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SOME VARIATIONAL EIGENVALUE PRINCIPLES

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In this note we describe two variational principles for eigenvalue problems associated with the generalized subdifferential of a locally Lipschitz functional. The results can be regarded as extensions of some contributions of G. Auchmuty [2] and J.P. Dias [4].

Let H be a real separable Hilbert space with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $\Phi : H \rightarrowtail \mathbb{R} \cup \{+\infty\}$ be a proper functional, i.e. its effective domain $D(\Phi) = \{ v \in H \mid \Phi(v) \in \mathbb{R} \}$ is nonempty. The generalized directional derivative of Φ at $x \in D(\Phi)$ in the direction $z \in H$ is defined by

$$\Phi^{\circ}(x;z) = \limsup_{\substack{x \to y \\ t \downarrow o}} \frac{\Phi(y + tx) - \Phi(x)}{t} .$$

We suppose henceforth that Φ is locally Lipschitz. It is readily seen that $\Phi^{\circ}(x;z) < \infty$ for all $x \in D(\Phi)$ while the map $z \rightarrowtail \Phi^{\circ}(x;z)$ is subadditive positively homogeneous and $\Phi^{\circ}(x;z) \leq C\|z\|$ where $C > 0$ depends only on z. Then, by the Hahn-Banach theorem, there exists at least one element $h \in H$ such that $\Phi^{\circ}(x;z) \geq (h,z)$ for all $z \in H$. The generalized subdifferential is the set-valued mapping $\partial\Phi : H \rightarrowtail H$ given by

$$\partial\Phi(x) = \{ h \in H \mid \Phi^{\circ}(x;z) \geq (h,z) \ \ \forall z \in H \}.$$

It follows that $\partial\Phi(x)$ is a nonempty bounded closed convex set in H while the graph $\{(x,z) \in D(J) \times H\}$ is strongly × weakly compact. A point $u \in D(\Phi)$ is stationary if $0 \in \partial\Phi(u)$.

Later on, let $\nabla Gu = G'u$ be the Fréchet differential of a function $G \in C^1(H,\mathbb{R})$. For any $r \in \mathbb{R}$ consider the level surface

$$\delta G_r = \{ v \in H \mid G(v) = r \}$$

and assume that δG_r is regular, i.e. $(G'v,v) \neq 0$ for any $v \in \delta G_r$. The tangent space to δG_r at u coincides with $N(G'u) = \ker G'u$ and so

$$T_u = T_u(\delta G_r) = \{ v \in H \mid (G'u, v) = 0 \}.$$

We say that $u \in \delta G_r$ is a critical point of Φ restricted to δG_r if there is a subgradient $h \in \partial\Phi(u)$ such that $(h, v) = 0$ for all $v \in T_u$. It is easily shown that any local extremum of $J|_{\delta G_r}$ is a critical point [5]. A multivalued variant of the Lagrange-multiplier rule shows that critical points of Φ on δG_r are solutions of the variational inclusion

(1) $$\partial\Phi(u) + \lambda G'u \ni 0, \quad G(u) = r,$$

for some real number λ. More precisely, $h + \lambda G'u = 0$ with $\lambda = -(h, w)$ where $w = \frac{u}{(G'u, u)}$. Moreover, equation (1) can be viewed as a nonlinear variational eigenvalue inequality. For the investigation of critical points of Φ on δG_r, we established a general form of the deformation lemma for set-valued mappings in [5].

To study thoroughly the equation (1) we assume further that Φ is convex. In this case, the subdifferential has the habitual form

$$\partial\Phi(u) = \{ h \in H \mid \Phi(u + v) \geq \Phi(u) + (h, v) \quad \forall\, v \in H \}$$

and we consider $J_\alpha = (I + \alpha\, \partial\Phi)^{-1}$ the resolvent of $\partial\Phi$, where $\alpha > 0$ and I denotes the identity map. Then the inclusion (1) is convertible into

$$u = (I + \partial\Phi)^{-1}(u - \lambda G'u) = J_1(u - \lambda G'u).$$

Recall that the Moreau-Yosida approximante

$$\Phi_\alpha(u) = \min_{v \in H} \left\{ \frac{1}{2\alpha} \|v - u\|^2 + \Phi(v) \right\}$$

attains its minimum for $v = J_\alpha u$ and we actually have

$$\Phi_\alpha(u) = \frac{\alpha}{2} \|A_\alpha u\|^2 + \Phi(J_\alpha u) \quad \text{with} \quad A_\alpha = \frac{1}{\alpha}(I - J_\alpha),$$

where $A_\alpha : H \rightarrowtail H$ is the so-called Yosida approximant of $\partial\Phi$. Besides, Φ_α is Fréchet differentiable and $\nabla\Phi_\alpha(u) = A_\alpha u$. As usual, let

$\Phi^* : H \rightarrowtail \mathbb{R}$ be the conjugate function of Φ defined by

$$\Phi^*(v) = \sup_{u \in H} \{ (v, u) - \phi(u) \}.$$

The approximante of the conjugate function and its derivative obey the following rules [1]

$$\Phi^*_\alpha(\alpha v) = \frac{\alpha}{2} \|v\|^2 - \Phi_{\alpha^{-1}}(v) \quad \text{and} \quad \nabla\Phi^*_\alpha(\alpha v) = J_{\alpha^{-1}} v.$$

In particular, for $\alpha = 1$ we have

$$\Phi_1^*(v) = \frac{1}{2}\|v\|^2 - \Phi_1(v) = \Lambda(v)$$

or

$$\Lambda(v) = \frac{1}{2}\left[\|v\|^2 - \|v - J_1 v\|^2\right] - \Phi(J_1 v) = (v, J_1 v) - \frac{1}{2}\|J_1 v\|^2 - \Phi(J_1 v)$$

and $\Lambda'(v) = J_1 v$. For sake of simplicity, let us denote

$$Tu = u - \lambda G'u \quad \text{and} \quad S = J_1 T - I.$$

Define now the functional $\Gamma : \mathbb{R}_+ \times \delta G_r \rightarrowtail \mathbb{R} \cup \{+\infty\}$ by

$$\Gamma(\lambda, u) = \Lambda \circ Tu - \frac{1}{2}(Tu, u)$$

and prove

THEOREM 1. A pair $(\lambda, u) \in \mathbb{R}_+ \times \delta G_r$ such that $\ker(I - \lambda G'u) = \{0\}$ is an eigensolution of (1) if and only if (λ, u) is a stationary point of Γ.

Proof. First, we notice that $u \in \delta G_r$ is an eigenfuction of (1) if and only if $u \in \operatorname{Ker} S$. As T is injective by hypothesis, we have

$$\ker S = \ker(T \circ S).$$

Since Λ and T are differentiable, we have $\nabla\Lambda(u) = J_1 u$ and $\nabla Tu = Tu$ for all $u \in H$, and thereby the chain rule yields

$$\nabla\Gamma(u) = TJ_1 Tu - Tu = TSu. \ ■$$

We consider now the unconstrained minimum problem

$$m = \inf_{u \in H} J(u)$$

where the functional $J : H \rightarrowtail \mathbb{R} \cup \{+\infty\}$ has the form

$$J(u) = \Phi(u) + \Phi^*(-\lambda G'u) + \lambda(G'u, u).$$

An element $u_0 \in H$ such that $J(u_0) = m$ is a minimizer of J on H.

THEOREM 2. If J is proper then $m \geq 0$; moreover and $J(u_0) = 0$ if and only if u_0 is a solution of (1).

Proof. The statements follow from the definition of conjugate function

$$\Phi(u) + \Phi^*(v) \geq (u, v) \quad \forall\, u, v \in H.$$

We take $v = -\lambda G'u$. Equality holds if and only if $v \in \partial\Phi(u)$. ■

This result is a basis for dual variational principles.

Finally, we mention a more restrictive subdifferential from below for a l.s.c. function $\phi : H \rightarrowtail \mathbb{R} \cup \{+\infty\}$ studied in [3]. For $u \in D(\phi)$ we denote by $\partial^{-}\phi(u)$ the set of all subgradients $h \in H$ such that

$$\liminf_{v \to u} \frac{\phi(v) - \phi(u) - (h, v - u)}{\|v - u\|} \geq 0.$$

We notice that ϕ is Fréchet differentiable at u if and only if ϕ and $-\phi$ have subgradient at u. Furthermore, in the certain classes of functions, the existence of a subgradient of ϕ at u involves some regularity properties of u.

The subdifferentials defined above provide an appropriate framework for some evolution equations and for the multiplicity of critical points with respect to an obstacle.

REFERENCES

[1] Attouch,H. and Wets,R. "Isometries for the Legendre-Fenchel transform", Trans. Amer. Math. Soc., 296 (1986), 33-60.

[2] Auchmuty,G. "Variational principles for operator equations and initial value problems", Nonlinear Anal., TMA, 12 (1988), 531-564.

[3] Degiovanni,M., Marino,A., and Tossques,M. "Evolution equations with lack of convexity", Nonlinear Anal., TMA, 9 (1985), 1401-1443.

[4] Dias,J.P. "Variational inequalities and eigenvalue problems for nonlinear maximal monotone operators", Amer. Math. J., 97 (1975), 903-914.

[5] Pascali,D., "Nonlinear variational eigenvalue inclusions", Proc. EQUADIFF 87, Lect. Notes Pures Math., Marcel Dekker, Inc., 1988, (to appear).

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