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CONVERGENT RAMSEY EQUILIBRIA

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I. Introduction

Professor Georgescu-Roegen's career left an early mark on capital theory and dynamic economics. His exploration of the von Neumann model of an expanding economy ([2]) was both a pioneering paper in activity analysis as well as the first paper to demonstrate the existence of an equilibrium in that framework by means of a separation theorem from convex analysis. Indeed, von Neumann's proof of the existence theorem for that model utilized a generalization of the Brouwer Fixed Point Theorem. Hence Professor Georgescu-Roegen's contribution on that score was a significant advance.

Nonlinear dynamical systems and the study of complex dynamics in economics has been intensively investigated in recent years. One goal of this line of research is to understand business cycles. It is worth recalling here that Professor Georgescu-Roegen ([3]) wrote on business cycles as a dynamic economic model as well. Our paper also fits into the new literature on equilibrium business cycles as a first step in studying the consequences of market imperfections in economic dynamics.

We study herein the orbits of a two dimensional dynamical system derived from the equilibrium equations governing the evolution of capital and consumption in a dynamic competitive equilibrium model based on the seminal study of Ramsey [5]. The detailed model is presented in Becker and Foias [1]; it is briefly reviewed below.

The underlying economic model is a standard neoclassical one good model of capital accumulation with discrete time. There are several infinitely lived agents maximizing lifetime utility subject to a sequence of budget constraints, one for each time. The agents are assumed to have different discount factors for future rewards: there is a most and a least patient household. Given a sequence of anticipated wages and rental rates for the single capital good, an agent chooses a pattern of consumption and capital holdings at each time such that consumption plus next period's capital equals wage and capital income. The sequential budget constraint combined with a restriction that capital is always nonnegative implies there is a borrowing constraint so that future income cannot be discounted to the present. A **Ramsey equilibrium** occurs at a path of rental and wage rates such that the demand for consumption and supply of next capital goods equals total output at each time.

We showed that the capital stock of all but the most patient household has a recurrent state at zero. If all those household's capital stock eventually remains at zero, then we say the **turnpike property** holds. We demonstrated that the turnpike property holds and all Ramsey equilibrium programs converge to a stationary Ramsey equilibrium with only the most patient consumer holding capital in the case where the total capital income is an increasing function of the total capital stock. This condition is a restriction on the production function. We found that in the absence of this monotonicity hypothesis, the equilibrium stocks may have a period two orbit.

In this paper, we introduce a dynamical system based on the Ramsey equilibrium necessary conditions in the case where the turnpike property has already taken hold on the economy. Our goal is to study the bifurcation behavior of this system as the income function g (defined below) varies. In this paper, we will assume that g is strictly increasing. The problem for future study is the case where this monotonicity property fails. The two period orbit example in [1] occurs in a case where g is not monotone increasing. The first step in our general program is to recover the convergence result when g is a monotone increasing.

We will also study the properties of the dynamical system along orbits that are not necessarily representing equilibrium trajectories. These paths are singular or degenerate solutions. In economic terms, those programs fail to satisfy a transversality condition.

The dynamical system is defined in Section II. The Convergence Theorem is demonstrated in Section III. The saddle point property for the fixed point of the dynamical system is investigated in Section IV. Concluding comments follow in Section V.

II. The Dynamical System

A Ramsey equilibrium for an economy with H households $(h = 1,2,\ldots,H)$ evolves in the case where the turnpike property holds according to the equations

(1) a)
$$\begin{aligned} x_{t+1} + c^1_{t+1} = g(x_t) &:= (1+r_{t+1})x_t + w_{t+1} \\ &= f'(x_t)x_t + \tfrac{1}{H}[f(x_t) - f'(x_t)x_t] \\ &= (1-\tfrac{1}{H})f'(x_t)x_t + \tfrac{1}{H}f(x_t) \ , \end{aligned}$$

b)
$$c^h_{t+1} = w_{t+1} \quad (h=2,\ldots,H), \text{ and}$$

(2)
$$u'(c^1_t) = \delta(1+r_{t+1})u'(c^1_{t+1})$$

where c^h_{t+1} is the consumption of agent h at time $t + 1$, x_{t+1} is the capital at $t + 1$ (held entirely by agent 1 due to the turnpike property), r_{t+1} is the rental on capital paid at $t + 1$, w_{t+1} is the wage in period $t + 1$, f is the production function, δ is the first household's discount factor, and u is that agent's one-period utility function. We call g the first household's income function. Equation (1) expresses the budget constraint for each agent. Equation (2) is called the <u>no-arbitrage</u> condition for the first player; it is a necessary condition for the maximization of $\sum_{t=1}^{\infty}\delta^{t-1}u(c_t)$ subject to (1a) and c^1_t , x_t nonnegative given x_0 . Equation (2) holds when the optimal c^1_t and x_t are positive at each t . The parameter $\delta \in (0,1)$ and function $u : \mathbb{R}_+ \to \mathbb{R}$ is assumed to be $C^{(2)}$ on $(0,\infty)$ with $u' > 0$, $u'(0+) = \infty$, $u'(\infty) = 0$, and $u'' < 0$. The production function $f : \mathbb{R}_+ \to \mathbb{R}_+$, is $C^{(2)}$ on $(0,\infty)$ with $f'(0+) = \infty$, $f'(\infty) < 1$, $f' > 0$, and $f'' < 0$. There is a unique maximum sustainable stock $K^M > 0$ such that $f(K^M) = K^M$. We also note that $g(0) = 0$.

In Ramsey equilibrium, $Hw_{t+1} = f(x_t) - (1+r_{t+1})x_t$ and $f'(x_t) = 1+r_{t+1}$. Hence, the consumption levels of $h = 2,3,\ldots,H$ are determined in (1b) from $(x_{t-1},c^1_t)^{\infty}_{t=1}$. We rewrite (1a) and (2) with the notation $c^1_t = y_t$ and $K^M = a$ as the system

(3)
$$x_{t+1} + y_{t+1} = g(x_t)$$
$$u'(y_t) = \delta f'(x_t)u'(y_{t+1})$$

where $x_t > 0$, $y_t > 0$ for $t = 0,1,\ldots$. The monotonicity properties of u and f imply the existence of a continuous function $F : (0,\infty) \times (0,\infty) \to (0,\infty)$ such that

(4) $$u'(y) = \delta f'(x)u'(F(x,y)) .$$

Moreover, F is strictly increasing in y for each x and strictly decreasing in x for each y. The properties of F are independent of assumptions governing g. We may write a Ramsey equilibrium, after the turnpike property takes hold, as the system

(5) $$\begin{aligned} y_{t+1} &= F(x_t, y_t) \\ x_{t+1} + y_{t+1} &= g(x_t) \end{aligned}$$

where (x_0, y_0) are given and $0 < x_t$, $y_t < a$ for all t.

A stationary Ramsey equilibrium is determined by the relation $\delta f'(\bar{x}) = 1$. Clearly $F(\bar{x},y) \equiv y$.

We want to obtain a dynamical system from (5) defined on $[0,a] \times [0,a]$. We will extend F to $([0,a] \times [0,a])\setminus\{(0,0)\}$ by the following definitions:

$$F(x,0) = \lim_{y \downarrow 0} F(x,y) , \quad x > 0 ;$$
$$F(0,y) = \lim_{x \downarrow 0} F(x,y) , \quad y > 0 .$$

Note that $F(0,y) = +\infty$ if $y > 0$. We observe the fact that $a \geq x > \bar{x}$ implies $F(x,y) \leq F(\bar{x},y) = y$. Hence $F(\bar{x},0) = 0$. Indeed, $F(x,0) = 0$ for all $x > 0$.

The dynamical system on $[0,a] \times [0,a]$ is

(6) $$\begin{aligned} y_\nu &= \min(F(x,y), g(x)) \\ x_\nu + y_\nu &= g(x) \end{aligned}$$

for $(x,y) \in (0,a) \times (0,a)$. If $x = 0$, $0 < y \leq a$, then $y_\nu = 0$ and $x_\nu = 0$. We define $F(0,0) \equiv \pi$ where π is arbitrarily chosen in $[0,\infty)$. If $y = 0$, $0 \leq x \leq a$, then $y_\nu = 0$ and $x_\nu = g(x)$. The initial data is (x_0, y_0) .

The sign of g' is crucial for the analysis of (6). The maintained assumptions on f do not imply $g' > 0$. In [1] the additional assumption $(f'(x)x)' > 0$ was used to imply $g' > 0$. In that case, the economy's capital stock was shown to converge (eventually) monotonically to $\bar{x}$. If $g' > 0$ did not hold, an example was found of a cycle of period two in the equilibrium capital stocks. With this background, we now state the fundamental

<u>Income Monotonicity Assumption</u>: $g' > 0$.

This postulate is to be taken as a blanket assumption governing the analysis for the remainder of this paper without further mention.

We conclude this section by recording the properties of g used in our analysis of the orbits of (6).

(g1) $0 \leq g(x) < f(x)$ for all $x \geq 0$.

Condition (g1) follows since concavity of f implies

$$f(x) = f(\varepsilon) + \int_\varepsilon^x f'(z)dz \geq f(\varepsilon) + (x-\varepsilon)f'(x) ,$$

and $f(x) > f'(x)x$, so

$$g(x) = (1 - \tfrac{1}{H})f'(x)x + \tfrac{1}{H}f(x) < f(x) \quad \text{for } x > 0 ,$$

(g2)

$$g(0) = 0 , \ g(a) < a .$$

The first part follows from $f(0) = 0$ and (g1). The inequality follows from $g(a) < f(a) = a$. As $f(\bar{x}) > f'(\bar{x})\bar{x}$, we also see that

$$g(\bar{x}) > \bar{x}f'(\bar{x}) = \delta^{-1}\bar{x} > \bar{x}$$

as $\delta^{-1} > 1$. Therefore:

(g3) there exists $x_\infty \in (\bar{x}, a)$ such that $x_\infty = g(x_\infty)$.

III. <u>The Convergence Theorem</u>

The main result in this paper is the following:

<u>Convergence Theorem</u>. The orbit $((x_t,y_t))$ of the dynamical system (6) is always convergent to some $(x_\infty,y_\infty) \in [0,a] \times [0,a]$. The sequences (x_t) and (y_t) are eventually monotonic. Further:

(i) If $x_\infty = 0$, then $y_\infty = 0$ and $(x_t,y_t) = (0,0)$ eventually.

(ii) If $y_\infty = 0$, then $x_\infty = g(x_\infty)$ and there is at least one such $x_\infty \in (\bar{x},a)$.

(iii) If $y_\infty > 0$, then $x_\infty > 0$ and $x_\infty = \bar{x}$, $y_\infty = g(\bar{x}) - \bar{x}$. In this last case the sequences (x_t) and (y_t) are monotonic, one nonincreasing the other nondecreasing.

We note that Ramsey equilibrium paths correspond to the orbits for which alternative (iii) is valid. The proof of the Convergence theorem follows after the presentation of a series of lemmas.

<u>Lemma 1.</u> If $x_t > \bar{x}$ and $x_t \geq t_{t-1}$, then $x_{t+1} > x_t$ and $y_t > y_{t+1}$.

<u>Proof</u>: If $x_t > \bar{x}$, then $y_t \leq g(x_{t-1}) < g(x_t)$ by income monotonicity and $F(x_t,y_t) < F(\bar{x},y_t) = y_t$. Hence

$$y_{t+1} = \min(F(x_t,y_t),\ g(x_t)) < y_t \ .$$

Therefore

$$x_{t+1} = g(x_t) - y_{t+1} > g(x_{t-1}) - y_t = x_t \ .$$

Q.E.D.

<u>Corollary 1.</u> If $x_{t_0} > \bar{x}$ and $x_{t_0} \geq x_{t_0-1}$ for some $t_0 \geq 1$, then

$x_{t_0-1} < x_{t_0} < x_{t_0+1} < \cdots$, hence $\lim_{t\to\infty} x_t = x_\infty$ exists;

$y_{t_0} > y_{t_0+1} > \cdots$, hence $\lim_{t\to\infty} y_t = y_\infty$ exists .

Moreover, $y_\infty = 0$ and $g(x_\infty) = x_\infty > \bar{x}$.

Proof: The monotonicity is an obvious consequence of Lemma 1. Therefore, the two limits x_∞ and y_∞ exist. So

$$y_\infty = \min(F(x_\infty, y_\infty), g(x_\infty))$$

$$x_\infty + y_\infty = g(x_\infty) .$$

Clearly $x_\infty > \bar{x}$. If $y_\infty > 0$ as well, then

$$y_\infty \leq F(x_\infty, y_\infty) < F(\bar{x}, y_\infty) = y_\infty ,$$

which is impossible. Hence $y_\infty = 0$ and $g(x_\infty) = x_\infty$.

Q.E.D.

Remark 1. If $x_t < \bar{x}$, then either $x_{t+1} = 0$ or $y_{t+1} = F(x_t, y_t) > y_t$.

Proof: If $x_{t+1} > 0$, then $y_{t+1} < g(x_t)$. Hence $y_{t+1} = F(x_t, y_t) > F(\bar{x}, y_t) = y_t$.

Q.E.D.

Lemma 2. If $x_t < \bar{x}$, $x_t \leq x_{t-1}$, then either $x_{t+1} = 0$ or $y_{t+1} = F(x_t, y_t) > y_t$ and $x_{t+1} < x_t$.

Proof: If $x_{t+1} > 0$ then the first relation follows from Remark 1. The income monotonicity assumption implies $g(x_t) \leq g(x_{t-1})$, so

$$x_{t+1} = g(x_t) - y_{t+1} < g(x_{t-1}) - y_t = x_t .$$

Q.E.D.

Corollary 2. If $x_{t_0} < \bar{x}$ and $x_{t_0} \leq x_{t_0-1}$, then eventually $x_t = 0$ and $y_t = 0$.

Proof: Given Lemma 2, if the conclusion was not valid, we must have for some t_0 that

$$x_{t_0-1} > x_{t_0} > \cdots \quad \text{hence } \lim_{t\to\infty} x_t = x_\infty \text{ exists,}$$

$$y_{t_0} < y_{t_0+1} < \cdots \quad \text{hence } \lim_{t\to\infty} y_t = y_\infty \text{ exists,}$$

and $x_\infty + y_\infty = g(x_\infty)$. Obviously $0 < y_\infty \leq a$; now passing to the limit we obtain

$$y_\infty = \min(\lim_{t\to\infty} F(x_t, y_t), g(x_\infty))$$

as well. If $x_\infty = 0$, then $y_\infty = 0$, which is impossible. Thus $x_\infty > 0$ and $y_\infty = \lim_{t\to\infty} F(x_t, y_t) = F(x_\infty, y_\infty)$. But this implies $x_\infty = \bar{x}$, which is impossible.

Q.E.D.

Up to this point we have shown that the limit (x_∞, y_∞) exists whenever either there is a $t_0 \geq 1$ such that $x_{t_0} > \bar{x}$ and $x_{t_0} \geq x_{t_0-1}$, or a t_0 such that $x_{t_0} < \bar{x}$ and $x_{t_0} \leq x_{t_0-1}$. The remaining case to be covered is: For any t one of the following three alternatives holds:

(a) if $x_t > \bar{x}$ then $x_t < x_{t-1}$

(b) if $x_t < \bar{x}$ then $x_t > x_{t-1}$, or

(c) $x_t = \bar{x}$.

In case (a) we have $x_t < x_{t-1} < \cdots < x_0$. In case (b) we have $x_t > x_{t-1} > \cdots > x_0$. Now cases (a) and (b) cannot occur simultaneously. Therefore, the sequences (x_t) are monotonic (perhaps not strictly monotonic) since we may eventually end up with case (c) or be in case (c) for all t . This preparation leads to the proof of our theorem.

Proof of Convergence Theorem: If the orbit satisfies the condition in Corollary 1, then the two sequences (x_t) and (y_t) are eventually strictly monotonic and $\bar{x} < x_\infty \leq a$, $y_\infty = 0$, $x_\infty = g(x_\infty)$. If the sequence (x_t) is as in Corollary 2, then the two sequences become eventually zero. Notice that these two cases lead to the alternatives (i) and (ii) respectively.

In the remaining cases we have either

(A) $\quad x_0 \leq x_1 \leq \cdots \leq x_t \leq \cdots \leq \bar{x}$, or

(B) $\quad x_0 \geq x_1 \geq \cdots \geq x_t \geq \cdots \geq \bar{x}$.

In either situation $\lim_{t\to\infty} x_t = x_\infty$ exists and $0 < x_\infty < a$. Moreover $y_{t+1} = g(x_t) - x_{t+1}$ converges to $g(x_\infty) - x_\infty$ as $t \to \infty$. Therefore the $\lim_{t\to\infty} y_t = y_\infty = g(x_\infty) - x_\infty$ exists. In case (A) we have from the income monotonicity property

$$y_{t+1} = \min(F(x_t, y_t), g(x_t)) \geq \min(F(\bar{x}, y_t), g(x_t))$$
$$= \min(y_t, g(x_t)) \geq \min(y_t, g(x_{t-1})) = y_t \ .$$

This implies $(y_t)_{t\geq 0}$ is also monotonic. In case (B) we likewise have

$$y_{t+1} = \min(F(x_t, y_t), g(x_t)) \leq \min(F(\bar{x}, y_t), g(x_t))$$
$$= \min(y_t, g(x_t)) \leq \min(y_t, g(x_{t-1})) = y_t \ .$$

Once again $(y_t)_{t\geq 0}$ is monotonic. If $y_\infty = 0$, then $g(x_\infty) = x_\infty$. If $g(x_\infty) \neq x_\infty$ then $F(x_\infty, y_\infty) = y_\infty$ and $0 < x_\infty < a$ implies $x_\infty = \bar{x}$. Therefore $y_\infty = g(\bar{x}) - \bar{x} > 0$, which is impossible. In this way, cases (A) and (B) fall under either of the alternatives (ii) or (iii).

Q.E.D.

IV. Local Dynamics

The Convergence Theorem treated the pair (x_0, y_0) as given data. In the underlying economic model, only x_0 is specified; typically y_0 is chosen so that the orbit $((x_t, y_t))$ converges to $(\bar{x}, \bar{y})$. We address this problem for the system (6) by means of a linear analysis of the dynamics near the fixed point $(\bar{x}, \bar{y})$ where $\bar{y} = g(\bar{x}) - \bar{x}$ and $F(\bar{x}, \bar{y}) = \bar{y}$.

For (x, y) near $(\bar{x}, \bar{y})$ we have the map

$$x_\nu = g(x) - F(x,y)$$
$$y_\nu = F(x,y) \ .$$

The differential of this map evaluated at $(\bar{x},\bar{y})$ is:

$$J = \begin{bmatrix} f'(\bar{x}) - F'_x(\bar{x},\bar{y}) & -F'_y(\bar{x},\bar{y}) \\ F'_x(\bar{x},\bar{y}) & F'_y(\bar{x},\bar{y}) \end{bmatrix} ,$$

where $F'_x(\bar{x},\bar{y}) = -\delta f''(\bar{x})u'(\bar{y})/u''(\bar{y}) = -\eta < 0$ and $F'_y(\bar{x},\bar{y}) = 1$. Let $g'(\bar{x}) = (1 - \frac{1}{H})f''(\bar{x})\bar{x} + f'(\bar{x}) = \gamma > 0$ by the Income Monotonicity assumption. The eigenvalues satisfy

$$\lambda^2 - (\gamma + \eta + 1)\lambda + \gamma = 0 \ .$$

A routine calculation shows both roots are real, $\lambda_1\lambda_2 = \gamma > 0$, and $\lambda_1 + \lambda_2 = \gamma + \eta + 1$. Therefore $0 < \lambda_1\lambda_2 = \gamma < \eta + \gamma = \lambda_1 + \lambda_2 - 1$ implies $\lambda_1 + \lambda_2 > 1$. As sign λ_1 = sign λ_2 and $\lambda_1 + \lambda_2 > 1$, it follows that $\lambda_1 > 0$ and $\lambda_2 > 0$. Neither root equals one. As $\lambda_1\lambda_2 < \lambda_1 + \lambda_2 - 1$, it follows that $\lambda_1(\lambda_2 - 1) < \lambda_2 - 1$. Thus, if $\lambda_2 > 1$, then $\lambda_1 < 1$ and conversely. Hence $(\bar{x},\bar{y})$ is a saddle point. According to the case (iii) of the Convergence Theorem, the trajectory $((x_t,y_t))_{t=0}^{\infty}$ is a Ramsey equilibrium if and only if it lies on the (global) stable manifold M of $(\bar{x},\bar{y})$. By the above local analysis, near $(\bar{x},\bar{y})$ the manifold M is one dimensional and transversal to the vertical direction at $(\bar{x},\bar{y})$. Indeed the tangent at $(\bar{x},\bar{y})$ to M is given by the equation

$$\eta x + (1 - \lambda_1)y = 0$$

where λ_1 is the smallest eigenvalue of J (i.e. $\lambda_1 \in (0,1)$) (see [4] Ch.II). Recalling again the case (iii) in the convergence proposition we easily see that M is the graph of a strictly decreasing function with a domain of definition containing a neighborhood N of $\bar{x}$. We can thus state the following.

Saddle Point Theorem. The fixed point $(\bar{x},\bar{y})$ of (6) is a saddle point, with an one-dimensional stable manifold M which is transversal to the vertical direction at $(\bar{x},\bar{y})$. An orbit $\{(x_t,y_t)\}_{t=0}^{\infty}$ is a Ramsey equilibrium if and only if it lies on M. Moreover for any x_0 in a small neighborhood N of $\bar{x}$ there exists one and only one y_0 (namely the one satisfying $(x_0,y_0) \in M$) such that the orbit $\{(x_t,y_t)\}_{t=0}^{\infty}$ is a Ramsey equilibrium.

V. Concluding Comments

The original Ramsey equilibrium system was not defined on all of $[0,a] \times [0,a]$. We extended this system to this domain and observed that a Ramsey equilibrium was an orbit contained in the interior of this region. We showed for the extended system under the Income Monotonicity hypothesis that orbits which are not monotonic converge to the boundary of the system's domain. Moreover, all other parts converge to $(x_\infty,y_\infty) = (\bar{x},\bar{y})$ and these are the only Ramsey equilibria. Morevoer they lie on an one dimensional graph passing through $(\bar{x},\bar{y})$.

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