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A NOTE ON THE WEIGHTED SHAPLEY VALUE

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Abstract

In this note we present a new axiomatization of the simple weighted Shapley value introduced by E. Kalai and D. Samet (1987, 1988) and we show that the weighted Shapley value satisfies a symmetry condition similar to that involved in the definition of the classic value due to L. S. Shapley (1953).

Keywords: Cooperative game, Shapley value, weighted Shapley value.

Introduction

Let N be a finite set whose elements are called *players* and whose subsets are called *coalitions*. We denote by $\mathbb{G}(N)$ the linear space of all *games* (in characteristic function form), that is the set of all functions $v:\mathcal{P}(N)\longrightarrow\mathbb{R}$ with $v(\emptyset)=0$ provided with the algebraic operations induced from $\mathbb{R}$. It has been shown by SHAPLEY (1953, p. 310) that the *unanimity games* i. e., the functions $w_S:\mathcal{P}(N)\longrightarrow\mathbb{R}$, $(S\in\mathcal{P}(N),\ S\neq\emptyset)$, defined by $w_S(T)=1$ if $S\subseteq T$ and $w_S(T)=0$, otherwise, form a basis in $\mathbb{G}(N)$ and that, for each $v\in\mathbb{G}(N)$, we have

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$$v = \sum_{S\neq\emptyset} \alpha_S(v)\cdot w_S \ , \tag{1}$$

where, for each nonempty coalition S, $\alpha_S(v)$ is defined by

$$\alpha_S(v) = \sum_{T\subseteq S} (-1)^{|S|-|T|}\cdot v(T). \tag{2}$$

KALAI and SAMET (1987, 1988) introduced the concept of "weighted Shapley value with respect to a weight system". In what follows we call *simple-weighted Shapley value* the function $\Phi\colon \mathbb{R}^N_{++}\times\mathbb{G}(N) \longrightarrow \mathbb{R}^N$ defined by

$$\Phi(x,v) = \sum_{S\neq\emptyset} \alpha_S(v)\cdot\Phi(x,w_S) \ , \tag{3}$$

where, for each nonempty coalition S and for any $x \in \mathbb{R}^N_{++}$, $\Phi(x,w_S)$ is the vector in $\mathbb{R}^N$ of coordinates

$$\Phi_i(x,w_S) = \begin{cases} \dfrac{x_i}{x(S)} & \text{if } i \in S, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

with $x(S) := \sum_{i\in S} x_i$. Note that $\Phi(x,v)$ is exactly the Shapley value with respect to the simple weight system $(x,(N))$ as defined by KALAI and SAMET (1988). Also, observe that, for any x in the diagonal $\mathbb{D}_N$ of $\mathbb{R}^N_{++}$, $\Phi(x,v)$ is exactly the "value of v" as defined by SHAPLEY (1953) because the function $\Phi(\cdot,v)$ is positively homogeneous.

It follows from Theorem 3 of KALAI and SAMET (1988) that a function $\Psi\colon \mathbb{G}(N) \longrightarrow \mathbb{R}^N$ satisfies the conditions of efficiency, additivity, (strict) positivity, dummy player and partnership consistency if and only if there exists a $x \in \mathbb{R}^N$ such that $\Psi(v) = \Phi(x,v)$, for any $v \in \mathbb{G}(N)$.

In this note we show that the simple-weighted Shapley value is characterized by a system of axioms which is much more similar to that involved in Shapley's (1953) classic definition of "non-weighted" value and we prove that the weighted Shapley value has a *symmetry* property analogous to that of the classic, "non-weighted" Shapley value.

The Main Result

In order to state the main result of this note we recall that a coalition S is called *carrier* of the game $v \in \mathbb{G}(N)$ iff $v(S \cap T) = v(T)$, for any $T \in \mathcal{P}(N)$.

THEOREM. *The simple weighted Shapley value* Φ *is the unique function* $\varphi: \mathbb{R}^N_{++} \times \mathbb{G}(N) \longrightarrow \mathbb{R}^N$ *which satisfies the following conditions:*

A. Strong Efficiency: If the coalition S *is a nonempty carrier of the game* $v \in \mathbb{G}(N)$, *then*

$$\sum_{i \in S} \varphi_i(x, v) = v(S). \tag{5}$$

B. Proportional Unanimity: For each nonempty coalition S *and for any two players* i, $j \in S$ *we have*

$$x_j \cdot \varphi_i(x, w_S) = x_i \cdot \varphi_j(x, w_S). \tag{6}$$

C. Linearity: For each $x \in \mathbb{R}^N_{++}$, $\varphi(x, \cdot)$ *is linear on* $\mathbb{G}(N)$.

Moreover, the weighted Shapley value Φ *is "symmetric", i. e. for each permutation* π *of* N *we have*

$$\Phi \circ \pi^* = \pi\Phi, \tag{7}$$

where, for any $z \in \mathbb{R}^N$, $(\pi z)(i) := z(\pi^{-1}i)$ *and* π^* *is the automorphism of* $\mathbb{R}^N_{++} \times \mathbb{G}(N)$ *defined by* $\pi^*(x, v) := (\pi x, \pi v)$ *with* $(\pi v)(S) := v(\pi^{-1}S)$, $(S \in \mathcal{P}(N))$, *as usual.*

PROOF: First we show that the weighted Shapley value satisfies the conditions (A), (B) and (C). To this end, note that, for each coalition $S \neq \emptyset$, the coefficient $\alpha_S(v)$ defined at (2) is linearly dependent on v. Therefore, according to (1), Φ is linearly dependent on v, i. e. Φ satisfies (C). If S is a nonempty carrier coalition of the game $v \in \mathbb{G}(N)$, then each player $i \in N\setminus S$ is a dummy player in v because, for any coalition T, we have

$$v(T\cup\{i\}) = v(S\cap(T\cup\{i\})) = v(S\cap T) = v(T).$$

Since, according to Theorem 3 of KALAI and SAMET (1988), Φ satisfies the dummy player and the efficiency conditions, it follows that

$$\sum_{i\in S} \Phi_i(x,v) = \sum_{i\in N} \Phi_i(x,v) = v(N) = v(S),$$

i.e. Φ satisfies (A). Obviously, Φ satisfies (B) because of (4).

Now, we show that Φ is symmetric. Let π be a permutation of N. It follows from (3) and (4) that, for each $i \in N$, we have

$$\Phi_i(x,v) = x_i \cdot \sum_{\{S;\, i\in S\}} \frac{\alpha_S(v)}{x(S)} . \tag{8}$$

Using (8) we obtain

$$[\Phi\circ\pi^*]_i(x,v) = \Phi_i(\pi x,\pi v) = (\pi x)_i \cdot \sum_{\{S;\, i\in S\}} \frac{\alpha_S(\pi v)}{(\pi x)(S)} .$$

Note that $(\pi x)(S) = x(\pi^{-1}S)$ and that $\alpha_S(\pi v) = \alpha_{\pi^{-1}S}(v)$, for any nonempty coalition S. Hence,

$$[\Phi\circ\pi^*]_i(x,v) = (\pi x)_i \cdot \sum_{\{S;\, i\in S\}} \frac{\alpha_{\pi^{-1}S}(v)}{x(\pi^{-1}S)}$$

$$= x_{\pi^{-1}i} \cdot \sum_{\{T;\, \pi^{-1}i\in T\}} \frac{\alpha_T(v)}{x(T)} = \Phi_{\pi^{-1}i}(x,v) = (\pi\Phi)_i(x,v),$$

for any $i \in N$, i. e. Φ satisfies (7).

It remains to prove that the function Φ defined by (3) and

(4) is the unique function from $\mathbb{R}^N_{++}\times\mathbb{G}(N)$ to $\mathbb{R}^N$ which satisfies (A), (B) and (C). To this end, suppose that Ψ is another function from $\mathbb{R}^N_{++}\times\mathbb{G}(N)$ to $\mathbb{R}^N$ which satisfies these conditions. We show that $\Phi(x,w_S) = \Psi(x,w_S)$, for each $x \in \mathbb{R}^N$ and for each nonempty coalition S. By the linearity of both functions Φ and Ψ this implies that $\Phi = \Psi$. Note that any coalition T with $S \subseteq T$ is a carrier of the unanimity game w_S. In particular, if j is a player and $j \notin S$, then $T = S\cup\{j\}$ is still a carrier of w_S and, according to condition (A) we have

$$\sum_{i\in S} \Psi_i(x,w_S) = w_S(S) = 1 = w_S(T) = \sum_{i\in T} \Psi_i(x,w_S),$$

that is $\Psi_j(x,w_S) = 0 = \Phi_j(x,w_S)$, whenever $j \notin S$. Now, suppose that $j \in S$. If S consists of a single player, then, applying again condition (A) to Ψ, we obtain that $\Psi_j(x,w_S) = 1 = \Phi_j(x,w_S)$. Assume that S contains more than one player. Then, according to condition (B), we have $\Psi_i(x,w_S) = (x_i/x_j)\cdot\Psi_j(x,w_S)$, for each $i \in S$. Using this fact combined with condition (A) applied to Ψ we get

$$1 = w_S(S) = \sum_{i\in S} \Psi_i(x,w_S) = \Psi_j(x,w_S)\cdot(x(S)/x_j),$$

showing that $\Psi_j(x,w_S) = x_j/x(S) = \Phi_j(x,w_S)$ for any $j \in S$. This completes the proof of the theorem.∎

REMARKS. (i) Note that conditions (A) and (C) are involved in the definition of SHAPLEY's (1953) concept of value. Condition (B) replaces Shapley's "symmetry" axiom. If one considers the restriction ψ of the function Φ to the set $\mathbb{E}_N\times\mathbb{G}(N)$, then condition (B) and Shapley's "symmetry" axiom are equivalent while they are applied to ψ.

(ii) Observe that the function π^* associated to the permutation π of N is an automorphism of $\mathbb{R}^N_{++}\times\mathbb{G}(N)$ provided with the topology of pointwise convergence. Combining this fact with Theorem 4 of KALAI and SAMET (1988) and with the Theorem proved above, it follows that even the weighted Shapley value with respect to an arbitrary (not necessarily simple) weight system is somewhat "symmetric". Precisely, if $\omega = (x,(S_1,\dots,S_m))$ is a weight system and if π is a permutation of N, then $\pi\omega := (\pi x,(\pi S_1,\dots,\pi S_m))$ is also a weight system and the λ-weighted Shapley value Φ_λ (see KALAI and SAMET (1988)) satisfies the following "symmetry condition":

$$\Phi_{\pi\omega}(\pi v) = \pi\Phi_\omega(v) \tag{7'}$$

for any game $v \in \mathbb{G}(N)$ and for any weight system ω.

(*iii*) The axioms (A), (C) and the symmetry condition (7) do not uniquely define a function $\Phi\colon \mathbb{R}^N_{++}\times\mathbb{G}(N) \longrightarrow \mathbb{R}^N$. Precisely, the function $\Xi\colon \mathbb{R}^N_{++}\times\mathbb{G}(N) \longrightarrow \mathbb{R}^N$ defined by

$$\Xi(x,v) = x_i\cdot\psi_i(v), \tag{9}$$

where $\psi(v)$ is the Shapley value of v, is a function satisfying (A), (C) and (7) but not satisfying (B). This is the unique function from $\mathbb{R}^N_{++}\times\mathbb{G}(N)$ to $\mathbb{R}^N$ which satisfies (A), (C), (7) and it is additive and positively homogeneous with respect to x.

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