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UNIQUENESS OF BOUNDED WEAK SOLUTIONS TO THE EQUATION $u'(t)=a(t)\ A\ u(t)$ IN HILBERT SPACES*

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Abstract.

In this work we consider ultra-weak solutions on the whole real line of the equation (with vector-valued functions) $u'(t)=a(t)\ A\ u(t)$, under a few assumptions for the scalar function $a(t)$ and the hermitian operator A in the Hilbert space H. It is proved that solutions which are norm-bounded over the real line vanish identically.

Introduction.

The question of uniqueness of bounded over R solutions (in strong or ultra-weak sense) for abstract differential equations was investigated in our papers [5], [6], [7] as well as in the monograph [8].

Further study and extensions were done in work by Levine [4], Goldstein-Lubin [3], Brézis-Goldstein [1]. We note that in [6], [7] only linear and time-independent equations were considered, while in [4], [3], [1] the time-dependent and the nonlinear case were successfully studied.

In the present article we consider a simple time-dependent equation, as appears in the title above; under (seemingly) new assumptions with respect to previously quoted work, we again establish the uniqueness, that is, here the vanishing of all the bounded solutions over the real line. The "nearest" (formally) result appears to be the Theorem 6 in [1].

Let H be a Hilbert space over the complex field, and A be a linear hermitian operator (usually unbounded), and with dense domain $D(A)\subset H$. We assume also the existence of a complete sequence $(e_k)_1^\infty$ of orthonormal eigen-vectors to A, corresponding to real eigen-values λ_k which belong to $R/\{0\}$ for all $k\in N$.

Next, let $a(t):R\rightarrow C$ (the complex field), be a continuous almost-periodic function such that

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$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \text{Re } a(s)ds \in R/\{0\} \tag{1.1}$$

Let also A^* be the adjoint operator to A (so that $A\subset A^*$) and the class $K_{A^*}(R)$ of vector-valued test-functions be defined as usual (see for instance [8]). We are now ready for the statement (and demonstration) of the following.

Theorem. Under the above assumptions, consider a strongly continuous function $u(t)$: $R\to H$ which is norm-bounded over R and verifies the integral identity

$$\int_R (u(t), \varphi'(t)+\bar{a}(t)A^*\varphi(t))_H \, dt=0, \quad \forall\varphi\in K_{A^*}(R). \tag{1.2}$$

Then $u(t)=\theta$, $\forall t\in R$.

Proof. We shall take in (1.2) functions $\varphi(t)$ of the special form:

$$\varphi(t)=f(t)\, e_k$$

where $f(t)$ is a (real) scalar-valued function in $C_0^1(R)$, while e_k is an eigen-vector of A. It follows accordingly that

$$\int_R (u(t), f'(t)e_k+\bar{a}(t)f(t)A^*e_k)_H \, dt=0, \quad \forall f\in C_0^1(R),$$

that is

$$\int_R \{f(t), (u(t),e_k)_H+ f(t)\,(a(t)u(t), \lambda_k e_k)_H\}\, dt=0. \tag{1.3}$$

If we use the notation: $u_k(t)=(u(t), e_k)_H$ it follows that

$$\int_R \{f'(t)u_k(t)+ \lambda_k f(t)a(t)u_k(t)\}\, dt=0$$

and accordingly

$$(d/dt)u_k(t)=\lambda_k a(t)u_k(t) \qquad \text{in the sense of } \mathcal{D}'(R) \tag{1.4}$$

From the continuity of both $u_k(t)$ and $a(t)$ we find that the relation (1.4) holds true also in the usual sense. Note that the function $u_k(t)$ is also bounded over R (for any k) and that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \text{Re}(\lambda_k\, a(t)dt \neq 0$$

holds true.

As well-known (see for instance Fink [2], p.102-103) there is uniqueness of bounded solutions for the equation (1.4); accordingly, all the functions u_k reduce to the null-function. Finally, from the completeness of the sequence $(e_k)_1^\infty$ it will follow that

$$||u(t)||^2 = \sum_{k=1}^{\infty} |u_k(t)|^2 = 0,$$

for all $t \in R$.

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