) stable quasidynamic structure by choosing $A_0 = A = \{M\}$, $\tilde{\alpha} \equiv \tilde{\beta} \equiv \tilde{\omega}$, $k = I$, and taking both $F_{M,\alpha}$, $F_{M,\beta}$ be be the filter F_M consisting only of M itself. Then $S(M) = \{M\}$, $\mathfrak{M} = \{M\}$, $F(M) = \{F_M\}$, $G_\alpha \equiv G = \tilde{F}(M)$, $F_\beta = \{F_M\}$ and the k-dominant initial (α,β)-function required by Definition 1 is $\tilde{I}$.

Remark 2. On any set possessing more than one element we can construct more than one (non-trivial) stable and more than one unstable quasidynamic structures. Let $m_1, m_2 \in M$, $m_1 \neq m_2$, and let F_M, $F_M^{(i)}$ be the filter consisting only of M and the ultrafilter consisting of all the subsets of M which contain m_i $(i = 1,2)$, respectively. We take $A_0 = A = \{M\}$, hence $S(M) = \{M\}$, $\mathfrak{M} = \{M\}$. If we also choose $F_{M,\alpha} = F_M$, $F_{M,\beta} = F_M^{(1)}$, $\tilde{\alpha} \equiv \tilde{\beta} \equiv \tilde{\omega}$, $k = I$, then it follows that $F(M) = \{F_M\}$, $G_\alpha = \tilde{F}(M)$, $F_\beta = \{F_M^{(1)}\}$ and the initial (α,β)-function f defined by $f(F(M)) = F_M^{(1)}$ satisfies the property required in Definition 1. Another stable quasidynamic structure can be obtained by setting $F_{M,\beta} = F_M^{(2)}$ and defining f by $f(F(M)) = F_M^{(2)}$. On the other hand, if in the above construction we take $F_{M,\alpha} = F_M^{(1)}$, $F_{M,\beta} = F_M^{(2)}$, and then $F_{M,\alpha} = F_M^{(2)}$, $F_{M,\beta} = F_M^{(1)}$, we obtain two distinct unstable quasidynamic structures.

Theorem 1. A quasidynamic structure $(S;G_\alpha,F_\beta;k,U)$ *on* M *is stable if and only if there are*

(i) a stable quasidynaimc structure $(S;G_\gamma,F_\beta;k,U)$ *on* M*;*

(ii) an (α,γ)*-function* $g : G_\alpha \to G_\gamma$ *such that*

(a) $g(G_\alpha(M_0)) \subseteq G_\gamma(M_0)$ *for every* $M_0 \in A_0$*;*

(b) $g \circ \tilde{I}$ *is I-dominant for* $\tilde{G}_\alpha$ *in* G.

Proof. Suppose that $(S;G_\alpha,F_\beta;k,U)$ is stable, if we take $(\gamma) \equiv \equiv (\alpha)$, $\tilde{\gamma} \equiv \tilde{\alpha}$ and $g = I$, then (i) and (ii) are satisfied.

Suppose now that (i) and (ii) are fulfilled and let $h : G_\gamma \to \to F_\beta$ be an initial (γ,β)-function which is k-dominant for G_γ in $\bigcup_{M\in\mathfrak{M}} F(M)$. It is easy to show that $f : G_\alpha \to F_\beta$ defined by $f = h \circ g$ is an initial (α,β)-function. Let us fix a family $F_\alpha(M_0) \in G_\alpha$. Since $g \circ \tilde{I}$ is I-dominant for $\tilde{G}_\alpha$ in G, according to (a) we have $g(F_\alpha(M_0)) = ((g\circ\tilde{I})\circ\tilde{I}^{-1})(F_\alpha(M_0)) =$ $= F_\gamma(M_0) \geq F_\alpha(M_0)$. This means that the corresponding elements $F_{M,\gamma}$ and $F_{M,\alpha}$ of $F_\gamma(M_0)$ and $F_\alpha(M_0)$ satisfy $F_{M,\gamma} \geq F_{M,\alpha}$ for every $M \in S(M_0)$. By virtue of the properties of filters (see, for instance, [1]) and those of k, $k(F_{M,\gamma})$ and $k(F_{M,\alpha})$ are filters on U. Moreover,

$$k(F_{M_0,\beta}) \geq k(F_{M,\gamma}) \quad \text{for every} \quad M \in S(M_0).$$

Now h is k-dominant for G_γ in $\bigcup_{M\in M} F(M)$, thus $k(h(F_\gamma(M_0))) =$ $= F_{M_0,\beta}$ is an upper bound for the set $k(F_\gamma(M_0))$, i.e.

$$k(F_{M_0,\beta}) \geq k(F_{M,\gamma}) \quad \text{for any} \quad M \in S(M_0).$$

Combining the above inequalities we obtain $k(F_{M_0,\beta}) \geq k(F_{M,\alpha})$ for any $M \in S(M_0)$, which means that $k(F_{M_0,\beta}) = k(f(F_\alpha(M_0)))$ is an upper bound for the set $k(F_\alpha(M_0))$, i.e. f is k-dominant for G_α in $\bigcup_{M\in M} F(M)$. Hence, $(S;G_\alpha,F_\beta;k,U)$ is stable.

<u>Theorem 2</u>. *A quasidynamic structure* $(S;G_\alpha,F_\beta;k,U)$ *on* M *is unstable if and only if there are*

(i) an unstable quasidynamic structure $(S;G_\gamma,F_\beta;k,U)$ *on* M*;*

(ii) a (γ,α)*-function* $g : G_\gamma \to G_\alpha$ *such that*

(a) $g(G_\gamma(M_0)) \subseteq G_\alpha(M_0)$ *for every* $M_0 \in A_0$*;*

(b) $g \circ \tilde{I}$ *is I-dominant for* $\tilde{G}_\gamma$ *in* G.

<u>Proof</u>. Suppose that $(S;G_\alpha,F_\beta;k,U)$ is unstable, if we take $(\gamma) \equiv (\alpha)$, $\tilde{\gamma} \equiv \tilde{\alpha}$ and $g = I$, then (i) and (ii) are satisfied.

Suppose now that (i) and (ii) are satisfied, but $(S;G_\alpha,F_\beta;k,U)$ is stable. Then the following assertions are true:

(1) There is a stable quasidynamic structure $(S;G_\alpha,F_\beta;k,U)$ on M.

(2) There is a (γ,α)-function $g : G_\gamma \to G_\alpha$ such that

(a) $g(G_\gamma(M_0)) \subseteq G_\alpha(M_0)$ for every $M_0 \in A_0$;

(b) $g \circ \tilde{I}$ is I-dominant for $\tilde{G}_\gamma$ in G.

By Theorem 1 $(S;G_\gamma,F_\beta;k,U)$ is stable, contrary to our assumption (i). Therefore, $(S;G_\alpha,F_\beta;k,U)$ is unstable.

III. APPLICATIONS

A. We give an example of how this stability language can be used in topology.

Theorem 3. A Hausdorff space X *is compact if and only if there is a stable quasidynamic structure* $(S;G_\omega,F_\beta;k,X)$ *on* X, *where* $G_\omega = G$ *and* F_β *is the subset of all the filters* F_{M_0} *from* F *with respect to which* $k|_{M_0}$ *has a limit value in* X.

Proof. Suppose that X is compact and let $A_0 = A = \{X\}$. Then S contains only the quasidynamic configuration $S(X) = \{X\}$ and we have $F(X) = \{F_X\}$, $G = \tilde{F}(X)$, $\mathfrak{M} = \{X\}$. From the properties of filters and compact spaces [1] it follows that any F_X has an adherent point $x \in X$, which implies in turn that there is a filter $F'_X \geq F_X$ for which x is a limit point, or, in other words, x is a limit value of I with respect to F'_X. If we take $k = I$, $\tilde{\beta} \equiv \tilde{\omega}$, then $F'_X \in F_\beta$ and the initial (ω,β)-function $f : G \to F_\beta$ defined by $f(F_X) = F'_X$ is, therefore, k-dominant for G in $F(X)$. According to Definition 1, the quasidynamic structure $(S;G_\omega,F_\beta;k,X)$ we have constructed is stable.

Suppose now that there is a stable quasidynamic structure $(S;G_\omega,F_\beta;k,X)$ on X, where G_ω and F_β are the sets specified in the statement of the theorem. Let f be the initial (ω,β)-function, k-dominant for G in $\bigcup_{M\in\mathfrak{M}} F(M)$, whose existence is ensured by Definition 1. We consider a quasidynamic configuration $S(M_0) \in S$ and an arbitrary filter F_X on X. Since $k|_M$ is a surjection for $M \in \mathfrak{M}$, $k|_M^{-1}(F_X)$ is a filter basis on M. Let F_M be the filter generated by it and let $F(M_0) = \{F_M\}_{M\in S(M_0)}$ be the element of G constructed in this way. Then $k(F_{M_0,\beta}) = k(f(F(M_0)) \geq k(F_M)$ for all $M \in S(M_0)$. Since the filter generated by the filter basis $k(k|_M^{-1}(F_X))$ is finer than F_X [1], we obtain

$$k(F_{M_0,\beta}) \geq F_X.$$

Now $k(F_{M_0,\beta}) = k|_{M_0}(F_{M_0,\beta})$ and from the definition of F_β it follows that $k(F_{M_0,\beta})$ has a limit point $x \in X$. Thus, according to the above relation, for any given filter on X there is a finer filter on X which has a limit point x. Then x is an adherent point of the given filter and by definition the Hausdorff space X is compact.

B. We consider a function $f : X \to Y$, where X and Y are two normed spaces, and a point $x_0 \in X$. Let $\|\cdot\|_X$, $\|\cdot\|_Y$ be the norms in X, Y respectively. We construct a quasidynamic structure on the set $M = \{(x,f(x)) : x \in X\}$ by choosing $A_0 = A = \{M\}$, $\tilde{\alpha} \equiv \tilde{\beta} \equiv \tilde{\omega}$, and $k = I$, in which case $S(M) = \{M\}$ and $\mathfrak{M} = \{M\}$. We also choose $F_{M,\alpha}$, $F_{M,\beta}$ to be filters on M generated by sets of the form $\{(x,f(x)) : \|f(x) - f(x_0)\|_Y < \varepsilon\}$, $\{(x,f(x)) : \|x - x_0\|_X < \delta\}$ respectively.

Theorem 4. *The function* f *is continuous at* x_0 *if and only if the quasidynamic structure constructed above is stable.*

Proof. The function f is continuous at x_0 if and only if for and $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x - x_0\|_X < \delta$ implies $\|f(x) - f(x_0)\|_Y < \varepsilon$. This function $\delta = \delta(\varepsilon)$ generates in an obvious manner the initial (α,β)-function with the property required by Definition 1.

C. Let $I = [0,T)$, $I_0 = [0,\tau)$ $(\tau < T)$ be intervals on R, X a certain set, $U = \{u : I \to X\}$ a set of functions, and $d : U \times U \times I \to R_+$, $d_0 : U \times U \times I_0 \to R_+$ two functions with the property that $d(u_1,u_2;t) = d_0(u_1,u_2,t_0) = 0$ if and only if $u_1 = u_2$.

Definition 3. An element $u_0 \in U$ is said to be (d_0,d)-stable in U if for any $t_0 \in I_0$ and any $\varepsilon > 0$ there is a $\delta(t_0,\varepsilon) > 0$ such that $d_0(u,u_0;t_0) < \delta$ implies $d(u,u_0;t) < \varepsilon$ for all $t \geq t_0$.

We put $M = U \times I$ and $A = \{U \times t\}_{t\in I}$, $A_0 = \{U \times t_0\}_{t_0\in I_0}$. If $M_i = U \times t_i$ $(i = 1,2)$, then we can take $p(M_1,M_2) = t_2 - t_1$ and construct a quasidynamic structure $(S;G_\alpha,F_\beta;k,U)$ on M, where $M = U \times t$, $M_0 = U \times t_0$, $S(M_0) = S(t_0) = \{U \times t : t \geq t_0, t \in I\}$, $S = \{S(t_0) : t_0 \in I_0\}$, $\mathfrak{M} = \{U \times t, \ t \in I\}$, G_α is the subset of all $F_\alpha(M_0) = F_\alpha(t_0)$ from G consisting of filters $F_{M,\alpha} = F_{t,\alpha}$ on $U \times t$ generated by a basis of the form $\{(u,t) : d(u,u_0;t) < \varepsilon_t\}$ and such that $\inf_{t\in S(t_0)} \varepsilon_t > 0$, $F_{M_0,\beta} = F_{t_0,\beta}$ is a filter on $U \times t_0$ generated by a basis of the form $\{(u,t_0) : d_0(u,u_0;t_0) < \delta_{t_0}\}$, $\tilde{\alpha} \equiv \tilde{\beta} \equiv \tilde{\omega}$, and $k : M \to U$ is defined by $k(u,t) = u$. We call this $(S;G_\alpha,F_\beta;k,U)$ the dynamic structure associated with u_0.

<u>Theorem 5.</u> *An element* $u_0 \in U$ *is* (d_0,d)*-stable in* U *if and only if its associated dynamic structure is stable.*

<u>Proof.</u> Suppose that u_0 is (d_0,d)-stable and let us consider a family $F_\alpha(t_0) \in G_\alpha$. If we put $\varepsilon = \inf_{t\in S(t_0)} \varepsilon_t$, then the family $F'_\alpha(t_0) \in G_\alpha$ consisting of the filters $F_{t,\alpha}$ constructed with $\varepsilon_t = \varepsilon$ for all $t \in S(t_0)$ obviously satisfies $F'_\alpha(t_0) \geq F_\alpha(t_0)$. From this relation, Definition 3 and the definition of k it follows that the initial (α,β)-function f defined by $f(F_\alpha(t_0)) = F_{t_0,\beta}$, where $F_{t_0,\beta} \in F_\beta$ is constructed with $\delta = \delta(t_0,\varepsilon)$ provided by Definition 3, is k-dominant for G_α in $\bigcup_{M\in\mathfrak{M}} F(M)$, thus, according to Definition 1, $(S;G_\alpha,F_\beta;k,U)$ is stable.

Suppose now that the associated dynamic structure is stable. If we consider families $F_\alpha(t_0) \in G_\alpha$ consisting of filters $F_{t,\alpha}$ with $\varepsilon_t = \varepsilon$ for all $t \in S(t_0)$, then the k-dominant initial (α,β)-function whose existence is ensured by Definition 1 generates a function $\delta = \delta(t_0,\varepsilon)$ satisfying the condition required in Definition 3 and therefore u_0 is (d_0,d)-stable in U.

<u>Remark 3.</u> The analysis of stability of motion fits the above scheme when d_0 and d are two norms by means of which the perturbations at times t_0 and t are measured.

<u>Remark 4.</u> Theorem 5 remains valid also for many other types of stability (see [2],[6]) if we suitably modify some definitions in the construction of the dynamic structure associated with u_0. We give below two examples.

(i) In the case of equiasymptotic stability [6] we should take $T = \infty$ and G_α should be the subset of all $F_\alpha(t_0)$ from G which consist of filters $F_{t,\alpha}$ on $U \times t$ generated by a basis of the same form as above but with $\lim_{t\to\infty} \varepsilon_t = 0$.

(ii) In the case of uniform stability the only required change concerns $\tilde{\alpha}$ and $\tilde{\beta}$. We should say that $F_\alpha^{(1)}(t_0^{(1)}) \tilde{\alpha} \tilde{\alpha} F^{(2)}(t_0^{(2)})$ if $\inf_{t\in S(t_0^{(1)})} \varepsilon_t^{(1)} = \inf_{t\in S(t_0^{(2)})} \varepsilon_t^{(2)}$, and $F_{t_0,\beta}^{(1)} \tilde{\beta} \tilde{\beta} F_{t_0,\beta}^{(2)}$ if $\delta_{t_0}^{(1)} = \delta_{t_0}^{(2)}$. The function $\delta = \delta(t_0,\varepsilon)$ occurring in the proof of Theorem 5 will then not depend on $t_0 \in I_0$.

Remark 5. The existence of a Lyapunov function f provides us with a means of satisfying the conditions of Theorem 1. The quasidynamic structure $(S;G_\gamma,F_\beta;k,U)$ is constructed in the same way as $(S;G_\alpha,F_\beta;k,U)$ with the only modification that d in the definition of $F_{t,\alpha}$ is replaced by f to obtain $F_{t,\gamma}$. The properties of f ensure the stability of $(S;G_\gamma,F_\beta;k,U)$ and the availability of the function g required by condition (ii). Thus, from Theorem 1 we can obtain Lyapunov's stability theorems as special cases. Movchan's results [4], which generalize Lyapunov's theorems and have been applied to the study of dynamic stability in continuum mechanics [5],[3], are also easily obtained by specialization from Theorem 1.

Remark 6. Similar comments can without difficulty be made about instability and Theorem 2.

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