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ON THE BOUNDEDNESS OF SOLUTIONS AND OSCILLATIONS OF SOME CLASSES OF DIFFERENTIAL SYSTEMS

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Summary. We consider a class of nonlinear differential systems $\dot{x}(t) = f(t,x(t)) + p(t)$ and associate to it the linear system $\dot{x}(t) = A(t)x(t) + l$. We obtain some conditions for the linear system assuring the existence and the unicity of a bounded solution of the initial nonlinear system whatever a bounded function p would be. We study also the case of the periodic and almost periodic oscillations.

§1. Introduction

The existence of the bounded solutions for differential systems is a topic of interest for almost all the specialists in the qualitative theory of differential equations. It would not be so easy to record even the most important results. Even in 1930, O. Perron [21] raised the following problem: What conditions are to be imposed on the solutions of the homogeneous linear system $\dot{y}(t) = A(t)y(t)$ in order to assure that the solutions of the linear system $\dot{x}(t) = A(t)x(t) + p(t)$ be bounded whatever would be a bounded function p ? Many mathematicians were concerned by this problem from 1930 until today. All of them (except O. Perron himself, as well as J. L. Massera and J. J. Schäffer [13-16], J. J. Schäffer [23-27], T. F. Bridgeland Jr. [2] and W. A. Coppel [3]) made the hypothesis that all the solutions of the homogeneous system be bounded. Similarly, one may consider the following problem: What conditions must verify the homogeneous system in order that the non-homogeneous system had a unique bounded solution whatever the bounded function p would be? An answer is given in [9] as follows: If the trivial solution of the homogeneous system is uniformly asymptotically stable, then the initial linear system has a unique bounded solution whatever a bounded function p would be.

In the present paper, we obtain a more general result. If the matrix A is positive and the system $\dot{y}(t) = A(t)y(t) + 1$ has a positive solution, then the initial linear system has a unique bounded solution whatever the bounded function p would be. But even more, this result is generalized for nonlinear systems.

The problem of the oscillations of the differential systems is connected to the existence of bounded solutions. J. L. Massera [12] established that, in the case of scalar differential equations, the existence of a bounded solution in the future implies the existence of a harmonic vibration. But this result does not hold any more in the case of the differential systems. A generalization for the differential systems with two variables was obtained by N. Levinson [11]. A similar result for an arbitrary differential system is established by N. Pavel [20].

The existence of bounded solutions is not enough for the existence of an almost periodic solution, as proved Z. Opial [18] by means of an example. This example was used by A. M. Fink and P. O. Frederickson [7] for the construction of another example where the solutions are even uniformly bounded in the future, but no one is almost periodic.

B. P. Demidovich [6], N. Gheorghiu [8], Z. Opial [19] assume the existence of bounded solutions of a nonlinear scalar equation and obtain conditions assuring the existence of an almost periodic solution. A generalization of this results for nonlinear systems is to be found in [28], [29] and [30].

In the present paper, we give sufficient conditions for the periodicity or almost periodicity of the unique bounded solution whose existence is proved by our main result.

§2. Statement of the Main Result

We will use the symbol V^n for the n-dimensional vector space with the euclidean norm $|\cdot|$. Also $|\cdot|$ will be used to denote value absolute. By the symbol M^n we denote the space of n by n matrices with the operator norm $|m| = \sup_{|v|=1} |mv|$. P^n is the set of the continuous and bounded in the past functions, i.e. a continuous function $x \in P^n$ if and only if $\forall a \in R$, $\exists M_a > 0$, so that $|x(t)| \leq M_a$, for any $t < a$.

By $|\cdot|_t$ we denote $\sup_{t \leq s} |x(s)|$. A function $y: R \to V^n$ is called positive if its components $(y_1, \ldots, y_n)$ are positive for any $t \in R$. By the symbol 1 we denote either the unit matrix, or even the number 1. C^n is the space of continuous functions $f: R \times V^n \to V^n$, with the topology of the uniform convergence. B^n is the space of the continuous and bounded functions $f: R \times V^n \to V^n$.

Consider the function $f \in C^n$ and the differential problem:

$$(F_0) \begin{cases} \dot{x}(t) = f(t, x(t)) \\ x \in P^n \end{cases}$$

Work conditions permit us to prove the existence and the uniqueness of the solutions not only for the problem (F_0) but for a more general problem:

$$(F_p) \begin{cases} \dot{x}(t = f(t, x(t)) + p(t) \\ x \in P^n \end{cases}$$

where $p \in P^n$.

We suppose that the function f, with its first derivatives $\frac{\partial f_i}{\partial x_j}$ is continuous on the space $R \times V^n$, and that there is a continuous function g,

$g:R \to M^n$ such that we have the inequalities:

$$(I) \quad \begin{array}{l} \frac{\partial f_i}{\partial x_i}(t,x) \le g_{ii}(t), \quad i=1,\dots,n, \ \forall (t,x) \\ \left|\frac{\partial f_i}{\partial x_j}(t,x)\right| \le g_{ij}(t), \quad \forall\ i \ne j, \quad \forall (t,x) \end{array},$$

One observes that, excepting the principal diagonal, the elements of the matrix g are positive, i.e. the matrix g is positive (see [1]).

To the problem (F_p) we associate the following problem:

$$(G) \quad \begin{cases} \dot{y}(t) = g(t)y(t) + 1 \\ x \in P^n \end{cases}.$$

The main hypothesis of this paper is that the problem (G) has a positive solution.

These conditions are not enough for the existence of at least one solution of the problem (F_p); we prove this by the following example:

The differential problem:

$$\begin{cases} \dot{x}(t) = -x(t) + t + 1 \\ x \in P^1 \end{cases},$$

has no solution, though the associated problem:

$$\begin{cases} \dot{y}(t) = -y(t) + 1 \\ y \in P^1 \end{cases}$$

has a positive solution $y(t) \equiv 1$.

The main result of this paper is the following theorem:

<u>Theorem 2.1.</u> If for a function $p \in P^n$ the problem (F_p) has at least a solution, then for any $p \in P^n$ the problem (F_p) has only one solution. In addition, if $p_1 \in P^n$, $p_2 \in P^n$ and x_1, x_2 are the solutions of the problems (F_{p_1}) and (F_{p_2}) then we have the inequality:

(1) $$|x_1(t)-x_2(t)| < |y(t)|\,|p_1-p_2|_t \quad .$$

By y we denote the positive solution of the problem (G).

Remark 2.1. Sometimes it is very easy to find a function $p \in P^n$ so that the problem (F_p) has at least one solution.

For example, if there is a vector $v_0 \in P^n$ such that the function $f(t,v_0)$ is bounded in the past, then the problem:

$$\begin{cases} \dot{x}(t) = f(t,x(t)) - f(t,v_0) \\ x \in P^n \end{cases}$$

has a solution $x(t) = v_0$.

Obviously, if the function $f(t,x)$ is periodic in t, or almost periodic in t, uniformly with respect to $x \in K$, K is a compact set of V^n, then we have $f(t,v_0) \in P^n$ for any vector v_0, and the problem (F_p), $p = f(t,v_0)$ has a solution $x(t) = v_0$.

§3. Some Properties of the (G) Systems

Consider the differential systems:

$$\dot{x}(t) = a(t)x(t)$$

$$\dot{y}(t) = b(t)y(t),$$

$a,b \in M^n$, and let $A(t,s)$, $B(t,s)$ be their fundamental matrices of solutions, respectively. Then the following theorem is known [31]:

Theorem A. The necessary and sufficient condition for the inequalities

$$|A_{ij}(t,s)| \le B_{ij}(t,s), \quad \forall\ i,j \ ,\forall\ s \le t$$

is to have the inequalities

$$\begin{cases} |a_{ij}(t)| \le b_{ij}(t) & \forall i \ne j \\ a_{ii}(t) \le b_{ii}(t). \end{cases}$$

Let G be the matrix function $G:R\times R\to M^n$, which is the solution of the system:

$$\text{(2)}\qquad \begin{cases} \dot G(t,s) = g(t)G(t,s), \\ G(s,s) = I. \end{cases}$$

Remark 3.1. If G_{ij} are the entries of matrix G, then $G_{ij}(t,s)\geq 0$, for any $t\geq s$ and $\forall$ i, j.

Indeed, the inequalities

$$|g_{ij}(t)|\leq g_{ij}(t)\ ,\qquad \forall i\neq j$$
$$g_{ii}(t)\leq g_{ii}(t),$$

imply, according to the Theorem A, that $|G_{ij}(t,s)|\leq G_{ij}(t,s)$, $\forall\ t\geq s$.

Theorem 3.1. If the problem (G) has at least one positive solution, then the problem (G) has only one solution and its form is the following:

$$\text{(3)}\qquad y(t)=\int_{-\infty}^{t} G(t,s).1\ ds\ .$$

In addition, if we denote by

$$y^a t) = \int_a^t G(t,s).1\ ds$$

then we have $\lim_{a\to-\infty} y^a = y$ in the space C^n.

Proof. If the differential system of the problem (G) has a positive solution y, then, by the variation of constants formula we get:

$$\text{(4)}\qquad y(t) = G(t,a)y(a) + y^a(t)\ .$$

For $a\leq t$ we can write the inequalities

$$\int_a^t |G(t,s)|ds \leq \int_a^t \sum_i \sum_j G_{ij}(t,s)ds\ = \sum_i y_i^a(t) \leq \sum_i y_i(t)\ ,$$

Therefore, the function $t \to \int_{-\infty}^{t} |G(t,s)|ds$ is bounded in the past because the solution y is bounded in the past. The integral $\int_{-\infty}^{t} |G(t,s)|ds$ is convergent and then it follows $\lim_{s \to -\infty} \inf |G(t,s)| = 0$.

The last relation implies the uniqueness of the positive solution of the problem (G).

Indeed, if we suppose that the problem (G) has two positive solutions y and z, then we find:

$$
(5) \qquad \begin{cases} \dot{y}(t) - \dot{z}(t) = g(t)[y(t) - z(t)], \\ y - z \in P^n. \end{cases}
$$

The equalities (5) lead to the following form of the difference $y - z$:

$$
(6) \qquad y(t) - z(t) = G(t,s)\ [y(s) - z(s)]
$$

and it follows:

$$
|y(t) - z(t)| \le |G(t,s)|\,|y(s) - z(s)| \le |G(t,s)|\,|y - z|_0, \ \forall t \ge s, \ s \le 0.
$$

Passing to the limit in the above inequality, for $s \to -\infty$ we get $y(t) = z(t)$ $\forall t \in R$.

The function $t \to \int_{-\infty}^{t} G(t,s) \cdot 1\ ds$ is well-defined and by a direct verification it follows that this function is a solution of the differential system of the problem (G).

We have $|y(t) - y^a(t)| \le |G(t,a)|\,|y|_0$, $\forall a \le 0$, $\forall t \ge a$ and for any t fixed, $y^a(t)$ is increasingly convergent to $y(t)$; but the function y is continuous and it follows, according to Dini's theorem [10 p.314], that y^a is uniformly convergent on compacts.

The following theorems will establish some sufficient conditions for the existence of a positive solution of the problem (G).

Let h be the function

(7) $$h(t) = \max_j \sum_i g_{ij}(t),$$

and let us consider the following problem:

(8) $$\begin{cases} \dot z(t) = h(t)z(t) + 1, \\ z \in P^1. \end{cases}$$

Theorem 3.2. If the problem (8) has a positive solution, then the problem (G) has a positive solution.

Proof. Let z be the positive solution of the problem (8). According to Theorem 3.1, the function z has the following form:

(9) $$z(t) = \int_{-\infty}^{t} H(t,s)\, ds$$

where $H(t,s) = \exp \int_s^t h(u)du$.

The inequalities

$$|G(t,s)| \le \sum_j \sum_i G_{ij}(t,s) \le n.\max_j \sum_i g_{ij}(t,s) \le n.H(t,s) \forall s \le t,$$

imply that the function $t \to \int_{-\infty}^{t} |G(t,s)| ds$ is bounded in the past and then the function (3) is a positive solution of the problem (G).

Consider the homogeneous differential system associated to the differential system of the problem (G):

(10) $$\dot y(t) = g(t)y(t).$$

Theorem 3.4. If the solution $y = 0$ of system (10) is uniformly asymptotically stable, then the problem (G) has a positive solution.

Proof. Because (10) is a linear differential system and the solution $y = 0$ is uniformly asymptotically stable it follows that this solution is even exponentially asymptotically stable. Therefore, there are two positive numbers α,β such that:

$$|G(t,s)| \leq \beta e^{-\alpha(t-s)}\ ,\ \forall t \geq s.$$

Consider the integral:

$$\left|\int_{-\infty}^{t} G(t,s).1\ ds\right| \leq \int_{-\infty}^{t} |G(t,s)|ds \leq \beta/\alpha,$$

and function (3) is bounded even on the real line and it is a positive solution of problem (G).

§4. The Existence of the Solutions of Linear Differential Systems

Consider the continuous function $K:\mathbb{R}\to M^n$ and suppose that the inequalities

$$\begin{cases} k_{ii}(t) \leq g_{ii}(t)\ , & \forall t \in \mathbb{R},\ \forall i, \\ |k_{ij}(t)| \leq g_{ij}(t), & \forall i \neq j,\ \forall t \in \mathbb{R}. \end{cases}$$

hold.

For any function $p \in P^n$ we consider the problem:

$$\text{(11)} \qquad \begin{cases} \dot{x}(t) = K(t)\ x(t) + p(t), \\ x \in P^n. \end{cases}$$

By $K(t,s)$ we denote the fundamental matrix of solutions of differential system of problem (11).

According to the Theorem A it follows:

$$|K_{ij}(t,s)| \leq G_{ij}(t,s),\ \forall s \leq t,\ \forall i,j,$$

and therefore, the inequality $|K(t,s)| \leq |G(t,s)|,\ \forall s \leq t.$

Theorem 4.1. If the problem (G) has a positive solution, then for any $p \in P^n$ the problem (11) has a unique solution which is of the following form:

$$\text{(12)} \qquad x(t) = \int_{-\infty}^{t} K(t,s)\ p(s)\ ds.$$

In addition, if we denote by

$$x^a(t)=\int_a^t K(t,s)\,p(s)\,ds$$

then we get the inequality:

(13)
$$|x(t)-x^a(t)|\le|y(t)-y^a(t)|\,|p|_a,\ \forall t\ge a.$$

If $a=t$ one obtains:

(14)
$$|x(t)|\le|y(t)|\,|p|_t\ .$$

Proof. From the inequality $|K(t,s)|\le|G(t,s)|$, $\forall s\le t$, we deduce the inequalities:

$$\int_{-\infty}^t|K(t,s)|ds\le\int_{-\infty}^t|G(t,s)|ds\le\sum_i y_i(t),$$

and therefore the function (12) is bounded in the past and it is a solution of the differential system of problem (11).

The uniqueness of solution follows by a similar technique as in the proof of the Theorem 3.1.

For any $t\ge a$ we have

$$|x_i(t)-x_i^a(t)|\le\left|\int_{-\infty}^a\sum_j K_{ij}(t,s)p_j(s)\,ds\right|\le$$

$$\le\left[\int_{-\infty}^a\sum_j G_{ij}(t,s)ds\right]\cdot|p|_a=\left[y_i(t)-y_i^a(t)\right]|p|_a$$

and the theorem is proved.

Remark 4.1. The inequalities (13) yield

(15)
$$\lim_{a\to-\infty}x^a(t)=x(t)$$

in the space C^n.

Indeed, with respect to the components, we have

$$|x_i(t)-x_i^a(t)|\le\left[y_i(t)-y_i^a(t)\right]\cdot|p|_0$$

$\forall a\le 0$, $\forall t\ge a$ and because the sequence y_i^a is convergent in the spa

3.1 follows.

Remark 4.2. If problem (G) has a solution which is not positive, then the problem (G) has no positive solution.

Remark 4.3. If the matrix $K(t)=K$ is a constant matrix, $K_{ij}\geq 0$, i,j, $i\neq j$ and $\operatorname{Re}\lambda_i<\alpha<0$, $p(t)>0$, then the problem (11) has a positive solution (λ_i are the characteristic roots of K).

Indeed, let us consider the associated problem:

$$\begin{cases} \dot{y}(t)=Ky(t)+1, \\ y\in P^n, \end{cases}$$

which has a positive solution because the homogeneous system $\dot{y}(t)=Ky(t)$ has the solution $y=0$ uniformly asymptotically stable (according to Theorem 3.4.).

Therefore the solution of the problem (11) has the form (12) and it is easy to see that it is a positive solution. Thus, we obtain for weaker conditions a result due to N. M. Medvedev [17], who supposed that $K_{ij}\geq 0$, $\forall i,j$.

§5. The Existence of Solutions of a Class of Nonlinear Differential Systems

Consider now the continuous function $K:R\times V^n\to M^n$ such that the inequalities:

$$\begin{cases} K_{ii}(t,v)\leq g_{ii}(t), & \forall i,\ \forall (t,v) \\ |K_{ij}(t,v)|\leq g_{ij}(t), & \forall i\neq j,\ \forall (t,v) \end{cases}$$

hold.

For each function $p\in P^n$ we consider the problem:

$$\text{(16)} \qquad \begin{cases} \dot{x}(t)=K(t,x(t))\,x(t)+p(t) \\ x\in P^n \end{cases}$$

For the problem (16) the following theorem holds:

Theorem 5.1. If the problem (G) has a positive solution y, then the problem (16) for any $p\in P^n$ has at least a solution. Any solution of problem (16) satisfies the inequality:

(17) $$|x(t)| \le |y(t)| \cdot |p|_t \ .$$

<u>Proof</u>. We can write the differential system of the problem (16) under the form of an operatorial equation.

For each $x \in C^n$ the problem:

(18) $$\begin{cases} (\dot{T}x)(t) = K(t,x(t))(Tx)(t) + p(t), \\ Tx \in \rho^n, \end{cases}$$

has a solution and this solution is unique (see Theorem 4.1). From (14) we deduce the inequalities:

$$|(Tx)(t)| \le |y(t)| \cdot |p|_t \quad , \ \forall x \in C^n \ .$$

Let us consider the set $M = \{x \in C^n / |x(t)| \le |y(t)| \cdot |p|_t, \ \forall t \in R\}$. In this way, we obtained an operator $T: C^n \to M$ given by (18), such that the image of each $x \in C^n$ is a unique function $Tx \in M$. The differential system (16) is equivalent to the operator equation $Tx = x$. One may observe that the set M is closed and convex. Any element of the set M is bounded in the past. We have in view to apply the Schauder-Tychonoff's theorem for the restriction of the operator T to the subset $M \subset C^n$. In order to prove that the operator T is continuous, let us denote by $T^a: C^n \to C^n$ the operators of the form:

(19) $$(T^a x)(t) = \int_a^t K(t,s;x)p(s) \ ds$$

where $K: \bar{R} \times R \times C^n \to M^n$ is a solution of the system:

(20) $$\begin{cases} \dot{K}(t,s;x) = K(t,x(t))K(t,s;x), \\ K(s,s;x) = 1. \end{cases}$$

We shall prove that the function K is continuous in x, therefore the operators T^a will be continuous.

Let $x_n(t)$ be a sequence convergent to the function x in the space C^n and consider the systems:

$$\begin{cases} \dot{K}(t,s;x_n) = K(t,x_n(t))K(t,s;x_n), \\ K(s,s;x_n) = I. \end{cases}$$

For each fixed j, we denote by $z(t,s;x_n)$ and $z(t,s;x)$ the columns of index j of matrices $K(t,s;x_n)$, respectively; then we have the following differential systems:

(21) $$\dot{z}(t,s;x_n) = K(t,x_n(t))\ z(t,s;x_n),$$

(22) $$\dot{z}(t,s;x) = K(t,x(t))\ z(t,s;x).$$

From (21) and (22) one obtains:

(23) $$\dot{z}(t,s;x_n) - \dot{z}(t,s;x) = K(t,x_n)[z(t,s;x_n) - z(t,s;x)] + \\ +[K(t,x_n) - K(t,x)]z(t,s;x).$$

The equalities (23) imply the following forms for the differences

(24) $$z(t,s;x_n) - \dot{z}(t,s;x) = \int_s^t K(t;u;x_n)[K(u,x_n) - K(u,x)]z(u,s;x)du,$$

because $z(s,s;x_n) = z(s,s;x)$.

The inequalities $|K(t,s;x_n)| \leq |G(t,s)|$, $\forall n \in N$, $\forall t \geq s$ and (24) implies the inequalities:

(25) $$|z(t,s;x_n) - z(t,s;x)| \leq |y(t)| \cdot |z(\cdot,s;x)|_t \sup_{\mu \leq t} |K(u,x_n) - K(u,x)|.$$

Because the sequence $\{x_n\}$ is convergent to x in C^n and $K(t,x)$ is a continuous function, from (25) it follows that the sequence $z(t,s;x_n)$ is convergent to $z(t,s;x)$, in C^n and this fact implies the continuity of the fundamental matrix $K(t,s;x)$ with respect to x.

The continuity of the matrix K implies the continuity of the operator T^a. According to Theorem 4.1. one obtains the following inequalities:

(26) $$|(Tx)(t) - (T^a x)(t)| \leq |y(t) - y^a(t)| \cdot |p|_a$$

$\forall t \geq a$ and because $\lim_{a \to -\infty} y^a(t) = y(t)$ in C^n, we find that $\lim_{a \to -\infty} T^a x = Tx$ in C^n, uniformly wit respect to $x \in C^n$.

Therefore, the operator T is continuous on C^n. The subset M is bounded in C^n and then the subset $\{(Tx)(t);\ x\in M\}$ is bounded in C^n. From (18) we have that the subset $\{(\dot{T}x)(t);\ x\in M\}$ is bounded in C^n.

We conclude that, for each $t\in R$, the set $\{(Tx)(t);\ x\in M\}$ is a relatively compact set of V^n and the set $\{Tx,\ x\in M\}$ is equicontinuous at the point t. According to Ascoli's theorem [10, p.307] it follows that the subset TM is relatively compact subset of C^n and therefore Schauder-Tychanoff's theorem holds. There exists a fixed point x, solution of the problem (16). If x is a solution of the problem (16), then $x\in M$ and we have the inequalities (17).

§6. Proof of the Theorem 2.1

We denote by p_0 the function of P^n for which the problem (F_{p_0}) has at least a solution x_0. Consider an arbitrary function $p\in P^n$. The problem (F_p) is equivalent to the problem:

$$(27)\qquad \begin{cases} \dot{x}(t)-\dot{x}_0(t)=K(t,x(t)-x_0))[x(t)-x_0(t)]+p(t)-p_0(t), \\ x-x_0\in P^n, \end{cases}$$

where the function K is given by

$$K(t,v)=\int_0^1\frac{\partial f}{\partial v}(t,x_0+sv)ds$$

obtained by the application of Hadamard's lemma [22, p.186] to the difference $f(t,x(t))-f(t,x_0(t))$.

The function K is continuous and because we have (1) one obtains:

$$\begin{cases} K_{ii}(t,v)\le g_{ii}(t)\ , & \forall i \\ |K_{ij}(t,v)|\le g_{ij}(t), & \forall i\ne j,\ \forall(t,v). \end{cases}$$

According to Theorem 5.1 it follows that the problem (27) has at least a solution.

Next, we shall prove the inequality (1) and by this the unicity of the solution of problem (F_p).

Let p_1 and p_2 be some functions of P^n and (F_{p_1}), (F_{p_2}) the associated problems, with their solutions x_1 and x_2, respectively. The equality $\dot{x}_2(t)-\dot{x}_1(t)=f(t,x_2(t))-f(t,x_1(t))+p_2(t)-p_1(t)$ implies

(28) $$\dot{x}_2(t)-\dot{x}_1(t)=K(t)[x_2(t)-x_1(t)]+p_2(t)-p_1(t)$$

where K is the function given by the application of the Hadamard's lemma to the difference $f(t,x_2(t))-f(t,x_1(t))$.

The function K is continuous and satisfies the inequalities (1). According to Theorem 4.1. we obtain the inequality:

$$|x_2(t)-x_1(t)|\leq|y(t)|\cdot|p_2-p_1|_t \quad .$$

If the problem (F_p) has two solutions x_1 and x_2, then the above inequality yields $x_1(t)=x_2(t)$, $\forall t\in R$. Therefore, for each function $p\in P^n$, the problem (F_p) has only one solution.

Remark 6.1. If we add to the hypotheses of Theorem 2.1 the following hypotheses:

1. the function $p\in B^n$, i.e. p is bounded on the real line R,
2. the problem (G) has a positive solution which is bounded on the whole real line R,

then the solution of the problem (F_p) is bounded on the real line R.

§7. Periodic and Almost Periodic Oscillations of the Problem (F_p)

Let us consider the function $f(t,x)$, ω-periodic in t. We have the following result:

Theorem 7.1. If the function f and p are ω periodic in t, then the solution of the problem (F_p) is ω-periodic.

Proof. The problem (F_p) with $p(t)=-f(t,o)$ has the solution $x(t)=0$ and by Theorem 2.1, it follows that the problem (F_p) has only one solution for each $p \in P^n$.

Let x be the solution of the problem (F_p) and let us denote $z(t) = x(t+\omega)$. Then it follows:

$$\begin{cases} \dot{z}(t) = \dot{x}(t+\omega) = f(t+\omega, x(t+\omega)) + p(t+\omega) = f(t, z(t)) + p(t) \\ z \in P^n \end{cases}$$

i.e., z is a solution of the problem (F_p). Because (F_p) has only one solution, we get that $x(t) = x(t+\omega)$, $\forall t \in \mathbb{R}$.

Theorem 7.2. If the function $f(t,x)$ is an almost periodic function in t, uniformly with respect to $x \in K \subset V^n$, where K is any compact set, and the solution y of the problem (G) is bounded on the real line, then the problem (F_0) has only one almost periodic solution.

Proof. According to Remark 2.1 and to Theorem 2.1 the problem (F_0) has only one solution x and in addition the inequality

$$|x(t)| \le |y(t)| \cdot |f(\cdot,0)|_t$$

holds.

Let $a \in \mathbb{R}$ be, for the moment, arbitrary and denote by $z(t) = x(t+a) = x(t)$; then we get:

$$\dot{z}(t) = K(t)z(t) + p(t)$$

where K is obtained as usual and by $p(t)$ we denote the difference:

$$f(t+a, x(t+a)) - f(t, x(t+a)).$$

Applying the Theorem 4.1. we find the inequality $|z(t)| \le |y(t)| \cdot |p|_t$. Let S be the sphere of V^n with the radius $r = \sup_{t \in \mathbb{R}} \{|y(t)| \cdot |f(\cdot,0|_t\}$ and $Y = \sup_{t \in \mathbb{R}} y(t)$. Now, let "a" be an ε/Y-almost period of the function f corresponding to the compact S. Then, from the above inequality we deduce the following inequalaity: $|z(t)| \le \varepsilon$, $\forall t \in \mathbb{R}$, i.e. the solution x is almost periodic.

§8. The Continuability and the Stability of the Solutions

In the hypotheses of Theorem 2.1, we shall prove that any solution of the differential equation:

$$(F') \qquad \dot{x}(t) = f(t, x(t))$$

can be continued in the future.

The following example shows that the solution of the differential equation (F′) can not be always continued in the future. The equation $\dot{x}(t) = -(x^3 + x)$ has the solutions of the form:

$$x(t) = x_0/\sqrt{(1+x_0^2)e^{t-t_0} - x_0^2}$$

and we observe that only the solution $x(t)=0$ is defined on the entire real line. This is the unique solution of the problem

$$\begin{cases} \dot{x}(t) = -(x^3+x), \\ x \in P^1, \end{cases}$$

because the associated system

$$\begin{cases} \dot{y}(t) = -y(t) + 1 \\ y \in P^1 \end{cases}$$

has a positive solution $y(t) = 1$.

Consider for each $p \in P^n$ the vector equation:

$$(F'_p) \qquad \dot{x}(t) = f(t, x(t)) + p(t) \ .$$

Theorem 8.1. If the hypotheses of the Theorem 2.1. are fulfilled, for each $p \in P^n$, any solution of the equation (F'_p) can be continued in the future.

Proof. Let x_p be the solution of the problem (F_p) and x a certain solution of the equation (F'_p). We have the equality:

$$\dot{x}(t) - \dot{x}_p(t) = f(t, x(t)) - f(t, x_p(t))$$

and it follows:

$$\dot{x}(t) - \dot{x}_p(t) = K(t)(x(t) - x_p(t)) \ .$$

We denote by $K(t,s)$ the fundamental matrix of solutions of this differential system and we get: $x(t)-x_p(t)=K(t,s)(x(s)-x_p(s))$. Because $|K(t,s)|\le|G(t,s)|$ it follows

(29) $$|x(t)-x_p(t)|\le|G(t,s)|\,|x(s)-x_p(s)|.$$

The inequality (29) shows us that the solution x is bounded on any finite interval of $[0,\infty)$, therefore it can be continued in the future.

Theorem 8.2. We suppose that the solution $y=0$ of the differential system (10) is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable), then the solution x_p of the problem (F_p), for the differential system of this problem, is stable (uniformly stable, asymptotically stable, uniformly asymptotically stable, respectively).

Proof. It is known that if $G(t,s)$ is the fundamental matrix of solutions of the homogeneous system (10), then there is a matrix $X:R\to M^n$ so that $G(t,s)=X(t)\cdot X^{-1}(s)$. We suppose that the solution $y=0$ is stable, i.e. $|X(t)|\le M$, $\forall t\in R$, $M>0$. According to (29) we obtain

(30) $$|x(t)-x_p(t)|\le M|X^{-1}(s)|\cdot|x(s)-x_p(s)|$$

where x is an arbitrary solution of the system (F_p'). We choose $\delta(\epsilon,s)==(M|X^{-1}(s)|)^{-1}$ and then it follows $|x(t)-x_p(t)|\le\epsilon$ for $|x(s)-x_p(s)|\le\delta(\epsilon,s)$, therefore the solution x_p is stable.

If the solution $y=0$ is uniformaly stable, i.e. $|G(t,s)|\le M$, $\forall t\ge s$, $M>0$, then from (30) we deduce the inequality:

$$|x(t)-x_p(t)|\le M|x(s)-x_p(s)|\le\epsilon$$

for $|x(s)-x_p(s)|<\epsilon/M=\delta(\epsilon)$, i.e. the solution x_p is uniformly stable.

If the solution $y=0$ is asymptotically stable we have simple stability and in addition $\lim\limits_{t\to\infty}X(t)=0$, therefore the solution x_p is stable and the equality (30) implies $\lim\limits_{t\to\infty}|x(t)-x_p(t)|=0$.

If the solution $y=0$ is uniformly asymptotically stable it follows that this solution is even exponentially stable, therefore there are two constants $\alpha > 0, \beta > 0$ such that

$$|G(t,s)| \leq \beta e^{-\alpha(t-s)}, \quad \forall t \geq s$$

and the inequality (30) yields:

$$|x(t) - x_p(t)| \leq \beta e^{-\alpha(t-s)} |x(s) - x_p(s)|$$

i.e; the solution x_p is uniformly asymptotically stable.

Commentary.

Finally, we make a comparison between the Theorem 2.1 and a similar theorem due to C. Corduneanu [4-5].

Let $a: R \to R$ be a function. We say that the function "a" satisfies the condition (A) if:

$$\exp \int_0^t a(u)du \quad \text{and} \quad \int_0^t \left[\exp \int_s^t a(u)du \right] ds$$

are bounded on the semi-axis $[0,\infty)$. We say that the function "a" satisfies the condition (B) if:

$\exp \int_0^t a(u)du$ is not bounded, but $\int_0^t \left[\exp \int_s^t a(u)du \right] ds$ is bounded on the semiaxis $[0,\infty)$.

Suppose that the differential system:

$$\dot{x}(t) = f(t,x(t)) \tag{31}$$

verifies the following conditions:

1. f_i are continuous functions on the set $(\bar{\Delta}): t \in R_+, |x_i| \leq M, M > 0$.
2. the derivatives $\frac{\partial f_i}{\partial x_i}$ are continuous functions on the set $(\bar{\Delta})$ and in addition we suppose that there are continuous functions a_i such that:

   $\frac{\partial f_i}{\partial x_i}(t,x) \leq a_i(t)$ if a_i has the property (A),

   $\frac{\partial f_i}{\partial x_i}(t,x) \geq a_i(t)$ if a_i has the property (B),

3. $|f_i(t, x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)| \leq r$ on the set $(\bar{\Delta})$.

Corduneanu's theorem. In the hypotheses 1, 2, 3, the differential system (31) with the initial conditions $x_i(0) = x_i^0$ $i = i_A$, has at least a solution on $[0,\infty)$ if x_i^0 for $i = i_A$ are small enough.

This theorem and the Theorem 2.1 are independent as it is possible to see from the following examples:

Let us consider the differential problem:

$$
(32) \qquad \begin{cases} \dot{x}_1(t) = (1-2e^{-t})x_1(t) + e^{-t}x_2(t) + 1 \\ \dot{x}_2(t) = e^{-t}x_1(t) + (1-2e^{-t})x_1(t)x_2(t) + 1 \\ x_1 \in P^1, \ x_2 \in P^1 \end{cases}
$$

and as associated problem even the problem (32) itself. The problem (32) has a positive solution $x_1(t) = x_2(t) = e^t \in P^1$. Therefore the Theorem 2.1 holds. On the other hand we have the equalities $f_1(t,0,x_2) = e^{-t}x_2 + 1$, $f_2(t,x_1,0) = e^{-t}x_1 + 1$ and therefore Corduneanu's theorem is not applicable.

Consider the differential system:

$$
(33) \qquad \begin{cases} \dot{x}_1(t) = -x_1(t) + 2x_2(t) + 1 \\ \dot{x}_2(t) = 2x_1(t) - x_2(t) + 1 \\ x_1 \in P^1, \ x_2 \in P^1 \end{cases}
$$

and as associated problem even the problem (32) itself. The differential system (33) has the solution $x_1(t) = x_2(t) = -1$ and according to Remark 4.2. the problem (G) has not a positive solution, therefore Theorem 2.1. is not applicable. If we take $a_i = -1$, then Corduneanu's theorem holds if the initial conditions are chosen.

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