LIBERTAS MATHEMATICA, VOL. 8, 1988

TWO INTERPENETRATING SUPERCONDUCTING FLUIDS IN A MAGNETIC FIELD

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1. Introduction. The paper aims to clarify the repartition of a magnetic field in the presence of two different interpenetrating superconductors. We made detailed energetical considerations [1] in searching which normal-abnormal interfaces are more favorable, cheaper to be formed from energetical point of view. The question will be: if one superconductor will form vortices of its own, destroying the vortices of the other one or, maybe, to form double interfaces (superposition of the vortices of the two types) will be cheaper from energetical point of view. We will define the conditions for each case to occur. Since our calculations were made for two general superconductors, without specifying their own properties, our conclusions could be used in the study of two fermi superconductors, two bose superconductors or, one fermi and one bose superconductor, in the presence of an external magnetic field. For example, A may be a boson superconductor described by a simple Abelian Higgs model [2],[3] and B may be a fermi superconductor described by the Ginsburg Landau model [1].

2. Energetical Considerations. Say A and B are two different interpenetrating-superconductors. First of all we must check the possibility of forming double (A+B) sn interfaces (in the same place).

Let $F_{s(A-B)h}$ be the free energy of the AB superconductor in the common magnetic field h, F_{nA0} and F_{nB0}, the normal free energies of A and B respectively in 0 (zero) magnetic field; ψ_A, ψ_B = the order parameters of A and B superconductor; m_A^*, m_B^* their effective masses, $\vec{A}$ = the common potential vector, h = the common magnetic field. Then, the free energy of the system may be written (as Ginsburg Landau free energy):

$$F_{s(AB)h} = F_{nA0} + \alpha_A|\psi_A|^2 + \frac{1}{2}\beta_A|\psi_A|^4 + \frac{1}{2m_A^*}|(-i\hbar\nabla - \frac{e_A^*\vec{A}}{c})\psi_A|^2 +$$

$$+F_{nB0} + \alpha_B|\psi_B|^2 + \frac{1}{2}\beta_B|\psi_B|^4 + \frac{1}{2m_B^*}|(-i\hbar\nabla - \frac{e_B^*\vec{A}}{c})\psi_B|^2 + \frac{h^2}{8\pi}.$$

The corresponding Ginsburg Landau equations are: (w.r.t. the three functions ψ_A, ψ_B,h)

$$\frac{1}{2m_A^*}\left(\frac{\hbar}{i}\nabla-\frac{e_A^*\vec{A}}{c}\right)^2\psi_A+\alpha_A\psi_A+\beta_A|\psi_A|^2\psi_A=0 \tag{1}$$

$$\frac{1}{2m_B^*}\left(\frac{\hbar}{i}\nabla-\frac{e_B^*\vec{B}}{c}\right)^2\psi_B+\alpha_B\psi_B+\beta_B|\psi_B|^2\psi_B=0 \tag{2}$$

$$j_s=\frac{c}{4\pi}\,\mathrm{curl}\,\vec{h}=\frac{e_A^*\hbar}{2m_A^* i}[\psi_A^*\nabla\psi_A-\psi_A\nabla\psi_A^*]-\frac{(e_A^*)^2}{m_A^*c}|\psi_A|^2\vec{A}+ \tag{3}$$

$$+\frac{e_B^*\hbar}{2m_B^* i}[\psi_B^*\nabla\psi_B-\psi_B\nabla\psi_B^*]-\frac{(e_A^*)^2}{m_B^*c}|\psi_B|^2\vec{A}\,.$$

The order parameters in terms of the coefficients α, β are

$$|\psi_{\infty A}|^2-\frac{\alpha_A}{\beta_A} \quad \text{and} \quad |\psi_{\infty B}|^2=-\frac{\alpha_B}{\beta_B} \quad (-\alpha=|\alpha|).$$

The range of spatial variation of ψ_A, ψ_B are given by:

$$\xi_A^2=-\frac{\hbar^2}{2m_A^*\alpha_A} \quad \text{and} \quad \xi_B^2=-\frac{\hbar^2}{2m_B^*\alpha_B}\,.$$

The penetration depth is from

$$\mathrm{curl}\,j=-\left[\frac{(e_A^*)^2}{m_A^*c}|\psi_{A\infty}|^2+\frac{(e_B^*)^2}{m_B^*c}|\psi_{B\infty}|^2\right]h,$$

given by

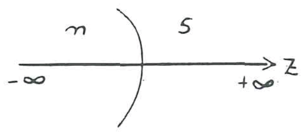
$$-\frac{1}{\lambda_{AB}^2}=\frac{4\pi(e_A^*)^2|\psi_{A\infty}|^2}{m_A^*c^2}+\frac{4\pi(e_B^*)^2|\psi_{B\infty}|^2}{m_B^*c^2}=\frac{1}{\lambda_A^2}+\frac{1}{\lambda_B^2}\,.$$

Thus, $\lambda_{AB}<\lambda_A$ and λ_B. $\frac{1}{\lambda_{AB}^2}=\frac{1}{\lambda_A^2}+\frac{1}{\lambda_B^2}$.

To find out which interfaces are more probable to be formed we must:

- calculate the sign of the surface energies σ_{ns} (to check if a normal-superconductor (ns) interface can be formed or not, that is to check if it is type II or type I superconductor) and,
- compare the possible σ_{ns} between them (if they are negative).

Before we calculate the sign of σ_{ns} we must check the dynamical equilibrium of the interface between s and n. We work in one dimension z:



The interface

The dynamical equilibrium of the sn interface asks that the Gibbs energies

$$G[-\infty] = G[+\infty].$$

If at $-\infty$ we ask it to be normal then at $+\infty$ superconductor we must have:

$$\begin{cases} \psi_A=0,\ \psi_B = 0, h = H_{c(AB)}, & \text{for } z\to-\infty \\ \psi_A=\psi_{A\infty},\ \psi_B=\psi_{B\infty},\ h=0, & \text{for } z\to+\infty \end{cases}$$

where $H_{c(AB)}$ is the critical field given by the minimum energy $\frac{H_c^2(AB)}{8\pi}$ we have to add to the AB superconductor energy to make it normal (the condensation energy). The minimum field that can destroy the two coexisting superconductors AB is $H_c(AB)=\sqrt{H_{cA}^2+H_{cB}^2}$ since, the (minimum) condensation energy we have to add to the AB superconductor energy to make it normal at $-\infty$ is

$$\frac{H^2_{c(AB)}}{8\pi} = \frac{H^2_{cA}}{8\pi} + \frac{H^2_{cB}}{8\pi}$$

To calculate the field H that gives dynamical equilibrium of the interface we have to equalize the Gibbs energy at $-\infty$ to that at $+\infty$: The Gibbs energy at $-\infty$ is:

$$G[-\infty]=G_{n(AB)}(H)=F_{n(AB)}(H_{c(AB)})-\frac{H\cdot H_{c(AB)}}{4\pi} = F_{n(AB)}(0) +$$

$$+\frac{H^2_{c(AB)}}{8\pi}-\frac{HH_{c(AB)}}{4\pi} = F_{nA0}+F_{nB0}+\frac{H^2_{c(AB)}}{8\pi}-\frac{H\cdot H_{c(AB)}}{4\pi}$$

(where G[ ] denotes the localization of G in space, and G() the magnetic field at which we calculate G).

The Gibbs energy at $+\infty$ is:

$G[+\infty] = F_{sA0}+F_{sB0}$(at $+\infty$ both A, B are superconductors in field $h = 0$).

From $G[-\infty] = G[+\infty]$ we obtain:

$$\frac{H^2_{cA}}{8\pi}+\frac{H^2_{cB}}{8\pi}+\frac{H^2_{c(AB)}}{8\pi}-\frac{HH_{c(AB)}}{4\pi}=0$$

which means $H=H_{c(AB)}$.

The external field that gives dynamical equilibrium of the AB interface is equal to the critical field.

For instance, in the case $\psi_A>\psi_B$, $\xi_A>\xi_B$ the picture is

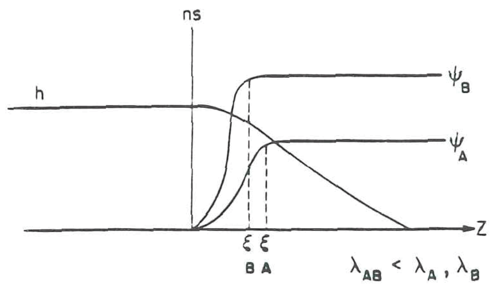


Fig. 1 The magnetic field repartition, the order parameters of A and B Superconductors and the normal - superconducting ns interface.

The surface energy in the external field that gives dynamical equilibrium ($H_{c(AB)}$) is:

$$\sigma_{ns}=\int_{-\infty}^{\infty} dz[G_{SH}-G_n]=\int_{-\infty}^{\infty} dz[F_{SH}-\frac{hH_{c(AB)}}{4\pi}-F_{sA0}-F_{sB0}]=$$

$$\int_{-\infty}^{\infty} dz\Big\{\alpha_A|\psi_A|^2+\frac{1}{2}\beta_A|\psi_A|^4+(2m^*_A)^{-1}|(-i\hbar\nabla-\frac{e^*_A\vec{A}}{c})\psi_A|^2+$$

$$\alpha_B|\psi_B|^2+\frac{1}{2}\beta_B|\psi_B|^4+(2m^*_B)^{-1}|(-i\hbar\nabla-\frac{e^*_B\vec{A}}{c})\psi_B|^2+$$

$$\frac{h^2}{8\pi}-\frac{hH_{c(AB)}}{4\pi}+\frac{H^2_{cA}+H^2_{cB}}{8\pi}\Big\}.$$

Multiplying eq. (2) by ψ^*_A and eq. (3) by ψ^*_B we obtain:

$$\int_{-\infty}^{\infty} dz\Big\{\alpha_A|\psi_A|^2+\beta_A|\psi_A|^4+\frac{1}{2m^*_A}|(\frac{\hbar\nabla}{i}-\frac{e^*_A\vec{A}}{c})\psi_A|^2\Big\}=0,$$

$$\int_{-\infty}^{\infty} dz\Big\{\alpha_B|\psi_B|^2+\beta_B|\psi_B|^4+\frac{1}{2m^*_B}|(\frac{\hbar\nabla}{i}-\frac{e^*_B\vec{A}}{c})\psi_B|^2\Big\}=0.$$

With these, the surface energy σ_{ns} will be:

$$\sigma_{ns}=\int_{-\infty}^{\infty} dz\Big\{-\frac{1}{2}\beta_A|\psi_A|^4-\frac{1}{2}\beta_B|\psi_B|^4-\frac{1}{8\pi}(h-H_c)^2\Big\}, \quad (4)$$

where, from now on we shall denote $H_{c(AB)}=H_c=\sqrt{H_{cA}^2+H_{cB}^2}$ and $\lambda_{AB}=\lambda$. We must solve the Ginsburg Landau equations in one dimension to find ψ_A, ψ_B, h and, introducing them in σ_{ns}, to find the sign of σ_{ns}. In solving these eqs. we shall make the $(K=\frac{\lambda}{\xi})$ approximation K_A^2, $K_B^2 >> 1$ and neglect terms in $\frac{1}{K_A^2}$, $\frac{1}{K_B^2}$.

From

$$H_c^2=H_{cA}^2+H_{cB}^2 = 4\pi[\frac{\alpha_A^2}{\beta_A}+\frac{\alpha_B^2}{\beta_B}] = \frac{\hbar^2c^2}{2}[\frac{1}{(e_A^*)^2\lambda_A^2\xi_A^2}+\frac{1}{(e_B^*)^2\lambda_B^2\xi_B^2}]$$

we see that H_c in the eq. (4) must be

$$H_c=\sqrt{4\pi(\frac{\alpha_A^2}{\beta_A}+\frac{\alpha_B^2}{\beta_B})} = \frac{\hbar c}{\sqrt{2}}\sqrt{\frac{1}{(e_A^*)^2\lambda_A^2\xi_A^2}+\frac{1}{(e_B^*)^2\lambda_B^2\xi_B^2}},$$

that is $H_c>H_{cA}$, $H_c>H_{cB}$.

We remember also that the common penetration depth of the two coexisting superconductors AB, λ is:

$$\frac{1}{\lambda^2}=\frac{1}{\lambda_A^2}+\frac{1}{\lambda_B^2}\ (\lambda<\lambda_A,\lambda_B).$$

We rewrite the Ginsburg Landau equations and the expression of surface energy in terms of nondimensional quantities ρ, h', $\mathcal{A}$ that are defined by:

$$h'=\frac{h}{\sqrt{2}\,H_c},\ \rho=\frac{z}{\lambda},\ \mathcal{A}=\frac{A}{\sqrt{2}\,H_c\lambda}.$$

Since $h'\equiv\frac{h}{\sqrt{2}\,H_c}=\frac{\text{curl } A}{\sqrt{2}\,H_c}=\text{Curl } \mathcal{A}$ (notation)

where the differentiation in Curl is w.r.t. ρ and not z,

$$A=\sqrt{2}\,H_c\lambda\mathcal{A}=\hbar c\lambda\sqrt{\frac{1}{(e_A^*)^2\lambda_A^2\xi_A^2}+\frac{1}{(e_B^*)^2\lambda_B^2\xi_B^2}}\cdot\mathcal{A}$$

Introducing also the nondimensional order parameter ψ'_A and ψ'_B by the relations

$$\psi_A\equiv\frac{\lambda_A}{\lambda}|\psi_{A\infty}|\psi'_A \quad\text{and}\quad \psi_B\equiv\frac{\lambda_B}{\lambda}|\psi_{B\infty}|\psi'_B$$

or, equivalently,

$$\frac{\lambda^2}{\lambda_A^2}\frac{|\psi_A|^2}{|\psi_{A\infty}|^2} \equiv |\psi'_A|^2 \quad \text{and} \quad \frac{\lambda^2}{\lambda_B^2}\frac{|\psi_B|^2}{|\psi_{B\infty}|^2} \equiv |\psi'_B|^2$$

where we know that $\frac{4\pi(e_A^*)^2|\psi_{A\infty}|^2}{m_A^2 c^2} = \frac{1}{\lambda_A^2}$ and $\frac{4\pi(e_B^*)^2|\psi_{B\infty}|^2}{m_B^* c^2} = \frac{1}{\lambda_B^2}$. Introducing the notation

$$K_{(A)} \equiv \lambda^2 e_A^* \sqrt{\frac{1}{(e_A^*)^2\lambda_A^2\xi_A^2} + \frac{1}{(e_B^*)^2\lambda_B^2\xi_B^2}},$$

$$K_{(B)} \equiv \lambda^2 e_B^* \sqrt{\frac{1}{(e_A^*)^2\lambda_A^2\xi_A^2} + \frac{1}{(e_B^*)^2\lambda_B^2\xi_B^2}},$$

(which reduce in the case of one superconductor to K_A or K_B) we can write the eq. (3) in terms of the nondimensional variables $\mathcal{A}$, h', ψ'_A, ψ'_B, ρ, and curl,. ∇, $\cdots$w.r.t.ρ as:

$$-\text{curl curl } \mathcal{A} = -\text{curl } h' = (|\psi'_A|^2 + |\psi'_B|^2)\mathcal{A} + \frac{i}{2K_{(A)}}(\psi_A'^*\nabla\psi'_A - \psi_A\nabla\psi_A'^*) + \tag{3.a}$$

$$+\frac{i}{2K_{(B)}}(\psi_B'^*\nabla\psi'_B - \psi_B\nabla\psi_B'^*).$$

We multiply eq. (1) and (2) by $\frac{\lambda}{\alpha_A\lambda_A|\psi_{A\infty}|}$ and $\frac{\lambda}{\alpha_B\lambda_B|\psi_{B\infty}|}$ respectively, using the fact that

$$\begin{cases} |\psi_{A\infty}|^2 = -\dfrac{\alpha_A}{\beta_A}, \\ |\psi_{B\infty}|^2 = -\dfrac{\alpha_B}{\beta_B}, \end{cases}$$

and we make the notations:

$$\frac{1}{\xi_A^2} \equiv (e_A^*)^2\lambda^2\left(\frac{1}{(e_A^*)^2\lambda_A^2\xi_A^2} + \frac{1}{(e_B^*)^2\lambda_B^2\xi_B^2}\right),$$

$$\frac{1}{\xi_B^2} \equiv (e_B^*)^2\lambda^2\left(\frac{1}{(e_A^*)^2\lambda_A^2\xi_A^2} + \frac{1}{(e_B^*)^2\lambda_B^2\xi_B^2}\right),$$

that reduce in the case of one superconductor to $\frac{1}{\xi_A^2}$ and $\frac{1}{\xi_B^2}$ respectively.

With these, we can write the eqs. (1), (2) in terms of nondimensional variables ψ'_A, ψ'_B, $\mathcal{A}$ and ∇ (w.r.t.ρ) as:

$$\frac{\xi_A^2}{\xi_A^2}\left(i\frac{\nabla}{k_{(A)}}+\mathcal{A}\right)^2\psi'_A=\psi'_A-\frac{\lambda_A^2}{\lambda^2}|\psi'_A|^2\psi'_A\ , \tag{1.a}$$

$$\frac{\xi_B^2}{\xi_B^2}\left(i\frac{\nabla}{k_{(B)}}+\mathcal{A}\right)^2\psi'_B=\psi'_B-\frac{\lambda_B^2}{\lambda^2}|\psi'_B|^2\psi'_B\ , \tag{2.a}$$

where we used the relations $|\psi_{A,B}|^2=-\frac{\alpha_{A,B}}{\beta_{A,B}}$ and $H^2_{cA,B}=\frac{4\pi\alpha^2_{A,B}}{\beta_{A,B}}$.

The double surface energy will be:

$$\sigma_{ns}=\int_{-\infty}^{\infty}dz\{-\tfrac{1}{2}\beta_A|\psi_A|^4-\tfrac{1}{2}\beta_B|\psi_B|^4+\frac{H_c^2}{4\pi}(h'-\frac{1}{\sqrt{2}})^2\}=$$

$$=\lambda\frac{H_c^2}{4\pi}\int_{-\infty}^{\infty}d\rho\{-\frac{1}{2}\frac{H^2_{cA}}{H_c^2}\frac{\lambda_A^4}{\lambda^4}\psi'^4_A-\frac{1}{2}\frac{H^2_{cB}}{H_c^2}\frac{\lambda_B^4}{\lambda^4}\psi'^4_B+(h'-\frac{1}{\sqrt{2}})^2\}\ .$$

But, from the explicit expressions of $H_{cA,B}$ and H_c we know that

$$\frac{H^2_{cA}}{H_c^2}=\frac{\hbar^2c^2}{2(e^*_A)^2\xi_A^2\lambda_A^2}\,\frac{2}{\hbar^2c^2}\,\frac{1}{\frac{1}{(e^*_A)^2\xi_A^2\lambda_A^2}+\frac{1}{(e^*_B)^2\xi_B^2\lambda_B^2}}$$

$$=\frac{\lambda^2}{\lambda_A^2}\frac{\xi^2_{(A)}}{\xi_A^2}$$

and, similarly

$$\frac{H^2_{cB}}{H_c^2}=\frac{\lambda^2}{\lambda_B^2}\frac{\xi^2_{(B)}}{\xi_B^2}$$

Thus, the surface energy is:

$$\sigma_{ns}=\lambda\frac{H_c^2}{4\pi}\int_{-\infty}^{\infty}d\rho\{-\frac{1}{2}\frac{\xi^2_{(A)}}{\xi_A^2}\frac{\lambda_A^2}{\lambda^2}\psi'^4_A-\frac{1}{2}\frac{\xi^2_{(B)}}{\xi_B^2}\frac{\lambda_B^2}{\lambda^2}\psi'^4_B+$$

$$+(h'-\frac{1}{\sqrt{2}})^2\}$$

We have to calculate σ_{ns} where the functions ψ'_A, ψ'_B and h' are solutions of the Ginsburg-Landau equations (3.a), (1.b), (2.b)

$$(i\frac{\nabla}{K_{(A)}} + \mathcal{A})^2\psi'_A = \frac{\xi^2_{(A)}}{\xi^2_A}\frac{\lambda^2_A}{\lambda^2}|\psi'_A|^2\psi'_A , \qquad (1.b)$$

$$(i\frac{\nabla}{K_{(B)}} + \mathcal{A})^2\psi'_B = \frac{\xi^2_{(B)}}{\xi^2_B}\frac{\lambda^2_B}{\lambda^2}|\psi'_B|^2\psi'_B . \qquad (2.b)$$

Being interested in surface energy, that is, working in one dimension only (the direction z normal to the interface sn), we can choose the order parameters for A and B superconductor as reals, putting:

$$\begin{cases} \psi'_A = f_A e^{i\varphi_A} = f_A \\ \psi'_B = f_B e^{i\varphi_B} = f_B \qquad (\varphi_A = \varphi_B = 0). \\ \mathcal{A} = \mathcal{A}_0 + \frac{\nabla\varphi_A}{K_A} + \frac{\nabla\varphi_B}{K_B} = \mathcal{A}_0. \end{cases}$$

Then, the three Ginsburg-Landau equations will be:

$$(i\frac{\nabla}{K_{(A)}} + \mathcal{A}_0)^2 f_A = \frac{\xi^2_{(A)}}{\xi^2_A} f_A - \frac{\xi^2_{(A)}}{\xi^2_A}\frac{\lambda^2_A}{\lambda^2} f^3_A \qquad (1.c)$$

$$(i\frac{\nabla}{K_{(B)}} + \mathcal{A}_0)^2 f_B = \frac{\xi^2_{(B)}}{\xi^2_B} f_B - \frac{\xi^2_{(B)}}{\xi^2_B}\frac{\lambda^2_B}{\lambda^2} f^3_B \qquad (2.c)$$

$$-\text{curlcurl}\,\vec{\mathcal{A}}_0 = -\text{curl}\,\vec{h}' = (f^2_A + f^2_B)\vec{\mathcal{A}}_0 \qquad (3.c)$$

(since in one dimension $\psi'_{A,B} = \psi'^*_{A,B}$), and the surface energy becomes:

$$\sigma_{ns} = \lambda\frac{H^2_c}{4\pi}\int_{-\infty}^{\infty} d\rho\{-\frac{1}{2}\frac{\xi^2_{(A)}}{\xi^2_A}\frac{\lambda^2_A}{\lambda^2} f^4_A - \frac{1}{2}\frac{\xi^2_{(B)}}{\xi^2_B}\frac{\lambda^2_B}{\lambda^2} f^4_A +$$

$$+(h' - \frac{1}{\sqrt{2}})^2\} .$$

From now on we note ρ by z and h′ by h.

In our one dimensional problem with z normal to the interface (sn) and a magnetic field parallel to the interface, it's natural to choose:

$$\vec{A} = A\vec{x};\ \vec{h} = h\vec{y};\ A = A(z);\ h = h(z);\ f_A = f_A(z);\ f_B = f_B(z).$$

Since $\mathcal{A} = (\mathcal{A}(z),0,0)$, $f_{A,B}$ are functions of z only, $\nabla^2 f = \frac{d^2 f}{dz^2}$, the purely imaginary terms in (1.c), (2.c) will be:

$$i[\nabla(\vec{A}f) + \vec{A}\nabla f] = i[\frac{\partial}{\partial x}(f\mathcal{A}_x) + \frac{\partial}{\partial y}(f\mathcal{A}_y) + \frac{\partial}{\partial z}(f\mathcal{A}_z) + \mathcal{A}_x\frac{\partial f}{\partial x} + \mathcal{A}_y\frac{\partial f}{\partial y} + \mathcal{A}_z\frac{\partial f}{\partial z}] = 0$$

So, the purely imaginary terms in (1.c), (2.c) disappear. On the other hand, the terms $\mathcal{A}_0^2 f_{A,B}$ from (1.c), (2.c) are, taking the square of the equation (3.c):

$$\mathcal{A}_0^2 f_{A,B}=\frac{(\text{curl h})^2}{(f_A^2+f_B^2)^2}f_{A,B}$$

and by this, the equations (1.c)(2.c) become:

$$-\frac{1}{K_{(A)}^2}\frac{d^2 f_A}{dz^2}+\frac{f_A}{(f_A^2+f_B^2)^2}\left(\frac{dh}{dz}\right)^2=\frac{\xi_{(A)}^2}{\xi_A^2}f_A-\frac{\xi_{(A)}^2}{\xi_A^2}\frac{\lambda_A^2}{\lambda^2}f_A^3 \qquad (1.d)$$

$$-\frac{1}{K_{(B)}^2}\frac{d^2 f_B}{dz^2}+\frac{f_B}{(f_A^2+f_B^2)^2}\left(\frac{dh}{dz}\right)^2=\frac{\xi_{(B)}^2}{\xi_B^2}f_B-\frac{\xi_{(B)}^2}{\xi_B^2}\frac{\lambda_B^2}{\lambda^2}f_B^3 \qquad (2.d)$$

Observation: Comparing

$$K_{(A)}=\lambda^2 e_A^*\sqrt{\frac{1}{(e_A^*)^2\lambda_A^2\xi_A^2}+\frac{1}{(e_B^*)^2\lambda_B^2\xi_B^2}}$$

to

$$\frac{\xi_A}{\xi_{(A)}}=\xi_A e_A^*\lambda\sqrt{\frac{1}{(e_A^*)^2\lambda_A^2\xi_A^2}+\frac{1}{(e_B^*)^2\lambda_B^2\xi_B^2}}$$

and to:

$$\frac{\xi_A}{\xi_{(A)}}\frac{\lambda}{\lambda_A}=\lambda^2 e_A^*\frac{1}{K_A}\sqrt{\frac{1}{(e_A^*)^2\lambda_A^2\xi_A^2}+\frac{1}{(e_B^*)^2\lambda_B^2\xi_B^2}}$$

we see that:

–in the first line, $K_{(A)}\gg\frac{\xi_A}{\xi_{(A)}}$ since $\lambda\gg\xi_A$ (we began from two hard type II superconductors A and B);

–in the second line, $K_{(A)}\gg\frac{\xi_A}{\xi_{(A)}}\frac{\lambda}{\lambda_A}$ since $K_A\gg 1$.

The same observations are true for the B superconductor too, such that, in the equation (1.d), (2.d) we can neglect the terms in $\frac{1}{K_{(A)}^2}$ and $\frac{1}{K_{(B)}^2}$ respectively, w.r.t. the other terms and, instead of

(1.d) and (2.d) we write the equations:

$$\frac{f_A}{(f_A^2+f_B^2)^2}\left(\frac{dh}{dz}\right)^2=\frac{\xi_{(A)}^2}{\xi_A^2}f_A-\frac{\xi_{(A)}^2}{\xi_A^2}\frac{\lambda_A^2}{\lambda^2}f_A^3 \qquad (1.e)$$

$$\frac{f_B}{(f_A^2+f_B^2)^2}\left(\frac{dh}{dz}\right)^2=\frac{\xi_{(B)}^2}{\xi_B^2}f_B-\frac{\xi_{(B)}^2}{\xi_B^2}\frac{\lambda_B^2}{\lambda^2}f_B^3 \qquad (2.e)$$

We want to express the eq. (3.c) in terms of the same functions f , f_B,h only. For this we transform the eq.(3.c) applying the curl:

$$-\text{curl curl}\,\vec{h} = \text{curl}\,[(f_A^2+f_B^2)\vec{\mathcal{A}}_0]=(f_A^2+f_B^2)\text{curl}\,\vec{\mathcal{A}}_0-\vec{\mathcal{A}}_0\times\text{grad}(f_A^2+f_B^2)=$$

$$=(f_A^2+f_B^2)\text{curl}\,\vec{\mathcal{A}}_0-\vec{\mathcal{A}}_0\times(2f_A\,\text{grad}\,f_A+2f_B\,\text{grad}\,f_B)=$$

$$=(f_A^2+f_B^2)\text{h}-\frac{2}{f_A^2+f_B^2}(f_A\,\text{grad}\,f_A+f_B\,\text{grad}\,f_B)\times\text{curl h}.$$

since, from (3.c) we have $\vec{\mathcal{A}}_0=-\dfrac{\text{curl}\,\vec{h}}{(f_A^2+f_B^2)}$.

Thus, instead of the equation (3.c) we obtain:

$$(f_A^2+f_B^2)\text{h}-\frac{2}{f_A^2+f_B^2}[(f_A\nabla f_A+f_B\nabla f_B)\times\text{curl h}]-\text{curl curl}\,\vec{h}.$$

We remember that in our one dimensional calculations we chose $\vec{h}=(h(z)\vec{j}$, $f_{A,B}=f_{A,B}(z)$ (say $\vec{i},\vec{j},\vec{\kappa}$ are the unitary vectors of the axes x, y, z) and so, curl $\vec{h}=-\frac{\partial h}{\partial z}\vec{i}$; curl curl $\vec{h}=-\frac{\partial^2 h}{\partial z^2}\vec{j}$ and the right parentheses in the above equation are $[\ldots]=-\frac{\partial h}{\partial z}\frac{\partial f}{\partial z}\vec{j}$.

Then, the equation (3.c) will be:

$$(f_A^2+f_B^2)\text{h}=-\frac{2}{(f_A^2+f_B^2)}[f_A\frac{df_A}{dz}+f_B\frac{df_B}{dz}]\frac{dh}{dz}+\frac{d^2h}{dz^2}. \tag{3.e}$$

Therefore, we have to solve the following problem: calculate the surface energy

$$\sigma_{ns}=\lambda\frac{H_c^2}{4\pi}\int_{-\infty}^{\infty}dz\{-\frac{1}{2}\frac{\xi^2_{(A)}}{\xi^2_A}\frac{\lambda^2_A}{\lambda^2}f_A^4-\frac{1}{2}\frac{\xi^2_{(B)}}{\xi^2_B}\frac{\lambda^2_B}{\lambda^2}f_A^4+$$

$$+(h-\frac{1}{\sqrt{2}})^2\}$$

where f_A, f_B, h are the solutions of the equations:

$$\frac{f_A}{(f_A^2+f_B^2)^2}(\frac{dh}{dz})^2=\frac{\xi^2_{(A)}}{\xi^2_A}f_A-\frac{\xi^2_{(A)}}{\xi^2_A}\frac{\lambda^2_A}{\lambda^2}f_A^3 \tag{1.e}$$

$$\frac{f_B}{(f_A^2+f_B^2)^2}(\frac{dh}{dz})^2=\frac{\xi^2_{(B)}}{\xi^2_B}f_B-\frac{\xi^2_{(B)}}{\xi^2_B}\frac{\lambda^2_B}{\lambda^2}f_B^3 \tag{2.e}$$

$$(f_A^2+f_B^2)\text{h}=-\frac{2}{(f_A^2+f_B^2)}[f_A\frac{df_A}{dz}+f_B\frac{df_B}{dz}]\frac{dh}{dz}+\frac{d^2h}{dz^2} \tag{3.e}$$

with the boundary conditions:

$$f_A(-\infty)=0 \qquad f_A(+\infty)=\frac{\lambda}{\lambda_A}<1 \qquad \frac{df_A}{dz}(+\infty)=\frac{df_B}{dz}(+\infty)=0$$

$$f_B(-\infty)=0 \qquad f_B(+\infty)=\frac{\lambda}{\lambda_B}<1 \qquad \frac{df_A}{dz}(+\infty)=\frac{df_B}{dz}(-\infty)=0$$

$$h(-\infty)=\frac{1}{\sqrt{2}} \qquad h(+\infty)=0$$

We observe that: $0<f_A<\frac{\lambda}{\lambda_A}<1$, $0<f_B<\frac{\lambda}{\lambda_B}<1$, $0<h<\frac{1}{\sqrt{2}}$. Knowing that h is a decreasing function in z, we can write the equations (1.e) and (2.e) as:

$$\frac{dh}{dz}=-(f_A^2+f_B^2)\frac{\xi_{(A)}}{\xi_A}\sqrt{1-\frac{\lambda_A^2}{\lambda^2}f_A^2} \tag{1.f}$$

$$\frac{dh}{dz}=-(f_A^2+f_B^2)\frac{\xi_{(B)}}{\xi_B}\sqrt{1-\frac{\lambda_B^2}{\lambda^2}f_B^2} \tag{2.f}$$

For instance, if A is the bose superconductor and B is the fermi one, with $e_A^*=\frac{1}{2}e_B^*$ (effective changes), then $\frac{\xi_{(A)}}{\xi_{(B)}}=\frac{e_B^*}{e_A^*}=2$ or, $\xi_{(A)}^2=4\xi_B^2$.

There are two sets of (f_A, f_B) that satisfy (1') and (2'):

$$f_A=f_B=0 \tag{4.1}$$

$$\frac{2}{\xi_A}\sqrt{1-\frac{\lambda_A^2}{\lambda^2}f_A^2}=\frac{1}{\xi_B}\sqrt{1-\frac{\lambda_B^2}{\lambda^2}f_B^2}\,.$$

that is

$$f_A^2=\frac{\lambda^2}{\lambda_A^2}[(1-\frac{\xi_A^2}{4\xi_B^2})+\frac{\xi_B^2}{4\xi_B^2}\frac{\lambda_B^2}{\lambda^2}f_B^2]. \tag{4.2}$$

But, f_A, f_B are physical quantities so, they must be continuous functions of z. We know that at $z\to-\infty$, the (AB) system must be normal ($f_A=0$, $f_B=0$) and at $z\to+\infty$, $f_A=\frac{\lambda}{\lambda_A}$, $f_B=\frac{\lambda}{\lambda_B}$; the solutions will be continuous from $-\infty$ to $+\infty$ only if $1-\frac{\xi_A^2}{4\xi_B^2}=0$ that is the double interface $(AB)_{sn}$ could exist only in the case that $\xi_A=2\xi_B$. For $\xi_A<2\xi_B$ or $\xi_A>2\xi_B$ such a double interface cannot exist. In the case $\xi_A=2\xi_B$ all the equations become very simple: $\frac{\xi_{(A)}}{\xi_A}=\frac{\xi_{(B)}}{\xi_B}=1$.

$$f_A^2=\frac{\lambda_B^2}{\lambda_A^2}f_B^2\,. \tag{4}$$

From (1.f),(2.f) we can write the eq. (3.e) as

$$h=-\frac{d}{dz}(1-\frac{\lambda_B^2}{\lambda^2}f_B^2)^{\frac{1}{2}}\,. \tag{3.f}$$

Differentiating this w.r.t. z

$$\frac{dh}{dz}=-\frac{d^2}{dz^2}(1-\frac{\lambda_B^2}{\lambda^2}f_B^2)^{\frac{1}{2}} \tag{5}$$

and using (2.f) for $\xi_A=2\xi_B$ we have:

$$(f_A^2+f_B^2)(1-\frac{\lambda_B^2}{\lambda^2}f_B^2)^{\frac{1}{2}}=\frac{d^2}{dz^2}(1-\frac{\lambda_B^2}{\lambda^2}f_B^2)^{\frac{1}{2}}\,.$$

Using (4) here we obtain a differential equation for f_B or, making the substitution

$$u\equiv(1-\frac{\lambda_B^2}{\lambda^2}f_B^2)^{\frac{1}{2}} \text{ that is } f_B^2=\frac{\lambda^2}{\lambda_B^2}(1-u^2), \tag{6}$$

we obtain a differential equation for u. We observe that $u(-\infty)=1$, $u(+\infty)=0 (0\leq u\leq 1)$ and

$$du=-\frac{\frac{\lambda_B^2}{\lambda^2}f_B df_B}{(1-\frac{\lambda_B^2}{\lambda^2}f_B^2)^{\frac{1}{2}}}<0. \tag{6.a}$$

Taking into account that $\lambda^2(\frac{1}{\lambda_A^2}+\frac{1}{\lambda_B^2})=1$ we obtain the differential equation for u: $[1-u^2]u=\frac{d^2u}{dz^2}$.

Multiplying this by $du=\frac{du}{dz}dz$ and integrating one time we find:

$$\frac{1}{2}(\frac{du}{dz})^2=\frac{u^2}{2}-\frac{u^4}{4}+\frac{c}{2}$$

or

$$(\frac{du}{dz})^2=u^2[1-\frac{u^2}{2}]+C$$

with C a constant of integration. From the boundary conditions we know that for $z\rightarrow+\infty$, $u=0$ and $\frac{du}{dz}=0$. From here, the constant C must be zero always.

On the other hand, from (6.a) $\frac{du}{dz}<0$ so, the differential equation for u (with the true sign) is:

$$\frac{du}{dz}=-u[1-\frac{u^2}{2}]^{\frac{1}{2}}\,. \tag{7}$$

The magnetic field is from (3′):

$$h=-\frac{du}{dz}=u\,[1-\frac{u^2}{2}]^{\frac{1}{2}}.$$

Now, we can easily calculate σ_{ns}, expressing it as a function of f_B and h only and then, as function of u only. Writing $\sigma_{ns} \equiv \lambda \frac{H_c^2}{4\pi} J_{AB}$, the integral J_{AB} is (using (6) and (7)):

$$J_{AB} = \int_{-\infty}^{\infty} dz \frac{du}{dz} \{\sqrt{2} - 2u[1 - \frac{u^2}{2}]^{\frac{1}{2}}\} =$$

$$= \int_0^1 \{\sqrt{2} - 2u[1 - \frac{u^2}{2}]^{\frac{1}{2}}\} du = \{-\sqrt{2} + \frac{4}{3}[1 - (1 - \frac{1}{2})^{3/2}]\} =$$

$$= -\sqrt{2}(1 + \frac{1}{3}) = \frac{4}{3}(\sqrt{2} - 1) \approx -0.55$$

The surface energy $\sigma_{ns} = \lambda \frac{H_c^2}{4\pi}(-0.55) < 0$. The first conclusion is that, if such double interface exists it will be formed since AB will be a type II superconductor (instead of 2 type II superconductors A and B, one has one type II superconductor (AB)). But this only for $\xi_A = 2\xi_B$.

Comparing now which interface is cheaper to form, $\sigma_{ns(AB)}$, σ_{nsA} or σ_{nsB} we have:

$$\frac{|\sigma_{nsAB}|}{|\sigma_{nsB}|} = \frac{\lambda H_c^2 \cdot 0.55}{\lambda_B H_{cB}^2 \cdot 0.55} = \frac{\lambda}{\lambda_B}[1 + (\frac{H_{cA}}{H_{cB}})^2].$$

But $H_{cA} = \frac{\hbar c}{2\pi\sqrt{2} e \xi_A \lambda_A}$ and $H_{cB} = \frac{hc}{2\pi\sqrt{2}\,(2e)\xi_B \lambda_B}$, so, $\frac{H_{cA}}{H_{cB}} = \frac{\lambda_B}{\lambda_A}$ (the interface could appear if $\xi_A = 2\xi_B$ but we don't know whether it is the cheaper one). So,

$$\frac{|\sigma_{nsAB}|}{|\sigma_{nsB}|} = \frac{\lambda}{\lambda_B}[1 + (\frac{\lambda_B}{\lambda_B})^2] = \frac{1}{\sqrt{1 + \frac{\lambda_B^2}{\lambda_A^2}}}[1 + (\frac{\lambda_B}{\lambda_A})^2] =$$

$$\sqrt{1 + (\frac{\lambda_B}{\lambda_B})^2} > 1.$$

Knowing that both $\sigma_{ns(AB)}$ and σ_{nsB} are negative we find that $\sigma_{ns(AB)} < \sigma_{nSB}$ so is cheaper to form $(AB)_{sn}$ interface than B_{sn} interface. A similar calculation shows that

$$\frac{|\sigma_{nsAB}|}{|\sigma_{nsB}|} = \frac{\lambda H_{cAB}^2 \cdot 0.55}{\lambda_A H_{cA}^2 \cdot 0.55} = \sqrt{1 + (\frac{\lambda_A}{\lambda_B})^2} > 1$$

Both σ's being negative we conclude that $(AB)_{sn}$ interface is cheaper to form that A_{sn} interface too. So, if $\xi_A=2\xi_B$, the magnetic field will appear as a superposition of A and B vortex field lines. When $\xi_A<2\xi_B$ or $\xi_A > 2\xi_B$ such a double interface cannot appear and we have to compare only σ_{nsB} with σ_{nsA}. Thus for $\xi_A \neq 2\xi_B$:

$$\frac{|\sigma_{nsA}|}{|\sigma_{nsB}|} = \frac{\lambda_A H^2_{cA} \cdot 0.55}{\lambda_B H^2_{cB} \cdot 0.55} = \frac{4\xi^2_B}{\xi^2_A} \frac{\lambda_B}{\lambda_A}$$

If $\sigma_{nsA}<\sigma_{nsB}$, only A vortex lines occur. If $\sigma_{snB}<\sigma_{snA}$ only B vortex lines will occur there and no A vortex. For instance, in some cases, the B fermi superconductor would destroy A bose vortices.

III. Conclusion.

We saw here that the internal conditions of each one of the two interpenetrating-superconductors (as: effective charge, effective mass, coherence length, penetration depth, etc.) decide if the global situation of the two coexisting superconductors in an external magnetic field would be:

– vortices of the A superconductor only.

– vortices of the B superconductor only, or,

– mixed vortices (A+B double interfaces).

For instance, the conclusions could be applied in a first approximation for neutron stars interiors where there are conditions for proton superconductor (a fermi superconductor) and π condensation [4],[5] (a bose superconductor) to occur in the magnetic field of the neutron star.

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