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# Generalized Periodic and Antiperiodic Solutions for the Heat Equation in $R^1$

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## 0. INTRODUCTION

The purpose of this paper is to prove, via $m$-dissipative operators, the existence and, in some cases, the uniqueness of solutions of the following initial-boundary-value problems

$(P_1)$

$$u_t(t,x) = u_{xx}(t,x) \qquad t>0,\ 0<x<1 \tag{0-1}$$

$$u(t,0) = u(t,1),\ u_x(t,0) = u_x(t,1) + \alpha u(t,0) \quad t \geq 0,\ \alpha \geq 0 \tag{0-2}$$

$$u(0,x) = u_0(x) \qquad 0 \leq x \leq 1 \tag{0-3}$$

$(P_2)$

$$u_t(t,x) = u_{xx}(t,x) \qquad t>0,\ 0<x<1 \tag{0-1}$$

$$u(t,0) = u(t,1) + \beta u_x(t,0),\ u_x(t,0) = u_x(t,1) \quad t \geq 0,\ \beta \geq 0 \tag{0-4}$$

$$u(0,x) = u_0(x) \qquad 0 \leq x \leq 0 \tag{0-3}$$

where $u_s$ denotes the partial derivative of $u$ with respect to $s$.

$(P_3)$

$$u_t(t,x) = u_{xx}(t,x) + f(t,u(t,x)) \qquad t>0,\ 0<x<1 \tag{0-5}$$

$$u(t,0) = -u(t,1),\quad u_x(t,0) = -u_x(t,1) \qquad t \geq 0 \tag{0-6}$$

$$u(0,x) = u_0(x) \qquad 0 \leq x \leq 1 \tag{0-3}$$

where $f : [0, +\infty) \times [0, 1] \longrightarrow R$ is a continuous function. The case that $u \mapsto f(t, u)$ is $\gamma - H\ddot{o}lder$ continuous with $0 < \gamma < 1$ is also dicussed.

The existence and uniqueness of solutions of $(P_1)$ and $(P_2)$ with $\alpha > 0$ and $\beta > 0$ is studied via $m$–dissipative operators in $X = C([0, 1]; R) \equiv C([0, 1])$, while $(P_3)$ is studied via compact analytic $C_0$–semigroup in $X$. In the case $\alpha = \beta = 0$, $(P_1)$ was studied by Pazy.[**3**, p234] Precisely, Pazy proved that the second order differential operator with periodic boundary conditions

$$A_p u = u'', \quad D(A) = \{u \in X_p, u', u'' \in X_p\} \tag{0-7}$$

where $X_p = \{u \in C([0, 1]), \quad u(0) = u(1)\}$ generates a compact analytic semigroup in $X_p$.

Denote by

$$X_{ap} = \{u \in C([0, 1]), \ \ u(0) = -u(1)\}$$

$$A_{ap} u = u'', \ \ D(A) = \{u \in X_{ap}, \ \ u', u'' \in X_{ap}\} \tag{0-7'}$$

Mainly,we prove in section 2 that the second order differential operator $A_{ap} u = u''$ with antiperiodic boundary conditions generates also a compact analytic semigroup in $X_{ap}$. In above notations, the subsripts $p$ and $ap$ are abbreviations of periodic and antiperiodic respectively.

In section 1, we prove that $Au = u''$ with generalized boundary conditions (0-2) or (0-4) is $m$–dissipative for every $\alpha > 0$ or $\beta > 0$.In section3, the existence and uniqueness of periodic/antiperiodic solutions for some ordinary second order differential equations are discussed, via the surjectivity of $m$–dissipative operators involving $A_p$ or $A_{ap}$.

**Remark 0.1** In the next sections we announce some preliminary results on this subject. More complete results will be published later. In the case of $L^2$ spaces, these problems have been recently studied in [1] and [2].

## I. GENERALIZED PERIODIC BOUNDARY VALUE PROBLEMS

### 1.1 The Problem $(P_1)$.

In this section we deal with the following perturbed periodic boundary value problem

$(P_1)$

$$u_t(t,x) = u_{xx}(t,x) \qquad t > 0,\ \ 0 < x < 1 \tag{0-1}$$

$$u(t,0) = u(t,1),\ \ u_x(t,0) = u_x(t,1) + \alpha u(t,0) \quad t \geq 0,\ \ \alpha \geq 0 \tag{0-2}$$

$$u(0,x) = u_0(x) \qquad 0 \leq x \leq 1 \tag{0-3}$$

Denote by $X = C([0,\ 1])$, and $A_\alpha u = u''$ with

$$D(A_\alpha) = \{u \in X,\ \ u',\ u'' \in X,\ u(0) = u(1),\ \ u'(0) = u'(1) + \alpha u(0) \quad \alpha \geq 0\} \tag{1-1}$$

The following result holds.

THEOREM 1.1. *The linear operator $A_\alpha$ defined above is $m$-dissipative in $X$.*

PROOF: We need to show that

(i) For every $\lambda > 0$ and $g \in X$, there is a unique solution $w$ of the problem

$$\begin{gathered} \lambda^2 w(x) - w''(x) = g(x) \\ w(0) = w(1), \quad w'(0) = w'(1) + \alpha w(0) \end{gathered} \tag{1-2}$$

i.e. $R(\mu I - A) = X, \quad \forall\mu > 0$, and

(ii) $\|w\| \le \frac{1}{\lambda^2}\|g\|$, or equivalently,

$$\|(\mu I - A)^{-1})\| \le \frac{1}{\mu}, \qquad \forall\mu > 0. \tag{1-3}$$

Let $g \in X$ and $\lambda > 0$. A direct computation yields a solution of (1.2), which is given by

$$w(x) = \frac{1}{2\lambda\sinh\frac{\lambda}{2}}\Big[\int_0^x \cosh\lambda(x-y-\tfrac{1}{2})\, g(y)\,dy + \int_x^1 \cosh\lambda(x-y+\tfrac{1}{2})\, g(y)\,dy$$

$$-\frac{\alpha\cosh\lambda(x-\frac{1}{2})}{\alpha\cosh\frac{\lambda}{2} + 2\lambda\sinh\frac{\lambda}{2}}\int_0^1 \cosh\lambda(y-\tfrac{1}{2})\, g(y)\,dy\Big] \tag{1-4}$$

So (i) is done. To prove (ii) we rewrite (1.4) as

$$w(x) = \frac{1}{2\lambda\sinh\frac{\lambda}{2}}\Big[\int_0^x E_1(\lambda,\alpha,x,x)\, g(y)\,dy + \int_x^1 E_2(\lambda,\alpha,x,y)\, g(y)\,dy\Big] \tag{1-5}$$

where

$$E_1(\lambda,\alpha,x,y) = \frac{\alpha\sinh\lambda(1-x)\sinh\lambda y + 2\lambda\sinh\frac{\lambda}{2}\cosh\lambda(x-y-\frac{1}{2})}{\alpha\cosh\frac{\lambda}{2} + 2\lambda\sinh\frac{\lambda}{2}} \ge 0,$$

and

$$E_2 = \frac{\alpha\sinh\lambda x\sinh\lambda(1-y) + 2\lambda\sinh\frac{\lambda}{2}\cosh\lambda(x-y+\frac{1}{2})}{\alpha\cosh\frac{\lambda}{2} + 2\lambda\sinh\frac{\lambda}{2}} \ge 0.$$

Therefore we have the following estimate

$$|w(x)| \le \frac{\|g\|}{2\lambda\sinh\frac{\lambda}{2}}\Big[\int_0^x E_1(\lambda,\alpha,x,y)\,dy + \int_x^1 E_2(\lambda,\alpha,x,y)\,dy\Big]$$

$$= \frac{\|g\|}{\lambda^2}\Big[1 - \frac{\alpha\cosh(x-\frac{1}{2})}{\alpha\cosh\frac{\lambda}{2} + 2\lambda\sinh\frac{\lambda}{2}}\Big]$$

$$\le \frac{\|g\|}{\lambda^2},$$

which completes the proof.

Now by Theorem 2.1 in [**4**, p66] we get

THEOREM 1.2. *For every $u_0 \in D(A_\alpha)$, the problem $(P_1)$ has a unique mild (integral) solution*

$$u(t) = S_\alpha(t)u_0, \qquad t \geq 0$$

*where $S_\alpha(t)$ is the $C_0$-strongly continuous semigroup generated by $A_\alpha$ on $\overline{D(A_\alpha)}$.*

### 1.2 The Problem $(P_2)$.

By the same approach as in section 1.1, we can discuss the initial- boundary-value problem $(P_2)$

$$u_t(t,x) = u_{xx}(t,x) \qquad t > 0, \ \ 0 < x < 1 \tag{0-1}$$

$$u(t,0) = u(t,1) + \beta u_x(t,0), \ \ u_x(t,0) = u_x(t,1) \quad t \geq 0, \ \ \beta \geq 0 \tag{0-4}$$

$$u(0,x) = u_0(x) \qquad 0 \leq x \leq 0 \tag{0-3}$$

Let $A_\beta u = u''$ with

$$D(A_\beta) = \{u \in X, \ \ u', u'' \in X, \ \ u(0) = u(1) + \beta u'(0), \ \ u'(0) = u'(1), \ \ \beta \geq 0\} \tag{1-5}$$

Then we have the following result.

THEOREM 1.3. *The operator $A_\beta$ defined by (1–5) is m–dissipative in $X$.*

PROOF: Similar to the proof of Theorem 1.1, we now consider, for $\lambda > 0$ and $g \in X$, the boundary-value problem

$$\begin{aligned} &\lambda^2 w(x) - w''(x) = g(x) \\ &w(0) = w(1) + \beta w'(0), \quad w'(0) = w'(1), \quad \beta \geq 0. \end{aligned} \tag{1-6}$$

The solution of (1.6) is given by

$$w(x) = \frac{1}{2\lambda \sinh \frac{\lambda}{2}}[\int_0^x \cosh \lambda(x-y-\tfrac{1}{2})\, g(y)\, dy + \int_x^1 \cosh \lambda(x-y+\tfrac{1}{2})\, g(y)\, dy$$

$$+ \frac{\lambda\beta \sinh \lambda(x-\frac{1}{2})}{2\sinh \frac{\lambda}{2} + \lambda\beta \cosh \frac{\lambda}{2}} \int_0^1 \sinh \lambda(y-\tfrac{1}{2})\, g(y)\, dy], \qquad (1\text{-}7)$$

with

$$|w(x)| \le \frac{\|g\|}{\lambda^2}, \qquad 0 \le x \le 1,$$

which implies that $A_\beta$ is dissipative and $\|(\mu I - A)^{-1}\| \le \frac{1}{\mu}$, $\forall \mu \ge 0$. Hence $A_\beta$ generates a $C_0$ -semigroup on $\overline{D(A_\beta)}$.

Therefore , the following holds.

THEOREM 1.4. *For every $u_0 \in D(A_\beta)$, $\beta \ge 0$, the problem $(P_2)$ has a mild (integral) solution*

$$u(t) = S_\beta(t)u_0,$$

*where $S_\beta(t)$ is the $C_0$–semigroup generated by $A_\beta$ on $\overline{D(A_\beta)}$.*

## II. ANTIPERIODIC BOUNDARY CONDITIONS IN $C[0,1]$

Consider the initial value problem with anti–periodic boundary conditions $(P_3)$

$$u_t(t,x) = u_{xx}(t,x) + f(t,u(t,x)) \qquad t>0,\ 0<x<1 \qquad (0\text{-}5)$$

$$u(t,0) = -u(t,1), \quad u_x(t,0) = -u_x(t,1) \qquad t \ge 0 \qquad (0\text{-}6)$$

$$u(0,x) = u_0(x) \qquad 0 \le x \le 1. \qquad (0\text{-}3)$$

We will prove, under suitable conditions, the existence of local and global classical solutions.The asymptotic behavior of the global solutions as $t \to +\infty$ is also discussed.

Let

$$X_{ap} = \{u \in C([0,1]),\ u(0) = -u(1)\} \qquad \|u\| = \max_{0\le x\le 1} |u(x)|$$

$$A_{ap}u = u'',\ \ D(A) = \{u \in X_{ap},\ u', u'' \in X_{ap}\} \tag{0-7}$$

THEOREM 2.1. *The linear operator $A_{ap}$ defined above is the infinitesimal generator of a compact analytic semigroup $T(t),\ t \ge 0$ on $X_{ap}$.*

PROOF: Let $g \in X_{ap}$ $\lambda = \rho e^{i\theta}$ with $\rho > 0$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Consider the boundary value problem

$$\begin{aligned} \lambda^2 u(x) - u''(x) &= g(x) \\ u(0) = -u(1), \quad & u'(0) = -u'(1). \end{aligned} \tag{2-1}$$

The unique solution of (2–1) is given by

$$u(x) = \frac{1}{2\lambda\cosh\frac{\lambda}{2}}[-\int_0^x \sinh\lambda(x-y-\tfrac{1}{2})g(y)\,dy + \int_x^1 \sinh\lambda(x-y+\tfrac{1}{2})g(y)\,dy]$$

$$= (R(\lambda^2; A_{ap})g)(x), \qquad \lambda \ne 0. \tag{2-2}$$

Denote by $\mu = \mathrm{Re}\lambda = \rho\cos\theta > 0$, applying elementary inequalities

$$|\cosh\frac{\lambda}{2}| \ge |\sinh\frac{\lambda}{2}| \ge \sinh\frac{\mu}{2},$$

$$|\sinh\lambda(x-y\pm\tfrac{1}{2})| \le |\cosh\lambda(x-y\pm\tfrac{1}{2})| \le \cosh\mu(x-y\pm\tfrac{1}{2}),$$

we get

$$|u(x)| \leq \frac{\|g\|}{2|\lambda \cosh \frac{\lambda}{2}|}\Big[\int_0^x |\sinh \lambda(x-y-\tfrac{1}{2})|dy + \int_x^1 |\sinh \lambda(x-y+\tfrac{1}{2})|dy\Big]$$

$$\leq \frac{\|g\|}{2|\lambda|\sinh \frac{\mu}{2}}\Big[\int_0^x \cosh \mu(x-y-\tfrac{1}{2})dy + \int_x^1 \cosh \mu(x-y+\tfrac{1}{2})dy\Big]$$

$$\leq \frac{\|g\|}{\cos\theta|\lambda|^2},$$

which implies that

$$\|R(\lambda,\ A_{ap})\| \leq \frac{1}{|\lambda|\cos(\frac{1}{2}\arg\lambda)}, \qquad \forall\lambda \neq 0.$$

where $\lambda \in \Sigma(\theta_0) = \{\lambda : \ |\arg\lambda| < \theta_0\}$ for fixed $\frac{\pi}{2} < \theta_0 < \pi$. Furthermore ,

$$\|R(\lambda,\ A_{ap})\| \leq \frac{M}{|\lambda|} \qquad \text{where } M = \frac{1}{\cos\frac{\theta_0}{2}}.$$

In fact we have just proved in above that $\rho(A_{ap}) \supset \Sigma(\theta_0)$. Then by Hille and Pazy [**3**, p60], $A_{ap}$ generates an analytic semigroup $T(t),\ \ t \geq 0$. Therefore, there exists a constant $C > 0$ such that

$$\|T(t+h) - T(t)\| \leq h\|A_{ap}T(t)\| \leq \frac{Ch}{t}, \qquad t > 0$$

Hence T(t) is continuous in the uniform operator topology for $t > 0$. And for $\lambda \in \Sigma(\theta_0)$, $R(\lambda,\ A_{ap})$ maps a bounded set in $X_{ap}$ into a bounded set in $D(A_{ap})$. Therefore, by Arzela–Ascoli theorem, $R(\lambda,\ A_{ap})$ is a compact operator, which implies that $T(t),\ \ t > 0$ is a compact semigroup.

In the Banach space $X_{ap}$, the problem $(P_3)$ can be written as:

$$\begin{aligned} u' &= A_{ap}u + Fu, \qquad t \geq 0 \\ u(0) &= u_0 \end{aligned} \tag{2-3}$$

where $F: C \longrightarrow C$ is given by

$$(Fu)(x) = f(u(x)), \qquad \forall u \in C = C[0,\ 1].$$

**Definition 2.1.** A function $u$ is said to be a mild solution of (2.1) (respectively, (2.3)) on an interval $[0,\ T_0]$, $T_0 > 0$ if $u$ is continuous from $[0,\ T_0]$ into $X_{ap}$ and

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(u(s))\,ds, \qquad 0 \leq t \leq T_0.$$

By Theorem 2.1 and Pazy [**3**, p235], we have

THEOREM 2.2. *For every continuous real valued function $f$ and every $u_0 \in X_{ap}$, there exists a $t_0 > 0$ such that the initial boundary value problem $(P_3)$ has a unique mild solution on $[0,\ t_0)$ and either $t_0 = \infty$ or if $t_0 < \infty$, then*

$$\lim_{t \uparrow t_0} \|u(t,x)\| = \infty.$$

Assume now that $f: R \longrightarrow R$ is *Hölder*-continuous.

$$|f(s) - f(t)| \leq L|s-t|^\alpha, \qquad 0 < \alpha < 1. \tag{2-4}$$

Define $F: X_{ap} \longrightarrow X_{ap}$ by $(Fu)(t) = f(u(t))$. Then $F$ is also *Hölder*-continuous, since $|f(u(t)) - f(v(t))| \leq L|u(t) - v(t)|^\alpha$ leads to

$$|(Fu)(t) - (Fv)(t)| \leq L|u(t) - v(t)|^\alpha \leq L\|u - v\|^\alpha,$$

i.e. $\qquad \|Fu - Fv\| \leq \|u - v\|^\alpha, \qquad 0 < \alpha < 1.$

THEOREM 2.3. *Let $f : R \to R$ be Hölder- continuous(as indicated by (2-4)). Then for every $u_0 \in X_{ap}$, there is $T = T(u_0, f) > 0$ such that $(P_3)$ has a unique solution $u \in C([0, T]; R) \cap C^1((0, T); R)$.*

PROOF: The abstract form of $(P_3)$ in $X_{ap}$ is

$$u' = A_{ap}u + Fu, \qquad u(0) = u_0$$

with $A_{ap}$–compact analytic semigroup generator, $0 \in \rho(A_{ap})$. $F$ is *Hölder*–continuous. So the result follows.

## III. APPLICATIONS TO SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

The $m$–dissipativity of the second order differential operator $Au = u''$ with generalized periodic boundary conditions as in $(P_1)$ and $(P_2)$ can be used to prove the existence and uniqueness of solutions of the second order equation

$$u''(t) = B(t)u(t) + \lambda u(t) + f(t) \qquad 0 \le t \le 1, \ \lambda > 0 \tag{3-1}$$

where $f \in C([0, 1]; R)$, with boundary conditions

$$u(0) = u(1), \quad u'(0) = u'(1) + \alpha u(0), \qquad \alpha \ge 0. \tag{3-2}$$

or

$$u(0) = u(1) + \beta u'(0), \quad u'(0) = u'(1), \qquad \beta \ge 0. \tag{3-3}$$

On the other hand we can also discuss the problem

$$u''(t) = B(t)u(t) + \lambda u(t) + f(t) \qquad 0 \leq t \leq 1, \ \ \lambda \geq 0 \qquad (3\text{-}4)$$

$$u(t+1) = -u(t) \qquad t \geq 0 \qquad (3\text{-}5)$$

with $t \longrightarrow B(t)x$ 1-periodic in $t$ and $u \longrightarrow B(t)u$ an odd function. The mapping $(t,x) \mapsto B(t)x$ is continuous.

THEOREM 3.1. *Let $B : R \longrightarrow R$ be a continuous, nondecreasing, odd function. Then*

$$u''(t) = Bu(t) + f(t) + \lambda u(t) \qquad \lambda > 0$$

$$u(0) = -u(1), \quad u'(0) = -u'(1)$$

*with $f$ 1-antiperiodic, has a unique 1-antiperiodic solution in $C^2([0,\ 1];\ R)$.*

PROOF: By Theorem 2.1, the operator

$$Au = u'', \qquad u \in X_{ap} = \{u \in C([0,1]),\ u(0) = -u(1)\}$$

generates an analytic compact semigroup. In particular, $A$ is $m$–dissipative on $X_{ap}$.

Let $\widetilde{B} : C([0,\ 1]) \longrightarrow C([0,\ 1])$ be defined by

$$(\widetilde{B}u)(t) = Bu(t).$$

For $u \in X_{ap}$, $u(0) = -u(1)$. Therefore

$$(\widetilde{B}u)(0) = Bu(0) = -Bu(1) = -(\widetilde{B}u)(1).$$

So $\widetilde{B} : X_{ap} \to X_{ap}$.

Since $B$ is monotone accretive, $\widetilde{B}$ is monotone accretive on $X_{ap}$. Then $-A + \widetilde{B}$ is $m$-accretive. Thus

$$R(\lambda I - A + \widetilde{B}) = X_{ap}, \qquad \lambda > 0.$$

So given $-f \in X_{ap}$, there exists $u \in X_{ap}'$ such that

$$\lambda u - u'' + \widetilde{B}u = -f$$

But that means $u$ is the solution of the problem

$$u''(t) = Bu(t) + \lambda u(t) + f(t), \qquad \lambda > 0$$
$$u(0) = -u(1), \quad u'(0) = -u'(1)$$

As $\lambda \downarrow 0$, we get the desired conclution.

With the same approach, the following holds.

THEOREM 3.2. *Suppose that:*

(1) $(t, x) \to B(t)x$ *from* $[0,\ 1] \times R \to R$ *is continuous;*

(2) *For each* $t \in [0,\ 1]$, $x \mapsto B(t)x$ *is nondecreasing.*

*Then for every* $f \in C([0,\ 1])$, $\alpha \geq 0$, $\beta \geq 0$ *and* $\lambda > 0$, *the problem (3.1)and (3.2)(respetively (3.1) and (3.3)) has a unique solution.*

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