

## COINCIDENCE DEGREE IN BIFURCATION THEORY

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The operator theory provides an abstract framework to investigate differential or integral equations. In this way, by an adequate choice of underlying function spaces, many problems can be rewritten as a semilinear operator equation

$$Lu = Fu \quad (1)$$

where both linear operator  $L$  and nonlinear perturbation  $F$  carry a real Banach space  $X$  into another  $Y$ . When  $L$  is defined on the whole of  $X$  and is invertible, the solving of (1) reduces to the determination of the fixed points of the map  $T = L^{-1}F$  and can be tackled by using degree theories for vector fields. However,  $L$  is often not invertible - indeed in some contexts invertibility is the exception. To study (1) when  $L^{-1}$  does not exist, J. Mawhin [8] introduced in 1972 the *coincidence degree* of the pair  $[L, F]$ . Moreover, this equation gains an outstanding operator feature whenever the hypotheses upon nonlinear mapping  $F$  interact with the topological and spectral properties of the linear operator  $L$ .

Our final purpose is to unify some recent trends concerning the concept of bifurcation, starting from an eigeninterval of a characteristic value of odd multiplicity for the nonlinear eigenvalue problem

$$Lu - \lambda Au + F(\lambda, u) + G(\lambda, u) = 0 \quad (2)$$

where the spectral parameter  $\lambda$  ranges over a real compact interval  $\Lambda$ ,  $F$  and  $G$  are nonlinear perturbations with  $G(\lambda, u) = o(\|u\|)$  for  $u \rightarrow 0$ , and  $F(\lambda, u) \leq k\|u\|$ , while  $L$  and  $A$  are selfadjoint operators,  $L$  being generally assumed noninvertible and  $A$  strongly positive definite in a Hilbert space. Our approach is based on some spectral analysis ele-

ments of linear polynomials equation in the form

$$Lu - \lambda Au = 0, \quad (3)$$

called also linear operator pencils ([4]), which actually corresponds to the linearized equation when  $F \equiv 0$ . The key is to find sets of hypotheses upon the pair  $(L, A)$  which assure that the characteristic values are isolated. Two alternatives are presented. We have paid a particular attention to estimate the eigeninterval lengths. Along this line, some recent results, due to A.P. Makhmudov and Z.S. Aliev [7], become transparent.

### 1. Prerequisites of the Coincidence Degree.

For a linear operator  $L: D(L) \subseteq X \rightarrow Y$ , we denote by  $\alpha(L)$  the nullity of  $L$  defined as the dimension of  $N(L) = \ker L = L^{-1}(0)$  and by  $\beta(L)$  the deficiency of  $L$  defined by

$$\text{codim } R(L) = \dim \text{coker } L = \dim Y/R(L) = \dim R(L)^\perp,$$

where  $R(L) = \text{Im } L = L(D(L))$ . A (possible unbounded) linear operator  $L$  is *Fredholm* if  $R(L)$  is a closed subspace and both  $\alpha(L)$  and  $\beta(L)$  are finite. The integer  $\text{Ind } (L) = \dim \ker L - \dim \text{coker } L$  is called the (*Fredholm*) *index* of  $L$ .

From now on, we suppose that  $L$  is a Fredholm operator of index zero. In this case there exist projectors  $P: X \rightarrow X$  and  $Q: Y \rightarrow Y$  such that

$$\text{Im } P = \ker L \quad \text{and} \quad \text{Ker } Q = \text{Im } L.$$

Such a pair of projectors  $(P, Q)$  is called *exact with respect to*  $L$ . Let  $J: \text{Im } Q \rightarrow \ker L$  be an algebraic isomorphism. Thereby the product  $J^{-1}P: X \rightarrow Y$  is compact. Because  $X$  and  $Y$  can be written as the direct topological sums

$$X = \ker P + \ker L \quad \text{and} \quad Y = \text{Im } L + \text{Im } Q,$$

the operator  $L + J^{-1}P: X \rightarrow Y$  is a bounded bijection and so a topological isomorphism by the closed graph theorem.

We notice that the restriction of  $L$  to  $\ker P \cap D(L)$ , denoted by  $L_P$ , is an isomorphism. So, its (algebraic) inverse  $K_P: R(L) \rightarrow D(L)$  makes sense, and we can define the *generalized inverse* of  $L$  by

$$K_{P,Q} = K_P(I - Q).$$

It is easy to check that  $JQ + K_{P,Q}$  is an algebraic isomorphism from  $Y$  into  $D(L)$  and, for every  $x \in D(L)$ ,

$$(JQ + K_{P,Q})^{-1}x = (L + J^{-1}P)x.$$

Moreover, with the equation (1) we associate the intermediate mapping

$M_J: X \rightarrow X$  defined by

$$M_J u = Pu + (JQ + K_{P,Q})Fu.$$

and we can state the

**Equivalence Theorem.** For all  $u \in D(L) \cap D(F)$ ,

$$(a) \quad Lu = Fu \iff u = M_J u;$$

$$(b) \quad (I - M_J)u = (JQ + K_{P,Q})(L - F)u.$$

For a detailed proof of this theorem we refer the reader to [5], Lemma III.4.

A mapping  $T: \bar{\Omega} \rightarrow Y$  is called *L-compact* if  $QT: \bar{\Omega} \rightarrow Y$  is continuous,  $QT(\bar{\Omega})$  is bounded and  $K_{P,Q}T: \bar{\Omega} \rightarrow Y$  is compact on every bounded domain  $\Omega \subset X$ .

**Remark 1.** Assume that  $F: \bar{\Omega} \rightarrow Y$  is *L-compact*. Then  $M_J$  is compact in  $\bar{\Omega}$ .

Under this hypotheses, the solution of the equation (1) reduces to the finding of fixed points of the compact mapping  $M_J$ .

Let us denote  $\Psi = J^{-1}P$ . Then the equation  $(L + \Psi)x = y$  is equivalent to the system

$$L_P(I - P)x = (I - Q)y, \quad J^{-1}Px = Qy.$$

We also easily check that  $(L + \Psi)^{-1}\Psi = JJ^{-1}P = P$  and

$$M = (L + \Psi)^{-1}(\Psi + F).$$

Now, if  $F: \bar{\Omega} \rightarrow Y$  is an *L-compact* mapping such that  $Lx \neq F(x)$  for all  $x \in D(L) \cap \partial\Omega$ , then the *coincidence degree* of the pair  $(L, F)$  on  $\Omega$  is defined by

$$d((L, F), \Omega) = \deg_{LS} \left[ I - (L + \Psi)^{-1}(\Psi + F), \Omega, 0 \right], \quad (4)$$

where  $\deg_{LS}$  stands for the Leray-Schauder degree. It is not hard to verify that the equation (4) does not depend on the particular choice of the isomorphism  $J$ . The coincidence degree, so defined, has most of expected properties, for example, if  $d((L, F), \Omega) \neq 0$ , then there exists at least one solution of (1) in  $\Omega$ .

## 2. LINEAR OPERATOR PENCILS

The investigation of bifurcate solutions for the operator equation (2) requires some spectral elements related to the first-order polynomial equation

$$Lu - \lambda Au = 0, \quad (3)$$

where  $L: D(L) \rightarrow Y$  and  $A: X \rightarrow Y$  are linear operators.

We say that  $\lambda \in \mathbb{R}$  is a *characteristic value* of the pair  $(L, A)$  if  $\ker(L - \lambda A) \neq \{0\}$  and  $\lambda$  is a *regular covalue* of  $(L, A)$  if the inverse  $(L - \lambda A)^{-1}$  exists on  $Y$  and it is continuous. A solution  $u \in X$  of the equation (3) corresponding to a characteristic value  $\lambda$  is called a *characteristic function* of (3).

In virtue of the equivalence theorem, we obtain

$$\ker(L - \lambda A) = \ker(I - M_J^*(\lambda))$$

where  $M_J^*(\lambda) = P + \lambda(JQ + K_{P,Q})A$ . We specify the structure of the set of characteristic values of  $(L, A)$  under the following hypotheses:

- (i)  $L$  is a Fredholm operator of index zero;
- (ii)  $A: X \rightarrow Y$  is a linear  $L$ -compact operator.

**Lemma 1.** A number  $\lambda \in \mathbb{R}$  is either a characteristic value or a regular covalue for  $(L, A)$ .

Indeed, for  $x \in D(L)$  we have  $[I - M_J^*(\lambda)]x = (JQ + K_{P,Q})(L - \lambda A)x$ . By (i),  $M_J^*(\lambda)$  is compact and so it has a continuous inverse when  $\lambda$  is not a characteristic value of  $(L, A)$ . In that case, we have

$$(L - \lambda A)^{-1} = [I - M_J^*(\lambda)]^{-1}(JQ + K_{P,Q})$$

whence  $(L - \lambda A)^{-1}$  is continuous together with  $K_{P,Q}$ . ■

We denote henceforth the set of all characteristic values by  $C(L, A)$ .

On  $\mathbb{R} \setminus C(L, A)$ , consider the *resolvent*  $R(\lambda) = (L - \lambda A)^{-1}$ . The map  $R(\lambda)$  satisfies the following resolvent identity

$$R(\lambda) - R(\mu) = (\lambda - \mu) R(\lambda) A R(\mu).$$

Indeed,

$$\begin{aligned} R(\lambda) - R(\mu) &= R(\lambda)(L - \mu A)R(\mu) - R(\lambda)(L - \lambda A)R(\mu) = \\ &= R(\lambda)[LR(\mu) - \mu AR(\mu) - LR(\mu) + \lambda AR(\mu)] = (\lambda - \mu)R(\lambda)AR(\mu). \end{aligned}$$

Moreover, to any  $\mu \in C(L, A)$  we attach the product  $A_\mu = (L - \mu A)^{-1}A$  called the *spectral operator* for  $(L, A)$ . Taking into account that

$$(L - \lambda A)^{-1}A = [I - M_J^*(\lambda)]^{-1}(JQ + K_{P,Q})A$$

we infer that  $A_\mu: X \rightarrow X$  is compact, due to (i)-(ii).

When  $L = I$  is the identity operator and  $A$  is compact it is well known that  $C(I, A)$  is a discrete set. However, if  $\ker L \cap \ker A \neq \{0\}$  then  $C(I, A)$  can be the whole real line. We have the following alternative:

**Proposition 1.**  $C(L, A)$  is either  $\mathbb{R}$  or an at most countable set with only one possible accumulation point at infinity.

Proof. If  $C(L, A) \neq \mathbb{R}$ , then there is  $\mu \in C(L, A)$  and we can write

$$L - \lambda A = (L - \mu A)[I - (\lambda - \mu)(L - \mu A)^{-1}A].$$

Thus, the characteristic values for  $(L, A)$  are just translates by  $\mu$  of usual characteristic values of the compact mapping  $A_\mu$ . Due to this the proof follows at once from the Riesz theory ([4]). ■

The translation  $C(L, A) = \mu + C(I, A_\mu)$  allows us to define the multiplicity of  $\lambda \in C(L, A)$  by

$$m(\lambda) = \dim \ker [I - (\lambda - \mu)A_\mu]^{n(\lambda)} \quad (5)$$

where  $\mu \in C(L, A)$  is arbitrary and  $n(\lambda)$  is the smallest integer such that

$$\ker [I - (\lambda - \mu)A_\mu]^n = \ker [I - (\lambda - \mu)A_\mu]^{n+1}.$$

Under the transversality condition

$$A(\ker(L - \lambda A)) \cap \text{Im}(L - \lambda A) = \{0\},$$

it is proved in (9) that the multiplicity (5) is equivalent to

$$m(\lambda) = \dim \ker (L - \lambda A).$$

In the frame of the coincidence degree theory we will introduce also an index for isolated zeros. As above, let  $F: \Omega \rightarrow Y$  be an  $L$ -compact mapping and let  $a \in \Omega$  be an isolated zero of the equation  $(L + F)u = 0$ . There is an open ball  $B(a, r_a)$  in  $X$  with center  $a$  and radius  $r_a$  such that

$$(L + F)^{-1}(0) \cap B(a, r_a) = \{0\}.$$

By using the excision property of the coincidence degree, the integer

$$i((L, F), a) = \lim_{r \rightarrow 0^+} d((L, F), B(a, r))$$

is well defined and it is called the coincidence index of  $(L, F)$  at  $a$ .

A consequence of this definition is the following generalization of Leray-Schauder formula for the index of linear operators [9]:

**Proposition 2.** Suppose that  $C(L, A) \neq \mathbb{R}$  for a pair  $(L, A)$ , satisfying (i)-(ii), and let  $(\lambda_1, \lambda_2) \cap C(L, A) = \{\lambda_0\}$ . Then

$$i((L, \lambda_1 A)) = (-1)^\nu i((L, \lambda_2 A)),$$

where  $\nu$  is the multiplicity  $m(\lambda_0)$ .

### 3. BIFURCATION RESULTS.

Let  $H$  be a real Hilbert space with the inner product  $(\cdot, \cdot)$  and the norm  $\|\cdot\|$ ,  $L, A: H \rightarrow H$  be two linear operators with  $L$  noninvertible and  $D(L) \subseteq D(A) \subseteq H$ , and let  $A = [a, b]$  be a compact interval. We in-

investigate the nontrivial eigensolutions, i.e. eigenvalue-eigenfunction pairs  $(\lambda, u) \in \Lambda \times H$ ,  $u \neq 0$ , of the parameter equation

$$Lu - Au + F(\lambda, u) + G(\lambda, u) = 0 \quad (6)$$

where the nonlinear L-compact perturbations  $F, G: \Lambda \times H \rightarrow H$  satisfy the following assumptions:

- (a)  $G(\lambda, u) = o(\|u\|)$  as  $u \rightarrow 0$  uniformly with respect to  $\lambda \in \Lambda$ ;
- (b)  $F(\lambda, \cdot)$  is bounded, i.e. there is a constant  $k > 0$  such that
 
$$\|F(\lambda, u)\| \leq k \|u\| \quad \forall (\lambda, u) \in \Lambda \times H. \quad (7)$$

In particular, we are interested in finding the bifurcation points, i.e.  $(\lambda_0, 0)$  accumulating nontrivial solutions of (1). On the product  $E = \Lambda \times H$ , we take the norm

$$\|(\lambda, u)\| = \left[ |\lambda|^2 + \|u\|^2 \right]^{\frac{1}{2}}$$

An eigensolution of (6) is a pair  $(\lambda, u) \in E$ . Clearly,  $(\lambda, 0)$  is a solution and  $S_0 = \{(\lambda, 0) \mid \lambda \in \Lambda\}$  is the line of the so-called trivial solutions. Let  $\lambda_0$  be an eigenvalue of (3) of odd multiplicity. The classical Krasnosel'skii theorem [8] asserts that if  $F$  is differentiable at  $u = 0$  and  $F'(\lambda_0, 0) = 0$ , then  $\lambda_0$  is a bifurcation point from the trivial solution line; in other words, for any  $\varepsilon > 0$ , there exists a nontrivial eigensolution  $(\lambda, u)$  such that  $\|(\lambda - \lambda_0, u)\| < \varepsilon$ . Sometimes,  $\lambda_0$  is also called a branch-point.

In the case  $F \equiv 0$ , it is well-known ([15]) that bifurcation points of the nonlinear equation

$$Lu - \lambda Au + G(\lambda, u) = 0$$

can occur among the characteristic values of the linearized equation (3). In general, if a perturbation  $F$  is involved, the usual linearization is no longer valid. When the characteristic values of  $(L, A)$  are isolated, we will find an eigeninterval centered in every characteristic value of odd multiplicity, whose length depends on the constant  $k$  in (7); which contains at least a bifurcation point of the initial equation (6). Without loss of generality, we may replace  $F + G$  by  $F$ .

Given  $\lambda_0 \in C(L, A)$  and  $\rho > 0$ , let  $J(\lambda_0, \rho) = \{\lambda \in \mathbb{R} \mid |\lambda - \lambda_0| < \rho\}$ . We also denote by  $B_r$  the intersection of the closed ball of radius  $r > 0$  in  $H$  with the domain  $D(A)$ ,  $B_r = \{v \in D(A) \mid \|v\| \leq r\}$ , and by  $\partial B_r$  its boundary. Assume that  $C(L, A) \neq \mathbb{R}$  and for  $\mu \in C(L, A)$  set

$$c(\mu) = \inf \{ \|Lv - \mu Av\| \mid v \in D(A), \|v\| = 1 \}.$$

**THEOREM 1.** Assume that (i)-(iv) in Sec.2 hold. Suppose that  $F$  is

an L-compact perturbation satisfying hypothesis (b) and let  $\lambda_0 \in C(L, A)$  be of odd multiplicity with

$$\text{dist}(\lambda_0, C(L, A) \setminus \{\lambda_0\}) > 2k.$$

Then, given any  $r > 0$ , the equation

$$Lu - \lambda Au + F(\lambda, u) = 0$$

admits an eigensolution  $(\lambda_r, u_r)$  with  $\lambda_r \in [\lambda_0 + k, \lambda_0 + k]$  and  $\|u_r\| = r$ .

**Proof.** Let  $\varepsilon = \text{dist}(\lambda_0, C(L, A) \setminus \{\lambda_0\}) - 2k > 0$ ,  $\underline{\lambda} = \lambda_0 - (k + \varepsilon)$ ,  $\bar{\lambda} = \lambda_0 + (k + \varepsilon)$  and notice that  $\text{dist}(\underline{\lambda}, C(L, A)) = \text{dist}(\bar{\lambda}, C(L, A)) = k + \varepsilon$ .

We assume by contradiction that

$$Lu - \lambda Au + F(\lambda, u) \neq 0$$

for all  $u \in \partial B_r$  and  $\lambda \in (\underline{\lambda}, \bar{\lambda})$ . Then there is a homotopy joining  $L - \underline{\lambda}A + F(\underline{\lambda}, \cdot)$  and  $L - \bar{\lambda}A + F(\bar{\lambda}, \cdot)$  and thereby, by the homotopy invariance property of the coincidence degree,

$$\deg(L - \underline{\lambda}A + F(\underline{\lambda}, \cdot), B_r) = \deg(L - \bar{\lambda}A + F(\bar{\lambda}, \cdot), B_r). \quad (8)$$

On the other hand, for all  $t \in [0, 1]$  and  $u \in B_r$ ,

$$\|Lu - \lambda Au + tF(\lambda, \cdot)\| \geq c(\underline{\lambda})\|u\| - t\|F(\lambda, \cdot)\| \geq (c(\underline{\lambda}) - k)\|u\| > \varepsilon\|u\|,$$

and, again by the homotopy invariance property, we obtain

$$\deg(L - \lambda A + F(\lambda, \cdot), B_r) = \deg(L - \underline{\lambda}A, B_r).$$

Now observe that, since  $\underline{\lambda} \notin C(L, A)$ , we have  $Lu - \underline{\lambda}Au = 0$  iff  $u = 0$ , and therefore  $i((L, \underline{\lambda}A)) = 0$ .

The argument used for  $L - \underline{\lambda}A + F(\lambda, \cdot)$  applies to  $L - \bar{\lambda}A + F(\lambda, \cdot)$  as well, so that from (8) we infer  $i((L, \underline{\lambda}A)) = i((L, \bar{\lambda}A))$ . Since  $[\underline{\lambda}, \bar{\lambda}] \cap C(L, A) = \{\lambda_0\}$  and  $\lambda_0$  is of odd multiplicity, Proposition 2 yields  $i((L, \underline{\lambda}A)) = -i((L, \bar{\lambda}A))$ . This contradiction proves the claim. ■

Degree theoretic arguments provide not only existence criteria for bifurcation points but also information regarding the behavior of solution branches. Set

$$S' = S \cup (C(\lambda, B) \times \{0\})$$

$$S = \{(\lambda, u) \in \mathbb{R} \times X \mid \mathfrak{Q}(\lambda, u) = 0, u \neq 0\}$$

where  $\mathfrak{Q}(\lambda, u) = Lu - \lambda Au + F(\lambda, u)$ . Let  $S_{\lambda_0}$  be the component (maximal connected subset) of  $S'$  meeting  $J(\lambda_0, k) \times \{0\}$ .

**THEOREM 2.** Under the hypotheses of Theorem 1,  $S_{\lambda_0}$  is an unbounded subset of  $J(\lambda_0, k) \times H$ .

The proof follows Rabinowitz global bifurcation theorem ([15], Th.3.2) by mean of straightforward adaptations.

#### 4. ENERGY SPACE

Later on, we confine ourselves to operators acting on a real Hilbert space  $H$  with the inner product  $(\cdot, \cdot)$  and corresponding norm  $\|\cdot\|$ . Let  $T: D(T) \subseteq H \rightarrow H$  be a linear symmetric, densely defined, and strongly monotone operator, namely there is a constant  $c$  such that

$$(Tu, u) \geq c_T^2 \|u\|^2 \quad \text{for all } u \in D(T). \quad (9)$$

We extend  $T$  to a maximal monotone operator, i.e. with no proper monotone extensions in  $D(T) \times H$ . This extension  $T_F: D(T) \rightarrow H$  is self-adjoint and coincides with the so-called Friedrichs extension, ([17], pp.827).

On the other hand, we define the energy space  $H_T$  as the completion of  $D(T)$  with respect to the so-called energy inner product  $(u, v)_T = (Tu, v)$ . Let us denote by  $H_T^*$  the dual space of  $H_T$ . The range of the energy extension  $T_E: H_T \rightarrow H_T^*$  leaves the original space  $H$ . The imbeddings  $H_T \subseteq H \subseteq H_T^*$  are continuous and  $T \subseteq T_F \subseteq T_E$ . In fact,  $T_E$  is the duality map of  $H_T$ , that means a linear homeomorphism with

$$(T_E u, u) = \|u\|_T^2 \quad \text{for all } u \in H_T.$$

It follows that  $\|u\| \leq \frac{1}{c_T} \|u\|_T$  for all  $u \in D(T)$  and  $T_F^{-1} f = T_E^{-1} f$  for all  $f \in H$ .

It can also be shown that  $H_T = D(T_F^{\frac{1}{2}})$  and  $\|u\|_T = \|T_F^{\frac{1}{2}} u\|$  for all  $u \in H_T$ .

**Proposition 3.** ([17], pp.127) If the imbedding  $H_T \subset H$  is compact, then the operator  $T_F^{-1}: H \rightarrow H$  is compact.

This result is crucial in our study of the eigenvalue problem

$$Tu = \lambda u + F \quad u \in D(T), \quad \lambda \in \mathbb{R},$$

in its generalized form

$$T_F u = \lambda u + F \quad u \in D(T_F), \quad \lambda \in \mathbb{R}.$$

We derive  $u = \lambda T_F^{-1} u + T^{-1} F$  and, in particular, when  $F = 0$ , by Proposition 3, we easily deduce

**Corollary 1.** Assume that  $H_T$  is compactly imbedded in  $H$ . Then

- (a) the operator  $T_F: D(T_F) \subseteq H \rightarrow H$  has a countable system of eigensolutions  $\{\lambda_n, u_n\}$  which contains all the eigenvalues of  $T_F$ ;
- (b) all eigenvalues of  $T_F$  have finite multiplicities. Furthermore,  $\lambda_n \geq c$  for all  $n \in \mathbb{N}$  and  $\lambda_n \rightarrow +\infty$  as  $n \rightarrow \infty$ ;
- (c) the eigenfunctions  $\{u_n\}$  form a complete orthogonal system in  $H_T$ .

Similar extensions for nonlinear operators, of course making appropriate changes in the definitions have been given by W.V.Petryshyn [13].

We suppose herefrom that  $D(L) \subseteq D(A) \subseteq H$  with  $D(L)$  dense in  $H$ . Having in mind the applications (see [12]), we consider that both  $L$  and  $A$  are selfadjoint and strongly monotone operators. We also assume that  $L - \lambda A$  is selfadjoint for every  $\lambda \in \Lambda$ . We construct the energy spaces  $H_L$  and  $H_A$  and denote henceforth by  $L$  and  $A$  the corresponding Friedrichs selfadjoint extensions.

**Proposition 4.** Suppose that  $H_L$  is compactly imbedded in  $H_A$ . Then

(a') the equation (3) has a countable set of characteristic values;

(b') all characteristic values have a finite multiplicities. The only accumulation point of  $C(L, \lambda)$  is infinity;

(c') the corresponding characteristic functions form a complete orthonormal systems in each of the spaces  $H$ ,  $H_L$ ,  $H_A$ .

**Proof.** Since  $A^{-1}$  is defined on all of  $H$  we can apply it to equation (3) and obtain a generalized equation

$$A^{-1}Lu - \lambda u = 0, \quad u \in H_A, \quad (10)$$

where  $T = A^{-1}L$  is a monotone operator in  $H_A$ . In fact,

$$(Tu, u)_A = (A^{-1}Lu, u)_A = (Lu, u) \geq c_L^2 \|u\|^2 \quad \forall u \in D(L).$$

This enable us to introduce the energy space with inner product

$$(u, v)_T = (Tu, v);$$

this space coincides with  $H_L$ , since

$$(u, v)_T = (A^{-1}Lu, v)_A = (Lu, v) = (u, v)_L.$$

The monotone operator  $T$  can be extended to a maximal one in  $H_A$ . Assuming that this extension has been carried out, by generalized characteristic values - functions of (3) we will understand those of (10). Moreover, the hypothesis that every bounded set in  $H_L$  is compact in  $H_A$  agrees to the condition that  $H_A$  is compactly imbedded in  $H$  in Corollary 1. Thereby, Proposition 4 follows easily from Corollary 1. ■

Results of this type are basically due to S.G. Mikhlin [10] and have extended to large classes of general operators by W.V. Petryshyn [14].

Finally, for any  $\lambda \in \mathbb{R}$  and  $L$  a selfadjoint operator, is well-known, the isolation distance formula

$$\text{dist}(\lambda, \sigma(L)) = \|(L - \lambda I)^{-1}\|^{-1}$$

where  $\sigma(L)$  denotes the spectrum of  $L$ . This relation is generalized in [7] to linear operator pencils involving a pair of selfadjoint operators  $(L, A)$

$$\|(L - \lambda A)^{-1}\| \leq \frac{1}{c_A^2} \frac{1}{\text{dist}(\lambda, C(L, A))}, \quad (11)$$

where the constant  $c_A$  is as in (6).

Indeed, we write

$$(L - \lambda A) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}LA^{-\frac{1}{2}} - \lambda I)A^{\frac{1}{2}}. \quad (12)$$

Since  $A^{-\frac{1}{2}}LA^{-\frac{1}{2}}$  is a selfadjoint operator, we have

$$\|(A^{-\frac{1}{2}}LA^{-\frac{1}{2}} - \lambda I)^{-1}\| = (\text{dist}(\lambda, C(L, A)))^{-1}.$$

But (12) implies  $(L - \lambda A)^{-1} = A^{-\frac{1}{2}}(A^{-\frac{1}{2}}LA^{-\frac{1}{2}} - \lambda I)^{-1}A^{-\frac{1}{2}}$ . Since  $\|A^{-\frac{1}{2}}\| \leq \frac{1}{c_A}$ , the estimate (11) follows. ■

Results of Sec.1 are particularized in (7) to the case  $Y = H$  and  $X = H_A$ , described above.

As in [1], with any  $\varepsilon > 0$  we associate the *approximate equation*

$$Lu - \lambda Au + F(\lambda, |u|^\varepsilon u) = 0 \quad (13_\varepsilon)$$

which is equivalent to the equation

$$u = M_\varepsilon(u)$$

where  $M_\varepsilon(u) = Pu + \lambda(JQ + K_{P,Q})Au + (JQ + K_{P,Q})F(\lambda, |u|^\varepsilon u)$ . Because  $P$  and  $JQ$  are operators of finite rank,  $K_{P,Q}A$  is compact and, by hypothesis b),  $F(\lambda, |u|^\varepsilon u) = o(\|u\|)$ , the classical bifurcation theory involving  $L$ -compact perturbations ([6] or [9]) ensures the existence of a solution pair  $(\lambda_{\varepsilon r}, u_{\varepsilon r}) \in \Lambda \times \partial B_r$  for any positive number  $\varepsilon > 0$ .

For brevity, we denote  $J_\lambda = \left[ \lambda - \frac{k}{c^2}, \lambda + \frac{k}{c^2} \right]$ , where  $c = c_A$ .

**Proposition 5.** Suppose that  $\{\varepsilon_n\}$  is a sequence of positive numbers converging to zero. If there exists a sequence  $\{(\lambda_{\varepsilon_n}, u_{\varepsilon_n})\}$  of solutions of (13<sub>ε</sub>) such that  $\{(\lambda_{\varepsilon_n}, u_{\varepsilon_n})\}$  converges to  $(\mu, 0)$  in  $E$ , then  $\mu \in J_\lambda$ .

**Proof.** We let  $\mu \neq \lambda$  and use the following notations

$$v_n = \frac{u_{\varepsilon_n}}{\|u_{\varepsilon_n}\|} \quad \text{and} \quad F_n = \frac{1}{\|u_{\varepsilon_n}\|} F(\lambda_{\varepsilon_n}, \|u_{\varepsilon_n}\|^{\varepsilon_n} u_{\varepsilon_n}).$$

Then  $Lv_n = \lambda_{\varepsilon_n} Av_n + F_n$  and  $\|F_n\| \leq k$ . For sufficiently large  $n$ ,

$$v_n = (L - \lambda_{\varepsilon_n} A)^{-1} F_n \quad \text{and so} \quad 1 \leq \|(L - \lambda_{\varepsilon_n} A)^{-1}\| k.$$

Since  $AR(\lambda)$  is closed and defined on  $H$ , it is also a bounded operator by the closed graph theorem. From the resolvent identity it follows that

$\|R(\lambda_{\epsilon_n})\| \rightarrow \|R(\mu)\|$  as  $n \rightarrow \infty$  and thereby  $1 \leq \|(L - \mu A)^{-1}\| K$ . Taking into account (11) we derive  $\text{dist}[\mu, C(L, A)] < \frac{k}{c^2}$  and hence  $\mu \in J_\lambda$ . ■

Now we conclude our main result

**THEOREM 3.** Let  $A$  be a linear monotone maximal operator. Assume that hypotheses (i)-(iv) in Sec.2 hold for  $Y = H$  and  $X = H_A$ , while  $F(\lambda, \cdot)$  is an  $L$ -compact perturbation fulfilling condition (b). Let  $\lambda_0 \in C(L, A)$  be of odd multiplicity satisfying

$$\text{dist}(\lambda_0, C(L, A) \setminus \{\lambda_0\}) > \frac{2k}{c^2}.$$

Then the equation

$$Lu - \lambda Au + F(\lambda, u) = 0$$

admits an unbounded continuum  $S_{\lambda_0}$  of nontrivial solutions emanating from the eigeninterval  $J_{\lambda_0}$ .

The proof makes use of the compact imbedding of  $H_A$  in  $H$ .

In conclusion, we mention briefly a basic simple example to which the above bifurcation theory can be applied.

We consider the nonlinear eigenvalue problem

$$\mathcal{L}u(t) = \lambda \int_0^1 K(t, s) u(s) ds + F(\lambda, u(t), u'(t)),$$

with a set of separated boundary conditions

$$\begin{cases} \alpha u(0) + \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

where  $(\alpha^2 + \beta^2)(\gamma^2 + \delta^2) > 0$ . Here  $\mathcal{L}u = -(p(t)u')' + q(t)u$  is a regular Sturm-Liouville operator, where  $p$  is a positive, continuously differentiable function, and  $q$  is a continuous function on  $[0, 1]$ . The kernel  $K \in L^\infty([0, 1] \times [0, 1])$  while  $F(\lambda, \cdot)$  is a bounded perturbation.

In the same spirit, existence results for positive solutions of semi-linear elliptic equations can be obtained. The complete treatment will appear in a future paper [12].

Proposition 5 stimulates a deeper investigation of the above equations by means of  $A$ -proper mappings. For branching of solutions from characteristic values, this approach is outlined in [16]. A definition of the topological index for  $A$ -proper operators is introduced in [11].

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