

Existence of Bounded Solutions for Systems of Discrete Equations

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The aim of this paper is to obtain some existence results for bounded solutions to discrete equations of the form

$$(1) \quad x_{n+1} = Ax_n + b_n,$$

in which $n \in Z_+$, where $Z_+ = 0, 1, 2, \dots$ and $\{x_n\}$ stands for the unknown process, with values in R^m , A is a square matrix of order m with real entries, and $\{b_n\}$ stands for a given discrete process, with values in the same space as $\{x_n\}$.

In [4], the existence of bounded solutions on Z to nonlinear discrete equations of the form

$$(2) \quad x_{n+1} = f(n, x_n)$$

has been discussed in detail, and in [1], and [3], the existence of bounded solutions has been assumed in order to prove other qualitative properties. Also in [4] the existence of bounded solutions on Z_+ or Z , to linear equation

$$(3) \quad x_{n+1} = Mx_n + b_n,$$

has been proved. Our interest here is to prove the existence of bounded solutions on Z_+ of discrete equations of the form (1), and obtain some formulas of representation of such solutions. To prove the existence we will use the method of iteration. In analyzing the iterations, we must refer to the individual elements in the iteration sequence. We will let x_0 denote the starting

guess, x_1 denote the first iteration and so on. The iterations will be as follow:

$$(4) \quad \left\{ \begin{array}{l} x_0 = \text{initial - guess,} \\ x_1 = Ax_0 + b_0, \\ x_2 = Ax_1 + b_1 = A^2x_0 + Ab_0 + b_1, \\ x_3 = Ax_2 + b_2 = A^3x_0 + A^2b_0 + Ab_1 + b_2, \\ \dots \\ x_n = Ax_{n-1} + b_{n-1} = A^n x_0 + A^{n-1}b_0 + A^{n-2}b_1 + \dots + b_{n-1}. \end{array} \right.$$

The last equation above can be also written when A^{-n} exists,

$$(5) \quad x_n = A^n [x_0 + A^{-1}b_0 + A^{-2}b_1 + \dots + A^{-n}b_{n-1}].$$

If $|A|^n$ approaches zero as n increases, and the series in the brackets converges, then x_n approaches zero as n increases. If $|A|^n$ diverges as n increases, then except for special choices of x_0 , $\{x_n\}$ will diverge (unbounded). In order to make the discussion easier, we will note that the matrix A is equivalent to a matrix of lower triangular form, via a linear nonsingular transformation. In other words, there exists a square nonsingular matrix P of order m , such that $P^{-1}AP = B$, has the form

$$(6) \quad B = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & \dots & a_{mm} \end{bmatrix},$$

where $a_{11}, a_{22}, a_{33}, \dots, a_{mm}$ are the eigenvalues of A . Therefore, we can assume A to have the form

$$(7) \quad A = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \dots & 0 \\ a_{21} & \lambda_2 & 0 & 0 & \dots & 0 \\ a_{31} & a_{32} & \lambda_3 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & \dots & \lambda_m \end{bmatrix}.$$

The above result can be found in [5].

We will make use of the fact that the determinant of a triangular (upper or lower) matrix A is the product of its diagonal entries. Since the diagonal entries of the matrix A are also its eigenvalues, in order to obtain the boundedness of $\{x_n\}$, one has to discuss distinct cases of the spectrum of A (the set of eigenvalues of A).

1 Spectrum of A is inside $|\lambda| = 1$

In this case, it can be easily shown that the solution $\{x_n\}$ constructed by (4) is bounded on Z_+ , anytime $\{b_n\}$ is a bounded process. This holds true regardless of the initial value x_0 chosen in constructing the solution.

Indeed, in case A has its spectrum inside the unit circle, one has according to the well known spectral results

$$(8) \quad \lim_{n \rightarrow \infty} |A^n|^{1/n} = r_A < 1, r_A = \max\{|\lambda_j|; 1 \leq j \leq m\}.$$

This implies that the series

$$(9) \quad \sum_{n=0}^{\infty} |A^n|$$

converges (the root test).

This property allows us to show immediately that the solutions $\{x_n\}$ constructed by (4) is bounded on Z_+ , as soon as $\{b_n\}$ is bounded there.

Assuming $|b_n| \leq B, n \geq 1$, one has from the last equation (4)

$$(10) \quad |x_n| \leq |A^n| |x_0| + B(1 + |A| + \dots + |A^{n-1}|),$$

or

$$(11) \quad |x_n| \leq |x_0| + B \sum_{n=0}^{\infty} |A^n|,$$

for any $n \in Z_+$ is large enough. In any circumstances, (11) can be written as

$$(12) \quad \sup |x_n| \leq C |x_0| + KB,$$

for two conveniently chosen positive constants C and K .

To summarize the discussion of this case, *the system (1) has only bounded solutions on Z_+ and the solution starting at x_0 satisfies the estimate (12).*

2 Spectrum of A is outside $|\lambda| = 1$

In this case, we will refer to the formula (5) for the construction of the process $\{x_n\}, n \geq 0$. Since $|A^n| \rightarrow \infty$ as $n \rightarrow \infty$ (because $|A^n|^{1/n} \rightarrow r_A > 1$ as $n \rightarrow \infty$), from (5) we see that x_n can be bounded only in case the quantity in the brackets tends to the zero element of R^m , as $n \rightarrow \infty$. In other words, only in case

$$(13) \quad -x_0 = A^{-1}b_0 + A^{-2}b_1 + \dots + A^{-n}b_{n-1} + \dots$$

First, let us notice that the series in the right hand side of (13) does converge. Indeed, A^{-1} is a matrix whose spectrum lies inside the circle $|\lambda| = 1$, because the eigenvalues of A^{-1} are the algebraic inverses of the eigenvalues of A . Hence, all of them lie inside $|\lambda| = 1$. According to the result in §1 above, the series in the right hand side of (13) must converge (in the matrix norm).

If we go back to (5), then we find for x_n the following expression

$$(14) \quad x_n = -A^{-1}(b_n + A^{-1}b_{n+1} + A^{-2}b_{n+2} + \dots).$$

Of course, the series in (14) converges in the matrix norm.

Therefore, there exists at most one bounded solution of (1) on Z_+ , in the case under consideration.

Actually, we can easily show that (14) represents a bounded solution of (1). Indeed, one easily obtains

$$(15) \quad Ax_n + b_n = A^{-1}(b_{n+1} + A^{-1}b_{n+2} + \dots) = x_{n+1},$$

if we take again (12) into consideration. Hence, we can formulate the following conclusion:

In case the spectrum of the matrix A lies outside the unit circle $|\lambda| = 1$, there exists a unique bounded solution (on Z_+) of the system (1). This solution is given by the formula (14).

3 A has spectrum both inside and outside $|\lambda| = 1$

In this case we refer to the lower triangular form (7) for the matrix A . Let us assume that $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$ are the eigenvalues inside $|\lambda| = 1$, and $\lambda_{k+1}, \lambda_{k+2}, \lambda_{k+3}, \dots, \lambda_m$ are those that lie outside the circle $|\lambda| = 1$. This can be accomplished, i.e. this labeling of the eigenvalues, in any case the matrix A has the property specified in this paragraph. The inspection of the proof in [5], concerning the reduction of A to the lower triangular form shows that we can proceed with the eigenvalues in whatever order we choose to. So, we will label them in such a way that the first k are inside $|\lambda| < 1$, while the remaining $m - k$ are outside the unit circle.

We can actually write the system (1) in the following equivalent form:

$$(16) \quad y_{n+1} = A_1 y_n + c_n,$$

and

$$(17) \quad z_{n+1} = A_{11} y_n + A_2 z_n + d_n,$$

where y and c are k -dimensional, while z and d are $(m-k)$ -dimensional: $x_n = (y_n, z_n)$, $b_n = (c_n, d_n)$. The matrices A_1 , A_{11} and A_2 are of the types $k \times k$, $(m-k) \times k$ and $(m-k) \times (m-k)$. The matrix A_1 and A_2 are lower triangular matrices:

$$(18) \quad A_1 = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ a_{21} & \lambda_2 & 0 & \dots & 0 \\ a_{31} & a_{32} & \lambda_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & a_{k3} & \dots & \lambda_k \end{bmatrix},$$

and

$$(19) \quad A_2 = \begin{bmatrix} \lambda_{k+1} & 0 & 0 & \dots & 0 \\ a_{(k+2)1} & \lambda_{k+2} & 0 & \dots & 0 \\ a_{(k+3)1} & a_{(k+3)2} & \lambda_{k+3} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & \lambda_m \end{bmatrix}.$$

The system $y_{n+1} = A_1 y_n + c_n$ has its eigenvalues inside $|\lambda| = 1$. Therefore, for any initial values y_0 one obtains a unique bounded solution (of course, we assume $\{c_n\}$ is bounded, which implies the boundedness of both $\{c_n\}$ and $\{d_n\}$).

We substitute a solution $\{y_n\}$ of the system in (16) into the system in (17) for z_n :

$$(20) \quad z_{n+1} = A_2 z_n + \bar{d}_n,$$

where $\bar{d}_n = d_n + A_{11} y_n$. Hence $\{\bar{d}_n\}$ is bounded. Since the spectrum of A_2 is outside $|\lambda| = 1$, there exists a unique bounded solution $\{z_n\}$ of the system (17) for each solution $\{y_n\}$.

Concluding the discussion of this case, we see that there exists a number k , $0 < k < m$, such that (1) has a unique solution $\{x_n\}$ which is bounded on Z_+ , and such that the initial data $x_{0,i}$, $i = 1, 2, \dots, k$, are assigned.

4 Spectrum of A is on $|\lambda| = 1$

We will consider, in this case only a simple example. We chose $A = I$ = the unit matrix. This matrix has only one(repeated) eigenvalue $\lambda = 1$.

The system (1) becomes

$$(21) \quad x_{n+1} = x_n + b_n,$$

and if we choose $b_n = c = \text{constant} \in R^m$, then we obtain $x_n = x_0 + nc$. Regardless of the choice of x_0 , for $c \neq 0$, the solution $\{x_n\}$ is unbounded.

5 Nonlinear case

It is possible to extend the result in §3 to the class of nonlinear discrete systems of the form

$$(22) \quad x_{n+1} = Ax_n + f(n, x_n), \quad n \in Z_+,$$

where $f(n, x_n)$ is a map from $Z_+ \times R^m$ into R^m , satisfying a Lipschitz condition with respect to the second argument:

$$(23) \quad |f(n, y) - f(n, z)| \leq L |y - z|.$$

If the Lipschitz constant L is sufficiently small, then the existence problem of bounded solution to (22) has the same kind of answer as in the linear case, as seen in §§1,2,3.

For instance, in case where A has the spectrum inside the unit disc, if we assume, besides (23) on $f(n, y)$, the boundedness of $\{f(n, 0)\}$ on Z_+ , one obtains the existence of bounded solutions to (22) regardless of the choice of x_0 .

The proof of the above assertion can be conducted by using the Banach fixed point method with space of bounded processes on Z_+ (with values in R^m). It suffices to consider the operation $\{u_n\} \rightarrow \{x_n\}$, where

$$(24) \quad x_{n+1} = Ax_n + f(n, u_n), \quad n \geq 0,$$

and apply the estimate obtained above (12).

References

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Note: General results, slightly different than those described above, concerning the system $x_{n+1} = A(n)x_n + b_n$, have been obtained by J. J. Schäffer in a series of papers in *Math. Annalen* (vols. 172,177,189).

