LIBERTAS MATHEMATICA, VOL. 8, 1988

Classroom Notes

A CONSTRUCTION WITHOUT FACTORIZATION FOR THE REAL NUMBERS

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The well-known constructions of real numbers, the Dedekind cuts method and the Cantor-Cauchy completion method, define real numbers as classes. In [1], P. Shiu defines real positive numbers as equivalence classes of infinite subsets of $\mathbb{N}$. Using natural numbers, P. Shiu lay a great part of his theory upon arithmetical properties, which, in this case, are helpful, but have no deep connection with the convergence of sequences.

The following construction starts from the main idea of [1] - real numbers can be constructed as series of rational numbers - but it avoids both arithmetical properties and partitioning. Working with equivalence classes implies proofs of independence of the representatives, which are often long and difficult.

Suppose that the rational number set is known, together with his algebraic and topological properties. Let (a_n) be a sequence of rational positive numbers converging (not necessarily decreasingly) to 0, with $a_0=0$ and $\lim_{n\to\infty}\sum_{i=1}^{n} a_i=\infty$.

$$\text{Denote } A=\left\{(a_{n(k)}) \mid \sum_{k=1}^{\infty} a_{n(k)} \text{ convergent and } n(0)=0\right\}.$$

We define $\mathbb{R}_+ \subset A$ as the set of the subsequences $(a_{n(k)})_{k\in\mathbb{N}}$ of (a_n), having the property:

$$n(k+1) = \min\left\{p>n(k) \mid \exists\, m\in\mathbb{N},\ \sum_{i=1}^{k} a_{n(i)}+a_p< \sum_{i=1}^{m} a_{n(i)}\right\},\ \forall k\in\mathbb{N}.$$

For example, putting $a_n=1/n$, let us form the (unique) sequence which represents number 2.

$1< 2 \Rightarrow a_{n(1)}=1$; $1+1/2 < 2 \Rightarrow a_{n(2)}=1/2$;

$1+1/2+1/3<2 \Rightarrow a_{n(3)}=1/3$; $1+1/2+1/3+1/4\geq 2$,

$1+1/2+1/3+1/5\geq 2$, $1+1/2+1/3+1/6\geq 2$,

$1+1/2+1/3+1/7< 2 \Rightarrow a_{n(4)}=1/7$, etc ...

Lemma. For every $(b_k)\in A$ there exists a unique $(a_{c(k)})\in\mathbb{R}_+$, such that

$$\lim_{k\to\infty}\sum_{i=1}^{k}(b_i-a_{c(i)})=0.$$

Proof. We define $c(k)$, $k\in\mathbb{N}$ in the following way:

$$c(k+1)=\min\left\{p>c(k)\mid \exists\ n\in\mathbb{N}\ \text{such that}\ \sum_{i=1}^{k}a_{c(i)}+a_p<\sum_{i=1}^{m}b_i\right\},\ \forall k\in\mathbb{N}.$$

We notice that, for every $k_0\in\mathbb{N}$ there exists $k>k_0$ and $m(k)\in\mathbb{N}$ such that:

$$\sum_{i=1}^{k}a_{c(i)}+a_{c(k+1)}<\sum_{i=1}^{m(k)}b_i\ \text{ and }\ \sum_{i=1}^{k}a_{c(i)}+a_{c(k)+1}\geq\sum_{i=1}^{m(k)}b_i.$$

Hence,

$$a_{c(k+1)}<\sum_{i=1}^{m(k)}b_i-\sum_{i=1}^{k}a_{c(i)}\leq a_{c(k)+1}\ ,$$

which implies that

$$\lim_{k\to\infty}\sum_{i=1}^{k}(b_i-a_{c(i)})+\sum_{i=k+1}^{m(k)}b_i=0.$$

As $\sum_{n=1}^{\infty}b_n$ is a convergent series, $\lim_{k\to\infty}\sum_{i=k+1}^{m(k)}b_i=0$ (Cauchy). Then, $\lim_{k\to\infty}\sum_{i=1}^{k}(b_i-a_{c(i)})=0$.

We shall show that $(a_{c(i)})\in\mathbb{R}_+$ by "reductio ad absurdum". We suppose that there could exist k, $m\in\mathbb{N}$ such that $c(k)-1>c(k-1)$ and

$$\sum_{i=1}^{k-1}a_{c(i)}+a_{c(k)-1}<\sum_{i=1}^{m}a_{c(i)}.$$

But, for a certain p,

$$\sum_{i=1}^{m}a_{c(i)}<\sum_{i=1}^{p}b_i.$$

Hence,

$$\sum_{i=1}^{k-1}a_{c(i)}+a_{c(k)-1}<\sum_{i=1}^{p}b_i,$$

which contradicts the definition of $c(k)$.

If we suppose that there would exist another $(a_{d(k)})\in\mathbb{R}_+$, such that $\lim_{k\to\infty}\sum_{i=1}^{k}(b_i-a_{d(i)})=0$, then:

$$\lim_{k\to\infty}\sum_{i=1}^{k}(b_i=\lim_{k\to\infty}\sum_{i=1}^{k}a_{d(i)}=\lim_{k\to\infty}\sum_{i=1}^{k}a_{c(i)}.$$

Hence $c(k+1)=\min\left\{p>c(k)\mid\exists\ m\in\mathbb{N}\ \text{such that}\ \sum_{i=1}^{k}a_{c(i)}+a_p<\sum_{i=1}^{m}a_{d(i)}\right\}$. And, by induction, it can be proved that $c(k)=d(k)$, $\forall k\in\mathbb{N}$.

We shall further define the sum and the product of two positive real numbers. Let $b=(b_n)\in\mathbb{R}_+$, $c=(c_n)\in\mathbb{R}_+$. We define $s=b+c$ $s=(a_{n(k)})$ as follows:

$$n(k+1)=\min\left\{n>n(k) \mid \exists\ m\in\mathbb{N},\ \sum_{i=1}^{k}a_{n(i)}+a_n<\sum_{i=1}^{m}(b_i+c_i)\right\}.$$

The product $p=bc$, $p=(a_{n(k)})$ is defined by

$$n(k+1)=\min\left\{n>n(k) \mid \exists\ m\in\mathbb{N},\ \sum_{i=1}^{k}a_{n(i)}+a_n<\Big(\sum_{i=1}^{m}b_i\Big)\Big(\sum_{i=1}^{m}c_i\Big)\right\}.$$

Using a similar proof to that of the Lemma, we get $s, p\in\mathbb{R}_+$.

Theorem 1. For every $b, c, d\in\mathbb{R}_+$ we have

1) $b+c=c+b$

2) $(b+c)+d=b+(c+d)$

3) $bc=cb$

4) $(bc)d=b(cd)$

and also

5) $\exists\ e\in\mathbb{R}_+$ such that $eb=b$ $\forall\ b\in\mathbb{R}_+$

6) $\forall\ b\in\mathbb{R}_+, \exists b^{-1}$ such that $bb^{-1}=e$.

Proof. All the six proofs are alike. As an example, let us show 5). We define $e=(a_{e(k)})_{k\in\mathbb{N}}$ by

$$e(k+1)=\min\left\{n>e(k) \mid \sum_{i=1}^{k}a_{e(i)}+a_n<1\right\}.$$

Let $b=(a_{b(k)})_{k\in\mathbb{N}}\in\mathbb{R}_+$. We denote $be=(a_{p(k)})_{k\in\mathbb{N}}$, with

$$p(k+1)=\min\left\{n>p(k) \mid \exists\ m\in\mathbb{N},\ \sum_{i=1}^{k}a_{p(i)}+a_n<\Big(\sum_{i=1}^{m}a_{e(i)}\Big)\Big(\sum_{i=1}^{m}a_{b(i)}\Big)\right\}$$

Then,

$$\sum_{i=1}^{k+1}a_{p(i)}<\Big(\sum_{i=1}^{m_0}a_{e(i)}\Big)\Big(\sum_{i=1}^{m_0}a_{b(i)}\Big)<\sum_{i=1}^{m_0}a_{b(i)}$$

for a suitable $m_0\in\mathbb{N}$. On the other hand,

$$\sum_{i=1}^{k} a_{p(i)} + a_{p(k+1)-1} > \left(\sum_{i=1}^{m} a_{e(i)}\right)\left(\sum_{i=1}^{m} a_{b(i)}\right),\ \forall\ m \in \mathbf{N},$$

and
$$\sum_{i=1}^{k} a_{p(i)} + a_{p(k+1)-1} \geq \sum_{i=1}^{m} a_{b(i)},\ \forall\ m \in \mathbf{N}.$$

Hence

$$p(k+1)=\min \left\{ n > p(k) \mid \exists\ m\in \mathbf{N}, \text{ such that } \sum_{i=1}^{m} a_{p(i)} + a_n < \sum_{i=1}^{m} a_{b(i)}, \right.$$

which shows, inductively, that $p(k)=b(k),\ \forall k\in \mathbf{N}$. In 6), for $b=(a_{b(k)})_{k\in \mathbf{N}}$, we define $b^{-1}=(a_{n(k)})_{k\in \mathbf{N}}$ as follows:

$$n(k+1)=\min \left\{ n > n(k) \mid \left(\sum_{i=1}^{k} a_{n(i)} + a_n\right)\left(\sum_{i=1}^{k+1} a_{b(i)}\right) < 1 \right\}.$$

Definition. Let $b=(a_{b(k)})$, $c=(a_{c(k)})$, $b, c \in \mathbf{R}_+$. By $b<c$ we mean that there exists $k\in \mathbf{N}$, such that $b(k) > c(k)$ and $b(i)=c(i),\ \forall i < k$.

Theorem 2. Let $b, c, d\in \mathbf{R}_+$. Then

1) One, and only one of $b< c$, $c < b$, $b=c$ holds.

2) $b < c,\ c < d \Rightarrow b < d$.

3) $b < c \Rightarrow b + d < c + d$

4) $b < c \Rightarrow bd < cd$.

Proof. The first two statements are obvious. The last two have simple proofs.

Theorem 3. Every non-empty bounded subset of $\mathbf{R}_+$ has a supremum. That is $\forall M\in \mathbf{R}_+$, $M\neq\emptyset$, $\exists\ x\in \mathbf{R}_+$ with $m\leq x\ ,\forall m\in M \Rightarrow \exists\ x^0 \in \mathbf{R}_+,\ x^0=\sup M$.

Proof. Let

$$M_1=\left\{x=(a_{x(i)})\in M \mid x(1)\leq y(1),\ \forall\ y=(a_{y(i)})\in M\right\},$$
$$M_2=\left\{x=(a_{x(i)})\in M_1 \mid x(2)\leq y(2),\ \forall\ y=(a_{y(i)})\in M_1\right\},\ \text{etc.}$$

We denote $x^0(i)=x(i)$, where $x= (a_{x(k)})_{k\in \mathbf{N}} \in M_i$, $\forall\ i\in \mathbf{N}$. Then $x^0=a_{x^0(i)})_{i\in \mathbf{N}}=\sup M$.

References:

1) P. Shiu, "A new construction of the real numbers". The Mathematical Gazette, 403, 1974, pg. 39 - 46.