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UNIQUENESS OF THE CAUCHY PROBLEM FOR SOME ABSTRACT DIFFERENTIAL EQUATIONS [1]

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Introduction. In this work we present some results about the Cauchy problem for differential equations in Hilbert spaces with unbounded operator coefficients. The equations are of the second order and there are natural applications to some partial differential equations.

1. We consider the second order differential equations in implicit form:

$$Bu''(t) = Au(t) + f(t) \tag{1.1}$$

where A, B are linear operators with domains D(A), D(B) in the Hilbert space H.

Given a real interval $[0, T]$ we assume that $f(\cdot) \in C([0, T]; H)$ and we look for solutions $u(t) \in C^2([0, T]; H)$ such that $u(t) \in D(A)$, $t \in [0, T]$, $Au(\cdot) \in C([0, T]; H)$; $u'(t) \in D(B)$, $Bu'(t) \in C([0, T]; H)$; $u''(t) \in D(B)$ and $Bu'' = Au + f$ on $[0, T]$. To solve a Cauchy problem means here: given $u_0 \in D(A)$ and $v_0 \in D(B)$, find a function $u(t)$ as above, with supplementary properties: $u(0) = u_0$, $u'(0) = v_0$.

Theorem 1.1. Assume that operators A and B are symmetric and that: $(Bh, h) > 0$ $\forall h \in D(B)$, $h \neq \theta$, $(Ah, h) \leq 0$, $\forall h \in D(A)$.

Then two solutions $u(t)$, $v(t)$ with same initial data u_o, v_o and same right-hand side $f(\cdot)$ coincide on $[0, T]$.

Proof. If $w(t) = u(t) - v(t)$ we obtain: $Bw''(t) = Aw(t)$, and $w(0) = w'(0) = \theta$. Consider the "energy" $E(t) = (Bw', w') - (Aw, w)$.

We shall use the following:

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Lemma 1. Let P be a linear symmetric operator with domain $D(P)$ in the Hilbert space H and let $\varphi(t) \in C^1([0, T]; H)$, $\varphi(t) \in D(P)$, $P\varphi \in C([0, T]; H)$. Then the derivative $\frac{d}{dt}$ $(P\varphi(t), \varphi(t))$ exists and $= 2\mathbb{R}e(P\varphi, \varphi') = (P\varphi, \varphi') + (\varphi', P\varphi)$.

In fact, we have

$$(P\varphi(t+h), \varphi(t+h)) - (P\varphi(t), \varphi(t)) = (P\varphi(t+h), \varphi(t+h) - \varphi(t)) +$$
$$+ (P\varphi(t+h) - P\varphi(t), \varphi(t)) = (P\varphi(t+h), \varphi(t+h) - \varphi(t)) +$$
$$+ (\varphi(t+h) - \varphi(t), P\varphi(t)).$$

Then we divide by h and note that $(P\varphi)(t+h) \to (P\varphi)(t)$ as $h \to 0$. We obtain the result readily.

We shall apply this result twice: to the function (Bw', w') and to (Aw, w) (it is easy to check that all conditions are satisfied). Accordingly we obtain relations

$$\frac{d}{dt}(Aw, w) = (Aw, w') + (w', Aw).$$

$$\frac{d}{dt}(Bw', w') = (Bw', w'') + (w'', Bw').$$

Therefore the derivative of the function "energy" exists too and we have

$$\begin{aligned} E'(t) &= (Bw', w'') + (w'', Bw') - (Aw, w') - (w', Aw) \\ &= (w', Bw'') + (Bw'', w') - (Aw, w') - (w', Aw) \\ &= (w', Aw) + (Aw, w') - (Aw, w') - (w', Aw) = 0. \end{aligned}$$

Hence $E(t) = E(0)$, $0 \le t \le T$. On the other hand

$$E(0) = (Bw'(0), w'(0)) - (Aw(0), w(0)) = 0,$$

hence $E(t) = 0$, $0 \le t \le T$. Accordingly, we get $(Bw'(t), w'(t)) = 0$ $\forall\, t \in [0, T]$. Hence $w'(t) = \theta$ $\forall\, t \in [0, T]$, and $w(t) = w(0) = \theta$ $\forall\, t \in [0, T]$. □

In the following theorem we shall eliminate the hypothesis: "$(Ax, x) \le 0$ $\forall\, x \in D(A)$"; the method of proof will be based on some convexity properties enjoyed by the function $(Bw(t), w(t))$.

The precise statement is as follows:

Theorem 1.2. Let us assume that operators A and B are symmetric in the Hilbert space H, with domains $D(A)$ and $D(B)$ and that $(Bx, x) > 0$ $\forall x \in D(B)$, $x \neq \theta$. Consider a function $w(t) \in C^2([0, T]; H)$, such that: $w(t) \in D(A) \cap D(B)$ $\forall t \in [0, T]$, $Aw(t)$ and $Bw(t)$ both are in $C([0, T]; H)$; $w'(t) \in D(B)$ and $Bw' \in C([0, T]; H)$; $w'' \in D(B)$ and $Bw''(t) = Aw(t)$, $t \in [0, T]$; $w(0) = w'(0) = \theta$. Then $w(t) = \theta$ $\forall t \in [0, T]$.

Proof. We define $F(t) = (Bw(t), w(t))$. We can apply Lemma 1 and get: $F'(t)$ exists and $= (Bw, w') + (w', Bw) = 2\mathbb{R}e(Bw, w')$.

Now, in order to differentiate F' we need an extension of Lemma 1, given now as:

Lemma 2. Let P be a linear symmetric operator with $D(P) \subset H$ and let $\varphi(t)$, $\psi(t)$ two functions in $C^1([0, T]; H)$, such that $\varphi(t), \psi(t) \in D(P)$ $\forall t \in [0, T]$ and $P\varphi \in C([0, T]); H)$. Then the derivative $\frac{d}{dt}(P\varphi(t), \psi(t))$ exists and $= (P\varphi, \psi') + (\varphi', P\psi)$.

We have in fact

$$\begin{aligned}(P\varphi(t+h), \psi(t+h)) - (P\varphi(t), \psi(t)) &= (P\varphi(t+h), \psi(t+h)-\psi(t)) \\ &+ (P\varphi(t+h) - P\varphi(t), \psi(t)) = (P\varphi(t+h), \psi(t+h) - \psi(t)) \\ &+ (\varphi(t+h) - \varphi(t), P\psi(t)).\end{aligned}$$

Using continuity of $P\varphi$ we get the result.

We shall apply this result to (Bw, w') and (w', Bw) (for (w', Bw): we write it as $\overline{(Bw, w')}$ and note that $\frac{d}{dt}\overline{(Bw, w')} = \overline{\frac{d}{dt}(Bw, w')}$; then

$$\overline{\frac{d}{dt}(Bw, w')} = \overline{(Bw, w'')} + \overline{(w', Bw')} = (w'', Bw) + (Bw', w').$$

Thus we get

$$F''(t) = (Bw, w'') + (w', Bw') + (w'', Bw) + (Bw', w')$$
$$= 2\mathbb{R}e(Bw, w'') + 2\mathbb{R}e(w', Bw') = 2\mathbb{R}e(w, Bw'') + 2(w', Bw')$$
$$= 2\mathbb{R}e(w, Aw) + 2(w', Bw') = 2(Aw, w) + 2(Bw', w').$$

Let us introduce again, as in previous Theorem, the function $E(t) = (Bw', w') - (Aw, w)$. Looking back at the proof in Th. 1.1, we see that $E(t) = E(0)$ $\forall\, t \in [0, T]$ (even if no negativity condition on (Ax, x) appears; this condition is only used at the end of that proof). Also, here again $E(0) = 0$. Hence $E(t) = 0$, $t \in [0, T]$ and $(Aw, w) = (Bw', w')$. It follows: $F''(t) = 4(Bw', w')$ and then $FF'' - (F')^2 = 4(Bw, w)(Bw', w') - 4[\mathbb{R}e(Bw, w')]^2$.

Note now that on $D(B)$ the expression $(x, y)_B = (Bx, y)$ is a scalar product. Hence we get $|(x, y)_B|^2 \leq (x, x)_B(y, y)_B$, $x, y \in D(B)$, which becomes $|(Bx, y)|^2 \leq (Bx, x)(By, y)$; then

$$|\mathbb{R}e(Bx, y)|^2 \leq |(Bx, y)|^2 \leq (Bx, x)(By, y), \quad x, y \in D(B)$$

and it follows that

$$[\mathbb{R}e(Bw, w')]^2 \leq (Bw, w)(Bw', w'), \quad \forall\, t \in [0, T]$$

and then

$$FF'' - F'^2 \geq 0 \quad \text{on} \quad [0, T].$$

Our next goal is to establish: $F(t) = 0$ in $[0, T]$. We only know that $F(0) = 0$. Assume $F(t) \not\equiv 0$ in $[0, T]$. It follows, $\exists\, t_o \in (0, T)$ where $F(t_o) > 0$. Hence, again from continuity of $F(t)$ we find an interval $(c, d) \subset (0, T)$ where $F(t) > 0$. Let $\alpha = \inf\{c' \in (0, T), F(t) > 0 \text{ in } (c', d)\}$. Then $\alpha \leq c$. Also $F(t) > 0$ in (α, d) and $F(\alpha) = 0$.

Consider now the function $t \to \ell n\, F(t)$ in (α, d). We see that

$$(\ell n\, F)'' = \frac{FF'' - F'^2}{F^2} \geq 0.$$

It follows that $t \to \ell n\, F(t)$ is a convex function in the interval (α, d).

If we take an interval $[\alpha_1, d_1] \subset (\alpha, d)$, and any $t \in (\alpha_1, d_1)$, we have

$$t = \frac{t-\alpha_1}{d_1-\alpha_1} d_1 + \frac{d_1-t}{d_1-\alpha_1}\alpha_1 \text{ and } \frac{t-\alpha_1}{d_1-\alpha_1} + \frac{d_1-t}{d_1-\alpha_1} = 1.$$

Accordingly we obtain

$$-\infty < \ell n\, F(t) \le \frac{t-\alpha_1}{d_1-\alpha_1} \ell n\, F(d_1) + \frac{d_1-t}{d_1-\alpha_1} \ell n\, F(\alpha_1)$$

Finally, let $\alpha_1 \downarrow \alpha$. As $\ell n\, F(\alpha_1) \to -\infty$ we get an obvious contradiction. Therefore $F(t) = (Bw(t), w(t)) = 0 \ \forall\, t \in [0, T]$ and consequently $w(t) = \theta$, $t \in [0, T]$. □

Our last discussion in this section is about an explicit second order differential equation, precisely $u''(t) = Au(t)$ where now A is a skew-symmetric operator. A new method is used in order to get the usual uniqueness result. We have therefore following:

Theorem 1.3. Let H be a Hilbert space and A a linear operator in H, with domain $D(A)$. Assume relation $(Ah, k) = -(h, Ak)$, $\forall\, h, k \in D(A)$. Let $u(t) \in C^2([0, T]; H)$, such that $u(t) \in D(A)$ $\forall\, t \in [0, T]$, $u'(t) \in D(A)$, $u(0) = u'(0) = \theta$, $u''(t) = Au(t)$ on $[0, T]$. Then $u(t) = \theta$, $t \in [0, T]$.

Proof. We introduce a function $w(t) = e^{-\lambda t}u(t)$ where λ is a real number. We have $w(0) = \theta$; $w'(t) = e^{-\lambda t}u'(t) - \lambda e^{-\lambda t}u(t)$ and $w'(0) = \theta$. We also have: $u(t) = e^{\lambda t}w(t)$, $u'(t) = \lambda e^{\lambda t}w(t) + e^{\lambda t}w'(t)$, $u''(t) = \lambda^2 e^{\lambda t}w(t) + 2\lambda e^{\lambda t}w'(t) + e^{\lambda t}w''(t)$. Thus, the relation $u''(t) = Au(t)$ becomes $\lambda^2 e^{\lambda t}w(t) + 2\lambda e^{\lambda t}w'(t) + e^{\lambda t}w''(t) = e^{\lambda t}\, Aw(t)$ and then $\lambda^2 w(t) + 2\lambda w'(t) + w''(t) - Aw(t) = \theta$, $0 \le t \le T$, or also $[w''(t) + \lambda^2 w(t)] + [2\lambda w'(t) - Aw(t)] = \theta$, $t \in [0, T]$.

Let us note the

Lemma 3. In any (pre)Hilbert space H, the equality $(x+y) = \theta \Rightarrow \mathbb{R}e(x, y) \le 0$. In fact, we have

$$0 = \|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathbb{R}e(x, y),$$

hence the result. Therefore we derive the inequality

$$\mathbb{R}e(w''+\lambda^2 w, 2\lambda w'-Aw) \le 0$$

and also

$$\mathbb{R}e\, 2\lambda(w'', w') + 2\lambda^3\mathbb{R}e(w, w') - \mathbb{R}e(w'', Aw) - \lambda^2\mathbb{R}e(w, Aw) \le 0. \qquad (*)$$

We now see that

$$2\mathbb{R}e(w'', w') = \frac{d}{dt}(w', w'); \quad 2\mathbb{R}e(w, w') = \frac{d}{dt}(w, w);$$

also from $(Ah, k) = (-h, Ak)$ it follows $(Ah, h) = -(h, Ah) = -(\overline{Ah, h})$, $2\mathbb{R}e(Ah, h) = (Ah, h) + (h, Ah) = 0$. Hence $\mathbb{R}e(w, Aw) = 0$.

From (*) it remains only

$$\lambda \frac{d}{dt}\|w'(t)\|^2 + \lambda^3 \frac{d}{dt}\|w(t)\|^2 - \mathbb{R}e(w'', Aw) \le 0, \quad t \in [0, T]. \qquad (**)$$

Let us consider the derivative $\frac{d}{dt}(w', iAw)$ applying Lemma 2 to $P = iA$. Thus $\frac{d}{dt}(iAw', w) = (iAw', w') + (w'', iAw)$. We get $\frac{d}{dt}(w', Aw) = (-Aw', w') + (w'', Aw)$. Taking real parts: $\frac{d}{dt}\mathbb{R}e(w', Aw) = \mathbb{R}e(w'', Aw)$. Therefore (**) becomes

$$\lambda \frac{d}{dt}\|w'(t)\|^2 + \lambda^3 \frac{d}{dt}\|w(t)\|^2 - \frac{d}{dt}\mathbb{R}e(w', Aw) \le 0, \quad t \in [0, T].$$

We see that, introducing the function $Q_\lambda(t) = \lambda\|w'(t)\|^2 + \lambda^3\|w(t)\|^2 - \mathbb{R}e(w', Aw)$ we have $Q'_\lambda(t) \le 0$, $0 \le t \le T$. Hence $Q_\lambda(t) \le Q_\lambda(0)$ $(t \ge 0)$ and also $Q_\lambda(0) = 0$. Thus, $Q_\lambda(t) \le 0$, $0 \le t \le T$.

Now, if $\lambda > 0$, $Q_\lambda(t) \le 0$ implies $\lambda^3\|w(t)\|^2 - \mathbb{R}e(w', Aw) \le 0$, $0 \le t \le T$ which becomes $\lambda^3 e^{-2\lambda t}\|u(t)\|^2 - \mathbb{R}e(e^{-\lambda t}u' - \lambda e^{-\lambda t}u, A(e^{-\lambda t}u)) \le 0$ and also

$$\lambda^3\|u(t)\|^2 - \mathbb{R}e(u' - \lambda u, Au) \le 0$$

and then

$$\lambda^3\|u(t)\|^2 - \mathbb{R}e(u', Au) \le 0.$$

Thus

$$\|u(t)\|^2 \le \frac{1}{\lambda^3}|(u'(t), Au(t))|, \ 0 \le t \le T.$$

This is true $\forall \lambda > 0$. If $\lambda \to +\infty$ we get obviously $u(t) = \theta$ on $[0, T]$. □

2. In this short section we present an "abstract" version of the result concerning unicity in the paper [3]. The method of proof here is the "eigen-vectors" method. This section is independent of [2].

The equation here studied (again a second order equation) has the following form:

$$u''(t) = -A(u(t) + 2h\, u'(t)) - 2ku'(t) + f(t), \text{ where } h > 0 \text{ and } k \ge 0 \text{ are constants.} \quad (2.1)$$

Two Hilbert spaces V and H are given, such that $V \subseteq H$ also in vector space sense, V is dense in H and the immersion ("identity mapping") of V into H is continuous. We shall define below a class of weak solutions of (2.1), where $f(t)$ is given in $L^2([0, T]; H)$ and one looks for $u(t) \in C^1([0, T]; V)$. The initial data $u(0)$ and $u'(0)$ are given in V.

The operator A comes from a sesqui-linear continuous form $a(u, v)$ on V, that is a mapping $u, v \to a(u, v)$, $V \times V \to \mathbb{C}$, linear in u, anti-linear in V, such that $|a(u, v)| \le c_1\|u\|_V\|v\|_V$. Precisely, if $N = \{u \in V,\ v \to a(u, v)$ is continuous in H-norm$\}$, we can write $a(u, v) = (Au, v)_H$ for $u \in N$, $v \in V$ which defines a linear, usually unbounded operator A with domain $D(A) = N \subset V$. Now, the weak form of the equation (2.1) is as follows:

$$\int_0^T (u'(t), \varphi'(t))_H dt - \int_0^T a(u(t), \varphi(t))dt - \int_0^T 2ha(u'(t), \varphi(t))dt - 2k\int_0^T (u'(t), \varphi(t))_H dt$$
$$+ \int_0^T (f(t), \varphi(t))_H dt = 0, \ \forall\, \varphi(t) = \eta(t)v.$$

where $\eta(t) \in C^1[0, T]$, $\eta(0) = \eta(T) = 0$ and $v \in V$. (2.2)

(this equation is formally deduced from (2.1) multiplying scalarly with $\varphi(t)$ and using some partial integration, as well as relation $a(u, v) = (Au, v)$). About the operator A and the sesqui-linear form a we make a few more assumptions:

i) $(Au, u) = a(u, u) \geq 0 \ \forall u \in D(A)$; $a(u, v) = \overline{a(v, u)}$, $\forall u, v \in V$.

ii) There exists a sequence $(e_n)_1^\infty \subset V$ which is complete both in V and in H, and an increasing sequence of real (positive) members

$$0 \leq \lambda_1 < \lambda_2 \dots \leq \lambda_n \dots \to \infty$$

is such a way that

$$a(e_n, v) = \lambda_n (e_n, v)_H \tag{2.3}$$

holds, $\forall v \in V$, $\forall n = 1, 2, 3, \dots$ (this implies: $e_n \in D(A)$, $(Ae_n, v) = (\lambda_n e_n, v)$, $\forall v \in V$, hence $Ae_n = \lambda_n e_n$, $n = 1, 2, \dots$).

Now if $u(t) \in C^1([0, T]; V)$ is a solution of (2.2), with $f \equiv \theta$ and $u(0) = u'(0) = \theta$ we must establish that $u(t) = \theta \ \forall t \in [0, T]$. Because of completeness of $(e_n)_1^\infty$ in H we have $u(t) = \sum_1^\infty (u(t), e_n)e_n = \sum_1^\infty u_n(t)e_n$ where the series is strongly (pointwise) convergent in H and also

$$|u(t)|_H^2 = \sum_1^\infty u_n^2(t), \ 0 \leq t \leq T. \tag{2.4}$$

We shall find ordinary differential equations verified by every function $u_n(t)$. In order to do that, we take "test-functions" $\varphi_n(t)$ of the form $\varphi_n(t) = \eta(t)e_n$, where $\eta(t) \in C^1[0, T]$ and $\eta(0) = \eta(T) = 0$.

We obtain from (2.2) (with $f \equiv \theta$):

$$\int_0^T (u'(t), e_n)_H \eta'(t)dt - \int_0^T a(u(t), e_n)\eta(t)dt - 2h\int_0^T a(u'(t), e_n)\eta(t)dt$$

$$- 2k\int_0^T (u'(t), e_n)_H \eta(t)dt = 0, \quad n = 1, 2, \dots \tag{2.5}$$

If $u_n(t) = (u(t), e_n)$, we have $u_n'(t) = (u'(t), e_n)_H$ and also

$$a(u(t), e_n) = \overline{a(e_n, u(t))} = \overline{\lambda_n(e_n, u(t))} = \lambda_n(u(t), e_n) = \lambda_n u_n(t)$$

$$a(u'(t), e_n) = \lambda_n u_n'(t), \quad n = 1, 2, \dots .$$

Therefore (2.5) becomes

$$\int_0^T \{u_n'(t)\eta'(t) - [\lambda_n u_n(t) - 2h\lambda_n u_n'(t) - 2ku_n'(t)]\eta(t)\}dt = 0,$$

$$\forall\ \eta \in C^1[0, T]\ \ \eta(0) = \eta(T) = 0. \tag{2.6}$$

This relation (2.6) has as a known consequence:

$$u_n''(t) \text{ exists and } = -\lambda_n u_n(t) - 2h\lambda_n u_n'(t) - 2ku_n'(t), \quad 0 \le t \le T$$

or

$$u_n''(t) + 2(h\lambda_n + k)u_n'(t) + \lambda_n u_n(t) = 0, \quad 0 \le t \le T.$$

Note also that $u_n(0) = u_n'(0) = 0$, $n = 1, 2, \dots$. Hence $u_n(t) = 0$ in $[0, T]$, $\forall$ $n = 1, 2, \dots$ and it follows that $u(t) = \sum_1^\infty u_n(t)e_n = \theta$ on $[0, T]$. □

3. In t his section we give a (hopefully) non-trivial example for the situation expressed in Th. 1.1. The Hilbert space considered H in $L^2[0, 1]$ (vector space of square - integrable complex - valued functions on $[0, 1]$). Next, in order to define operators A and B we first take:

$$D(A) = \{\varphi \in C^2[0, 1], \varphi'(0) = \varphi'(1) = 0\}$$

and on this set we put: $A\varphi = \dfrac{d^2\varphi}{dx^2}$. We shall see that A is symmetric and $(A\varphi, \varphi) \le 0$ $\forall$ $\varphi \in D(A)$.

In fact:

$$(A\varphi, \varphi) = \int_0^1 \varphi''(x)\overline{\varphi}(x)dx = \int_0^1 \frac{d}{dx}(\varphi'(x)\overline{\varphi}(x))dx - \int_0^1 \varphi'(x)\overline{\varphi}'(x)dx$$

$$= \varphi'(1)\overline{\varphi}(1) - \varphi'(0)\overline{\varphi}(0) - \int_0^1 |\varphi'(x)|^2dx = -\int_0^1 |\varphi'(x)|^2dx \le 0$$

(it is null for $\varphi = \text{const}$).

Also

$$(A\varphi, \psi) = \int_0^1 \varphi''(x)\overline{\psi}(x)dx = -\int_0^1 \varphi'(x)\overline{\psi}'(x)dx =$$

$$-\int_0^1 \frac{d}{dx}((\varphi(x)\overline{\psi}'(x))dx + \int_0^1 \varphi(x)\overline{\psi}''(x)dx = \int_0^1 \varphi(x)\overline{\psi}''(x)dx = (\varphi, A\psi)$$

for $\varphi, \psi \in D(A)$.

Next we shall define an operator B as a multiplication operator by a conveniently chosen function.

Precisely, we consider a real-valued measurable function $p(x)$ defined on $[0, 1]$, such that $p(x) > 0$ a.e. on $[0, 1]$. (For instance, one could take $p(x) = \frac{1}{x}$ for $0 \le x \le 1$ and x irrationnal, $p(x) = 0$ for x rational; this is a non-negative unbounded function.) Then, define $D(B) = \{\varphi \in L^2[0, 1],\ p.\varphi \in L^2[0, 1]\}$.

On this set we put $B\varphi = p.\varphi$. Now, B is symmetric, as p is real-valued. Furthermore we have $(B\varphi, \varphi) = \int_0^1 p(x)|\varphi(x)|^2dx \ge 0$; also $(B\varphi, \varphi) = 0 \Rightarrow p(x)|\varphi(x)|^2 = 0$ a.e : as $p(x) > 0$ a.e, hence $\varphi = \theta$ in $L^2[0, 1]$. Thus $(B\varphi, \varphi) > 0$ if $\varphi \in H$, $\varphi \ne \theta$. We now consider solutions of the partial differential equation

$$p(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \text{ for } 0 \le x \le 1,\ 0 \le t \le T. \tag{3.1}$$

We asume: $u(x, t) \in C^2([0, 1] \times [0, T])$; $u_x(0, t) = u_x(1, t) = 0$, $0 \le t \le T$

$$p(x)u_t \in L^2[0, 1] \text{ (with respect to } x). \tag{3.2}$$

We now associate, as usual, to such a solution $u(x, t)$, a function $\vec{u}(t)$ from $0 \leq t \leq T$ to $L^2[0, 1]$, in the following way:

$$\vec{u}(t)(x) = u(x, t).$$

That $\vec{u}(t) \in C^1([0, T]; L^2[0, 1])$ is seen as in [4] - (26.3). The partial derivative $u_t(x, t)$ defines $\vec{v}(t)(x) = u_t(x, t)$ which is the strong (L^2) derivative of $\vec{u}(t)$. Also, in the same way, we see that $u_{tt}(x, t)$ defines $\dot{w}(t) = (u_{tt}(t))(x)$ which is $L^2[0, 1]$-continuous and also is $\vec{u}''(t)$ (is strong sense). Thus relation $\vec{u}(t) \in C^2([0, T]; H)$ is proved.

Next, $\vec{u}(t) \in D(A)$ $\forall$ t : this means $u(x, t) \in D(A)$ $\forall$ t; but u is assumed in $C^2[0, 1]$ with respect to x, and we also assume $u_x(0, t) = u_x(1, t) = 0$, $t \in [0, T]$. Also $Au = u_{xx}(x, t) \in C([0, 1]\times[0, T])$ and this again implies $\int_0^1 |u_{xx}(x, t+h) - u_{xx}(x, t)|^2 dx \to 0$ as $h \to 0$, that is $Au \in C([0, T]; H)$. Next, condition $\int_0^1 p^2(x)|u_t(x, t)|^2 dx < +\infty$,means that $\vec{u}'(t) \in D(B)$, $0 \leq t \leq T$. Then $B\vec{u}'(t) = p(x)u_t(x, t)$; this is $L^2[0, 1]$-continuous for $t \in [0, T]$, as seen in the following lines:

Note first that

$$\int_0^1 |p(x)u_t(x, t+\delta) - p(x)u_t(x, t)|^2 dx = \int_0^1 |p(x)|^2 |u_t(x, t+\delta) - u_t(x, t)|^2 dx. \qquad (3.3)$$

The expression under integral sign converges, for almost all $x \in [0, 1]$ to 0, as $\delta \to 0$. We can write also

$$u_t(x, t+\delta) - u_t(x, t) = \int_t^{t+\delta} u_{tt}(x, \xi)d\xi,$$

which implies (if, for instance, $\delta > 0$)

$$|u_t(x, t+\delta) - u_t(x, t)| \leq \int_t^{t+\delta} |u_{tt}(x, \xi)|d\xi \leq (\int_t^{t+\delta} |u_{tt}(x, \xi)|^2 d\xi)^{1/2} \cdot \sqrt{\delta}$$

$$|u_t(x, t+\delta) - u_t(x, t)|^2 \leq \delta \int_t^{t+\delta} |u_{tt}(x, \xi)|^2 d\xi.$$

The expression under integral sign in (3.3) is therefore estimated by

$$|p(x)|^2 \delta \int_t^{t+\delta} |u_{tt}(x, \xi)|^2 d\xi = \delta \int_t^{t+\delta} |p(x)|^2 |u_{tt}(x, \xi)|^2 d\xi. \tag{3.4}$$

But from the equation (3.1) we find that $p(x)u_{tt}(x, t) = u_{xx}(x, t)$. Thus (3.4) is

$$\delta \int_t^{t+\delta} |u_{xx}(x, \xi)|^2 d\xi \le C^2\delta^2 \le C^2 \text{ if } \delta < 1$$

(which we can assume, and where $C = \sup_{[0, 1] \times [0, T]} |u_{xx}(x, \xi)| < \infty$.)

Thus we can apply Lebesgue's dominated convergence theorem and derive $L^2[0, 1]$ - continuity of the function $B\vec{u}'(t)$.

Finally, the condition $\vec{u}''(t) \in D(B)$ is easy to obtain: $\vec{u}''(t)$ is represented by $u_{tt}(x, t)$ and it belongs to $D(B)$ iff $p(x)u_{tt}(x, t) \in L^2[0, 1]$ which is true because $p(x)u_{tt}(x, t) = u_{xx}(x, t) \in C([0, 1] \times [0, T])$. Therefore, using Theorem 1.1 we get now that the solutions of (3.1) and (3.2) with initial data $u(x, 0) = 0$, $u_t(x, 0) = 0$, $0 \le x \le 1$ are null in $[0, 1] \times [0, T]$.

4. Now we shall first give two (different) examples for the situation covered by Theorem 1.2.

Take $\Omega \subset \mathbb{R}^n$ a bounded open set with regular boundary and then $H = L^2(\Omega)$. We shall take operator B the identity mapping in H. Then define operator A on

$$D(A) = \{\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega\} \tag{4.1}$$

by

$$A\varphi = -\Delta\varphi. \tag{4.2}$$

It is known that A is symmetric and $(A\varphi, \varphi)_{L^2(\Omega)} \ge 0$ $\forall\ \varphi \in D(A)$. Consider solutions of the partial differential equation

$$u_{tt}(x, t) + \Delta u(x, t) = 0 \text{ for } x \in \Omega,\ 0 \le t \le T. \tag{4.3}$$

We assume that

$$u(x, t) \in C^2(\overline{\Omega} \times [0, T]) \text{ and } u(x, t) = 0 \text{ for } x \in \partial\Omega,\ 0 \le t \le T. \tag{4.4}$$

The associated vector function $\vec{u}(t)$ belongs to $D(A)$ $\forall t \in [0, T]$. It also belongs to $C^2([0, T]; H)$ as easily seen (its strong derivatives are represented by $u_t(x, t)$ and $u_{tt}(x, t)$). The strong continuity of $A\vec{u}(t)$ follows because of equality $A\vec{u}(t) = \vec{u}''(t)$ which is obvious. Therefore, if we also assume that $u(x, 0) = u_t(x, 0) = 0$, $x \in \Omega$, we derive that $u(x, t) = 0$ in $\bar{\Omega} \times [0, T]$.

Next we give another application to Theorem 1.2 by a small change in the example (3.1). Precisely, we now consider solutions of the equation

$$p(x)u_{tt}(x, t) = -u_{xx}(x, t), \quad 0 \le x \le 1, \; 0 \le t \le T. \tag{4.5}$$

We assume also:

$$\left.\begin{array}{l} u(x, t) \in C^2([0, 1] \times [0, T]),\; u_x(0, t) = u_x(1, t) = 0,\; 0 \le t \le T \\ \\ p(x)u(x, t) \in L^2[0, 1] \text{ and } \sup\limits_{0 \le t \le T} p^2(x)|u(x, t)|^2 = g(x) \in L^1[0, 1] \\ \\ p(x)u_t(x, t) \text{ belongs to } L^2_x[0, 1]. \end{array}\right\} \tag{4.6}$$

Again consider $H = L^2([0, 1])$; $A = -\dfrac{d^2}{dx^2}$ defined on the same $D(A)$ as in Example of section 3. (A is symmetric and $A \ge 0$). Then $(B\varphi)(x) = p(x)\varphi(x)$ where $p(x)\varphi(x)$ where $p(x)$ is a real-valued measurable function on $[0, 1]$ such that $p(x) > 0$ a.e on $[0, 1]$, and $D(B) = \{\varphi \in H, p\varphi \in H\}$. We have seen that B is symmetric, and that $(B\varphi, \varphi) > 0$ if $\varphi \in H$, $\varphi \ne \theta$.

Next consider the vector-valued function $\vec{u}(t)$ represented by $u(x, t)$, $(0 \le t \le 1 \to L^2[0, 1])$. Again, we know that $\vec{u}(t) \in C^2[0, T]; H)$. Now $\vec{u}(t) \in D(A)$ is obvious from definition. Next $\vec{u}(t) \in D(B)$ means $p(x)u(x, t) \in L^2[0, 1]$ which is in (4.6); continuity of $A\vec{u}(t)$ which is represented by $-\dfrac{\partial^2 u(x, t)}{\partial x^2}$ is obvious again $(u \in C^2[0, 1] \times [0, T])$; continuity of $B\vec{u}(t)$ represented by $p(x)u(x, t)$ is easy:

$$\| B\vec{u}(t+h) - B\vec{u}(t)\|_H^2 = \int_0^1 |p(x)u(x, t+h)-p(x)u(x, t)|^2 dx.$$

For almost all x (where $p(x) < +\infty$), the expression under the integral sign $\to 0$ as $h \to 0$. Furthermore (for small h),

$$|p(x)u(x, t+h) - p(x)u(x, t)|^2 \leq 2|p(x)|^2|u(x, t+h)|^2 + 2|p(x)|^2|u(x, t)|^2$$

$$\leq 4g(x) \in L^1[0, 1] \quad (0 \leq t \leq T;\ h < 0 \text{ if } t = T).$$

Hence we can apply the dominated convergence theorem.

Now $\vec{u}'(t)$ is represented by $u_t(x, t)$; from (4.6) it follows that $\vec{u}'(t) \in D(B)$, $0 \leq t \leq T$. As for H-continuity of $B\vec{u}'(t)$, it reduces to $L^2[0, 1]$-continuity of $p(x)u_t(x, t)$, which was proved in section 3.

The relation $\vec{u}''(t) \in D(B)$ means $p(x)u_{tt}(x, t) \in L^2[0, 1]$ that is $-u_{xx}(x, t) \in L^2[0, 1]$ which is obvious. The equality $B\vec{u}''(t) = A\vec{u}(t)$ becomes $p(x)u_{tt}(x, t) = -u_{xx}(x, t)$ which is (4.5). Therefore, if $u(x, 0) = u_t(x, 0) = 0$, $0 \leq x \leq 1$ we obtain, applying Th. 1.2 that $u(x, t) = 0$ $\forall\, t \in [0, T]$ in L^2-sense: $\int_0^1 |u(x, t)|^2 dx = 0$. Using continuity of u we find $u(x, t) = 0$ $\forall\, x \in [0, 1]$, $t \in [0, T]$.

Finally we give a simple (natural) example for the situation in Th. 1.3. The Hilbert space H is again $L2[0, 1]$ over the complex numbers. On the set $D = \{\varphi \in C^2[0, 1],\ \varphi(0) = \varphi(1) = 0\}$ consider the operator A given by $A\varphi = i\dfrac{d^2\varphi}{dx^2}$, $\forall\, \varphi \in D$. (It is known to be a skew-symmetric operator). Then consider solutions $u(x, t) \in C^3([0, 1] \times [0, T])$ of the equation $\dfrac{\partial^2 u}{\partial t^2} = i\dfrac{\partial^2 u}{\partial x^2}$, $0 \leq x \leq 1$, $0 \leq t \leq T$, such that $u(0, t) = u(1, t) = 0$, $0 \leq t \leq T$.

The function $\vec{u}(t)$ associated to $u(x, t)$ belongs to $C^2([0, T]; H)$; also $\vec{u}(t) \in D(A)$, $0 \le t \le T$ is obvious. Next $\vec{u}'(t)$ represented by $u_t(x, t)$ belongs to $C^2[0, 1]$ as $u \in C^3$; the relations $u_t(0, t) = u_t(1, t) = 0$ follow from $u(0, t) = u(1, t) = 0$ $\forall t$ taking derivative with respect to t. The equality $\vec{u}''(t) = Au(t)$ reduces to $u_{tt}(x, t) = iu_{xx}(x, t)$. Thus, if we assume that $u(x, 0) = u_t(x, 0) = 0$, $0 \le x \le 1$ it follows that $u(x, t) = 0$, $0 \le t \le T$, $0 \le x \le 1$.

RÉSUMÉ

Dans ce travail on présente certains résultats concernant l'unicité dans le problème de Cauchy pour équations différentielles dans les espaces de Hilbert avec coefficients opérateurs non-bornés.

Les équations considérées ont la forme: $Bu''(t) = Au(t)$ ou $u''(t) = -A(u(t) + 2hu'(t)) - 2ku'(t)$, avec des hypothèses convenables sur les opérateurs A et B.

Des applications à l'unicité du problème mixte - Hadamard - pour certaines équations aux dérivées partielles apparaissent dans les sections finales du travail.

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